Math 434/734

Homework 7

1. Consider the tranformation

$$\begin{aligned} u &= x^3 - y \\ v &= 3x^3 + 2y \end{aligned}$$

Show that the Jacobian is zero on the y-axis. Compute the partial derivatives

$$\frac{\partial x}{\partial u}, \quad \frac{\partial x}{\partial v}, \quad \frac{\partial y}{\partial u}, \quad \frac{\partial y}{\partial v}.$$

However, show that the transformation is invertible over the entire xy-plane.

We have

$$\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = \begin{pmatrix} 3x^2 & -1 \\ 9x^2 & 2 \end{pmatrix}$$

which has determinant $6x^2 + 9x^2 = 15x^2$. This is nonzero, as long as $x \neq 0$. So we can apply the inverse function theorem when $x \neq 0$. This is off the y-axis.

The map is, however, one to one on the whole plane $\mathbb{R}^2$. Assume $f(x_1, y_1) = f(x_2, y_2)$ then

$$\begin{aligned} x_1^3 - y_1 &= x_2^3 - y_2 \\ 3x_1^3 + 2y_1 &= 3x_2^3 + 2y_2 \end{aligned}$$

We multiply the first equation by -3 and add to the second equation to get

$$5y_1 = 5y_2 \Leftrightarrow y_1 = y_2.$$

Then the first equation gives $x_1^3 = x_2^3 \Leftrightarrow x_1 = x_2$ as the function x^3 is one-to-one.

The map is also onto $\mathbb{R}^2$: Let (u, v) be given. We need to find a pair (x, y) with $f(x, y) = (u, v)$. We need to solve the system

$$\begin{aligned} u &= x^3 - y \\ v &= 3x^3 + 2y \end{aligned}$$

Again we multiply the first equation by -3 and add to the second equation to get $v - 3u = 5y \Leftrightarrow y = (u - 3v)/5$. Then $x = \sqrt[3]{u + y}$.

2. Consider the tranformation

$$\begin{aligned} u &= e^x \cos y \\ v &= e^x \sin y. \end{aligned}$$

Prove that the Jacobian is never zero, and, therefore, this tranformation is invertible in some neighborhood of each point. Show that it is not one-to-one (and this not invertible) on the whole of $\mathbb{R}^2$.

We have

$$\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = \begin{pmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{pmatrix}$$

which has determinant

$$e^{2x} \cos^2 y + e^{2x} \sin^2 y = e^{2x} > 0.$$

Therefore, we can apply the inverse function theorem in the neighborhood of every point. However, the map in not (globally) one-to-one: $f(x, y) = f(x, y + 2\pi)$ for all x, y, since both cos and sin are periodic with period 2π.

3. Show that for the polar coordinates r and θ and $r \neq 0$, i.e. off the origin $(0, 0)$, given by

$$x = r \cos\theta, \quad y = r \sin\theta$$

we have

$$\frac{\partial r}{\partial x} = \cos\theta, \qquad \frac{\partial \theta}{\partial x} = -r^{-1} \sin\theta, \tag{1}$$

$$\frac{\partial r}{\partial y} = \sin\theta, \qquad \frac{\partial \theta}{\partial y} = r^{-1} \cos\theta. \tag{2}$$

We easily compute the Jacobian matrix

$$A = \begin{pmatrix} x_r & x_\theta \\ y_r & y_\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & -r \sin\theta \\ \sin\theta & r \cos\theta \end{pmatrix}$$

which has determinant $r \cos^2\theta + r \sin^2\theta = r$. This is invertible if $r \neq 0$. The inverse matrix gives the Jacobian

$$A^{-1} = \begin{pmatrix} r_x & r_y \\ \theta_x & \theta_y \end{pmatrix} = \frac{1}{r} \begin{pmatrix} r \cos\theta & r \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -r^{-1} \sin\theta & r^{-1} \cos\theta \end{pmatrix}$$

4. Lef f be a continuously differentiable function defined on $\mathbb{R}$ with $|f'(x)| < 1$ for all $x \in \mathbb{R}$. Define

$$g(u, v) = (u + f(v), v + f(u)).$$

Prove that g is invertible in some neighborhood of each point in $\mathbb{R}^2$.

We have to compute the Jacobian determinant and show that it is nonzero for each point in $\mathbb{R}^2$. We use the notation $g(u, v) = (g^1(u, v), g^2(u, v))$ and have

$$\begin{pmatrix} g^1_u & g^1_v \\ g^2_u & g^2_v \end{pmatrix} = \begin{pmatrix} 1 & f'(v) \\ f'(u) & 1 \end{pmatrix}.$$

The determinant is

$$\det J = 1^2 - f'(u)f'(v).$$

Since $|f'(x)| < 1$ we have $|f'(u)| < 1$ and $|f'(v)| < 1$ and these imply

$$|f'(u)f'(v)| = |f'(u)||f'(v)| < 1 \Longrightarrow 0 < 1 - |f'(u)f'(v)| \leq |1 - f'(u)f'(v)| = |\det J|,$$

where we used the triangle inequality in the form $|a| - |b| \leq |a - b|$. So $|\det J| > 0$ and this implies the Jacobian is nonzero.

5. Use the chain rule to show that if we write a function $f(x,y)$ in polar coordinates $g(r,\theta)$, i.e. $f(x,y)=g(r,\theta)$ and we assume that the mixed partial derivatives are equal, then

$$\frac{\partial^2 f}{\partial x^2}=\cos^2\theta\frac{\partial^2 g}{\partial r^2}-2r^{-1}\sin\theta\cos\theta\frac{\partial^2 g}{\partial r\partial\theta}+r^{-2}\sin^2\theta\frac{\partial^2 g}{\partial\theta^2}+r^{-1}\sin^2\theta\frac{\partial g}{\partial r}+2r^{-2}\sin\theta\cos\theta\frac{\partial g}{\partial\theta}.$$

Show that for the Laplacian of f

$$\Delta f=\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}$$

we have in polar coordinates:

$$\Delta f=\frac{\partial^2 g}{\partial r^2}+r^{-1}\frac{\partial g}{\partial r}+r^{-2}\frac{\partial^2 g}{\partial\theta^2}.$$

We use subscripts to denote partial derivatives in the subscript variable. By the chain rule we have

$$f_x=g_r r_x+g_\theta\theta_x.$$

We use the chain rule to g_r and g_θ and the product rule to get

$$\frac{\partial^2 f}{\partial x^2}=(f_x)_x=(g_r r_x+g_\theta\theta_x)_x=((g_r)_r r_x+(g_r)_\theta\theta_x)r_x+g_r(r_x)_x+((g_\theta)_r r_x+(g_\theta)_\theta\theta_x)\theta_x+g_\theta(\theta_x)_x$$

$$=g_{rr}r_x^2+g_{r\theta}\theta_x r_x+g_r r_{xx}+g_{\theta r}r_x\theta_x+g_{\theta\theta}\theta_x^2+g_\theta\theta_{xx}=g_{rr}\cos^2\theta+2g_{r\theta}\cos\theta(-r^{-1}\sin\theta)+g_{\theta\theta}(-r^{-1}\sin\theta)^2$$

$$+g_r r_{xx}+g_\theta\theta_{xx}=g_{rr}\cos^2\theta-2r^{-1}\sin\theta\cos\theta g_{r\theta}+r^{-2}\sin^2\theta g_{\theta\theta}+g_r r_{xx}+g_\theta\theta_{xx},$$

using (1) and (2) and the equality of the mixed partial derivatives of g. To finish we must show that

$$r_{xx}=\frac{\partial^2 r}{\partial x^2}=r^{-1}\sin^2\theta,\quad \theta_{xx}=\frac{\partial^2\theta}{\partial x^2}=2r^{-2}\sin\theta\cos\theta.$$

We have using the chain rule

$$r_{xx}=(r_x)_x=(r_x)_r r_x+(r_x)_\theta\theta_x=(\cos\theta)_r r_x+(\cos\theta)_\theta(-r^{-1}\sin\theta)=-\sin\theta(-r^{-1})\sin\theta=r^{-1}\sin^2\theta$$

and

$$\theta_{xx}=(\theta_x)_x=(\theta_x)_r r_x+(\theta_x)_\theta\theta_x==(-r^{-1}\sin\theta)_r\cos\theta+(-r^{-1}\sin\theta)_\theta(-r^{-1}\sin\theta)$$

$$=r^{-2}\sin\theta\cos\theta+r^{-2}\cos\theta\sin\theta=2r^{-2}\cos\theta\sin\theta.$$

We show in similar fashion that

$$\frac{\partial^2 f}{\partial y^2}=\sin^2\theta\frac{\partial^2 g}{\partial r^2}+2r^{-1}\sin\theta\cos\theta\frac{\partial^2 g}{\partial r\partial\theta}+r^{-2}\cos^2\theta\frac{\partial^2 g}{\partial\theta^2}+r^{-1}\cos^2\theta\frac{\partial g}{\partial r}-2r^{-2}\sin\theta\cos\theta\frac{\partial g}{\partial\theta}. \tag{3}$$

The result on the Laplacian Δf follows by adding the two equations for f_{xx} and f_{yy}, the trig identity $\cos^2\theta+\sin^2\theta=1$ and the cancellation of the terms containing $g_{r\theta}$ and g_θ.

We now show (3). We have

$$f_{yy} = (f_y)_y = (g_r r_y + g_\theta \theta_y)_y = (g_r)_y + r_y + g_r r_{yy} + (g_\theta)_y \theta_y + g_\theta \theta_{yy} = g_{rr} r_y^2 + g_{r\theta} \theta_y r_y + g_r r_{yy} + g_{\theta r} r_y \theta_y$$

$$+ g_{\theta\theta} \theta_y^2 + g_\theta \theta_{yy} = \sin^2\theta g_{rr} + 2 g_{r\theta} r^{-1} \sin\theta\cos\theta + r^{-2}\cos^2\theta g_{\theta\theta} + g_r (\sin\theta)_y + g_\theta (r^{-1}\cos\theta)_y.$$

We finally have

$$(\sin\theta)_y = (\sin\theta)_r r_y + (\sin\theta)_\theta \theta_y = \cos\theta (r^{-1}\cos\theta) = r^{-1}\cos^2\theta,$$

$$(r^{-1}\cos\theta)_y = (r^{-1}\cos\theta)_r r_y + (r^{-1}\cos\theta)_\theta \theta_y = -r^{-2}\cos\theta\sin\theta + r^{-1}(-\sin\theta) r^{-1}\cos\theta$$
$$= -2r^{-2}\sin\theta\cos\theta.$$

6. Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $(x, y, z) = F(u, v, w) = (u^2 + v^2 + w^2, uv + uw + vw, uvw)$. Find the points $(u, v, w) \in \mathbb{R}^3$ where we can apply the inverse function theorem. Compute the partial derivatives:

$$\frac{\partial u}{\partial x}, \quad \frac{\partial u}{\partial y}, \quad \frac{\partial u}{\partial z}.$$

We calculate the Jacobian determinant, using properties of determinants:

$$\begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix} = \begin{vmatrix} 2u & 2v & 2w \\ v+w & u+w & u+v \\ vw & uw & uv \end{vmatrix} = 2 \begin{vmatrix} u & v & w \\ v+w & u+w & u+v \\ vw & uw & uv \end{vmatrix}$$

$$= 2 \begin{vmatrix} u & v & w \\ u+v+w & u+v+w & u+v+w \\ vw & uw & uv \end{vmatrix} = 2(u+v+w) \begin{vmatrix} u & v & w \\ 1 & 1 & 1 \\ vw & uw & uv \end{vmatrix}$$

where we added the first row to the second row and then factored out $u + v + w$,

$$= 2(u+v+w) \begin{vmatrix} u & v-u & w-u \\ 1 & 0 & 0 \\ vw & (u-v)w & (u-w)v \end{vmatrix} = -2(u+v+w) \begin{vmatrix} v-u & w-u \\ (u-v)w & (u-w)v \end{vmatrix}$$

where we subtracted the first column from the other two and then expanded along the second row,

$$= -2(u+v+w)(v-u)(u-w) \begin{vmatrix} 1 & -1 \\ -w & v \end{vmatrix} = -2(u+v+w)(v-u)(u-w)(v-w).$$

Notice that this technique provided the determinant in a factored form and we can figure out when the Jacobian determinant is zero or not:

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = 0 \Leftrightarrow u + v + w = 0 \text{ or } u = v \text{ or } u = w \text{ or } v = w.$$

Away from these planes in uvw-space we can invert F by applying the inverse function theorem. If DF is given by the matrix

$$A = \begin{pmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{pmatrix} = \begin{pmatrix} 2u & 2v & 2w \\ v+w & u+w & u+v \\ vw & uw & uv \end{pmatrix}$$

then $D(F^{-1})$ is given by the matrix

$$A^{-1} = \begin{pmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{pmatrix}.$$

Rather than invert the whole matrix A we use the classical adjoint to compute u_x, u_y, u_z.

$$u_x = \frac{1}{\det A} \begin{vmatrix} u+w & u+v \\ uw & uv \end{vmatrix} = \frac{-u(uv+wv-wv-wu)}{2(u+v+w)(v-u)(u-w)(v-w)} = \frac{-u^2}{2(u+v+w)(v-u)(u-w)}$$

$$u_y = \frac{-1}{\det A} \begin{vmatrix} 2v & 2w \\ uw & uv \end{vmatrix} = \frac{2u(v^2-w^2)}{2(u+v+w)(v-u)(u-w)(v-w)} = \frac{u(v+w)}{(u+v+w)(v-u)(u-w)}$$

$$\begin{aligned} u_z &= \frac{1}{\det A} \begin{vmatrix} 2v & 2w \\ u+w & u+v \end{vmatrix} = \frac{2(uv+v^2-uw-w^2)}{-2(u+v+w)(v-u)(u-w)(v-w)} \\ &= \frac{(u+v+w)(v-w)}{(u+v+w)(v-u)(u-w)(v-w)} = \frac{1}{(u-v)(u-w)} \end{aligned}$$

7. (a) Let V be the solid enclosed by a surface S. Let u and v by any functions with continuous second partial derivatives. Show that with $\mathbf{n}$ the outward unit normal vector on S we have:

$$\int_S (u\nabla v) \cdot \mathbf{n}\, dS = \int_V (u\Delta v + \nabla u \cdot \nabla v) dV.$$

This is sometimes called the first form of Green's theorem.

(b) Show that if $\Delta u = 0$, i.e. u is a harmonic function, then

$$\int_S u\nabla u \cdot \mathbf{n}\, dS = \int_V |\nabla u|^2 dV.$$

(c) Show the second form of Green's theorem, i.e. that

$$\int_S (u\nabla v - v\nabla u) \cdot \mathbf{n}\, dS = \int_V (u\Delta v - v\Delta u) dV.$$

(d) Show that if $u = v = 0$ on the surface S, then

$$\int u(\Delta v)\, dV = \int_V v(\Delta u)\, dV.$$

This shows, in the terminology of linear algebra, that Δ is a self-adjoint operator for functions on V that vanish on the boundary of V.

(a) We use the divergence theorem:

$$\int_S (u\nabla v) \cdot \mathbf{n}\, dS = \int_V \text{div } (u\nabla v)\, dV.$$

So we compute the divergence of $u\nabla v$:

$$u\nabla v = u\left(\frac{\partial v}{\partial x}\mathbf{i} + \frac{\partial v}{\partial y}\mathbf{j} + \frac{\partial v}{\partial z}\mathbf{k}\right) = (uv_x)\mathbf{i} + (uv_y)\mathbf{j} + (uv_z)\mathbf{k}.$$

$$\text{div } (u\nabla v) = \frac{\partial(uv_x)}{\partial x} + \frac{\partial(uv_y)}{\partial y} + \frac{\partial(uv_z)}{\partial z} = u_x v_x + uv_{xx} + u_y v_y + uv_{yy} + u_z v_z + uv_{zz}$$

using the product rule. Now we rearrange terms to get

$$\text{div } (u\nabla v) = u(v_{xx} + v_{yy} + v_{zz}) + u_x v_x + u_y v_y + u_z v_z = u\Delta v + \nabla u \cdot \nabla v.$$

(b) We apply (a) to $u = v$ and see that in the right-hand side only $\nabla u \cdot \nabla u = |\nabla u|^2$ remains under the assumption $\Delta u = 0$.

(c) We write the result of (a) with u and v interchanged:

$$\int_S (v\nabla u) \cdot \mathbf{n}\, dS = \int_V (v\Delta u + \nabla v \cdot \nabla u) dV.$$

Now we subtract the above equation from the result in (a). We notice that the terms with gradients $\nabla u \cdot \nabla v$ are equal, as the dot product is commutative. So they cancell when we subtract and we get

$$\int_S (u\nabla v - v\nabla u) \cdot \mathbf{n}\, dS = \int_V (u\Delta u - v\Delta u)\, dV.$$

(d) If $u = v = 0$ on the surface S, the surface integral

$$\int_S (u\nabla v - v\nabla u)\mathbf{n}\, dS = 0,$$

as the vector field to integrate over S is zero on the surface. Then (c) gives:

$$\int_V u(\Delta v) - v(\Delta u)\, dV = 0 \Leftrightarrow \int_V u(\Delta v)\, dV = \int_V v(\Delta u)\, dV.$$

Remark: If we let $\langle f, g\rangle = \int_V fg\, dV$ for f and g continuous functions on V, one can show that this is an inner product and the result of (d) shows that $\langle u, \Delta v\rangle = \langle \Delta u, v\rangle$, on the subspace of function that vanish on S and have continuous second partial derivatives, which is the definition of self-adjoint operator.

8. *Spherical coordinates.* The map T

$$x = r\cos\theta\cos\phi, \quad y = r\sin\theta\cos\phi, \quad z = r\sin\phi$$

defines spherical coordinates on the xyz-plane. Here r represents the polar distance, θ represent longitude and ϕ represents latitude. So $\theta \in [0, 2\pi]$ and $\phi \in [-\pi/2, \pi/2]$.

(a) Compute $\dfrac{\partial(x, y, z)}{\partial(r, \theta, \phi)}$.

(b) Show for a solid R in $r\theta\phi$-space the formula for integration in spherical coordinates is:

$$\int_{T(R)} f(x,y,z)\,dxdydz = \int_R f(r\cos\theta\cos\phi, r\sin\theta\cos\phi, r\sin\phi)r^2\cos\phi\,dr\,d\theta\,d\phi.$$

We have, factoring $r\cos\phi$ from the second column and r from the third and then expanding the determinant:

$$\begin{vmatrix} x_r & x_\theta & x_\phi \\ y_r & y_\theta & y_\phi \\ z_r & z_\theta & z_\phi \end{vmatrix} = \begin{vmatrix} \cos\theta\cos\phi & -r\sin\theta\cos\phi & -r\cos\theta\sin\phi \\ \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\phi & 0 & r\cos\phi \end{vmatrix}$$

$$= r^2\cos\phi \begin{vmatrix} \cos\theta\cos\phi & -\sin\theta & -\cos\theta\sin\phi \\ \sin\theta\cos\phi & \cos\theta & -\sin\theta\sin\phi \\ \sin\phi & 0 & \cos\phi \end{vmatrix} = r^2\cos\phi(\cos^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + \cos^2\theta\sin^2\phi$$

$$+\cos^2\phi\sin^2\phi) = r^2\cos\phi(\cos^2\theta + \sin^2\theta)(\cos^2\phi + \sin^2\phi) = r^2\cos\phi.$$

(b) The pullback of the form $f(x,y,z)\,dxdydz$ is

$$f(T(r,\theta,\phi))\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)}drd\theta d\phi = f(r\cos\theta\cos\phi, r\sin\theta\cos\phi, r\sin\phi)r^2\cos\phi\,dr\,d\theta\,d\phi.$$

9. Show that the two equations $x+y=uv$ and $xy=u-v$ determine the functions x and y implicitly as functions of u and v, if $x \neq y$. Find

$$\frac{\partial x}{\partial u}, \quad \frac{\partial x}{\partial v}, \quad \frac{\partial y}{\partial u}, \quad \frac{\partial y}{\partial v}.$$

To be able to solve for x and y, according to the implicit function theorem, we must show that the Jacobian of the two equations with respect to x and y is nonzero for $x \neq y$. We write the equations as

$$F(x,y,u,v) = x+y-uv = 0, \quad G(x,y,u,v) = xy-u+v = 0.$$

We have

$$\frac{\partial(F,G)}{\partial(x,y)} = \begin{vmatrix} F_x & F_y \\ G_x & G_y \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ y & x \end{vmatrix} = x-y.$$

To compute the partial derivatives with respect to u we differentiate the equations implicitly, thinking that x and y are functions of u, v. This gives the system

$$\frac{\partial x}{\partial u} + \frac{\partial y}{\partial u} = v, \quad y\frac{\partial x}{\partial u} + x\frac{\partial y}{\partial u} = 1.$$

The system can be solved with Cramer's rule:

$$\frac{\partial x}{\partial u} = \frac{\begin{vmatrix} v & 1 \\ 1 & x \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ y & x \end{vmatrix}} = \frac{xv-1}{x-y}, \quad \frac{\partial y}{\partial u} = \frac{\begin{vmatrix} 1 & v \\ y & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ y & x \end{vmatrix}} = \frac{1-yv}{x-y}.$$

To compute the partial derivatives with respect to v we differentiate the equations implicitly, thinking that x and y are functions of u, v. This gives the system

$$\frac{\partial x}{\partial v} + \frac{\partial y}{\partial v} = u, \quad y\frac{\partial x}{\partial v} + x\frac{\partial y}{\partial v} = -1.$$

The system can be solved with Cramer's rule:

$$\frac{\partial x}{\partial v} = \frac{\begin{vmatrix} u & 1 \\ -1 & x \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ y & x \end{vmatrix}} = \frac{ux+1}{x-y}, \quad \frac{\partial y}{\partial v} = \frac{\begin{vmatrix} 1 & u \\ y & -1 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ y & x \end{vmatrix}} = \frac{-1-yu}{x-y}.$$