Math 434/734

Homework 6

1. Exercise 1, p. 84 and 1(d), p. 103. For each of the following systems solve for the variables in the right-hand side in terms of the variables in the left-hand side, write the system in matrix form, compute the rank r and find all the exterior powers of the matrices involved.

(a) $\begin{array}{c} u = 3x + 2y + 1 \\ v = 2x + y - 3 \end{array}$.

We write

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ -3 \end{pmatrix},$$

$$f : \mathbb{R}^2 \to \mathbb{R}^2, f(x,y) = A \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ -3 \end{pmatrix}, \quad A = \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix}.$$

We subtract from the second equation the first multiplied by 2/3 to get

$$v - 2u/3 = y - (2/3)2y - 3 - 2/3 \Leftrightarrow v - 2u/3 + 11/3 = -y/3 \Leftrightarrow y = -3v + 2u - 11.$$

Then

$$3x = u - 2y - 1 = u - 2(-3v + 2u - 11) - 1 = -3u + 6v + 21 \Leftrightarrow x = -u + 2v + 7.$$

We see that for every pair (u,v) we can find a unique pair (x,y) with $f(x,y) = (u,v)$. So the map is one-to-one and the rank is 2. Notice that $\dim V = 2$, $\dim N(T) = 0$ and $\dim R(T) = 2$ for $T(\vec{v}) = A\vec{v}$ so that the equation

$$\dim V = \dim N(T) + \dim R(T) \tag{1}$$

is satisfied.

We also see that the pullback of a form of top order in the image space $\mathbb{R}^2$ is nonzero:

$$(dudv)^* = (3dx + 2dy)(2dx + dy) = 3dxdy - 4dxdy = -dxdy.$$

This is enough to say that the rank $r = 2$.

As far as the exterior powers are concerned we get:

$$A^{(1)} = A^T = \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix}, A^{(2)} = (\det \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix}) = (3 - 4) = (-1).$$

This last is a 1×1 matrix and gives the pullback of two forms, agreeing with the pullback of $dudv$ computed above.

(b) $\begin{array}{c} x = 2t + 1 \\ y = t + 2 \\ z = 4t - 3 \end{array}$. We can solve for t using any of the variables x, y, z as follows:

$$\begin{array}{c} t = (x-1)/2 \\ t = y - 2 \\ t = (z+3)/4 \end{array}$$

The system is consistent if the right-hand sides of the above equations agree: $y-2 = (x-1)/2 = (z+3)/4$. These equations determine a line in 3-space. The image is a line and the rank $r = 1$. The same result we can see by pulling back form from $\mathbb{R}^3$ to $\mathbb{R}$: $dx = 2dt$, $dy = dt$, $dz = 4dt$. Obviously the pullbacks

$$(dxdydz)^* = 0, \quad (dxdy)^* = (dxdz)^* = (dydz)^* = 0,$$

while the pullbacks of the one-forms are non zero. We write the system in matrix form

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = f(t) = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} t + \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}, \quad B = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix}.$$

The map is one-to-one and Eq. (1) becomes $1 = 0 + 1$. The exterior powers are:

$$B^{(1)} = B^T = (2, 1, 4)$$

and there are no other exterior powers.

(c) $\begin{array}{c} u = 2x + y - z \\ v = -4x + 3y + 2z - 4 \end{array}$.

We have in matrix form

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 2 & 1 & -1 \\ -4 & 3 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} 0 \\ -4 \end{pmatrix}.$$

We eliminate x from the second equation by multiplying the first by 2 and adding to the second. This gives:

$$v + 2u = 5y - 4 \Leftrightarrow y = (v + 2u + 4)/5.$$

Then we can solve for either x or z:

$$x = (3u/5 - v/5 + z - 4/5)/2, \quad z = v/5 - 3u/5 + 2x + 4/5.$$

However, we cannot solve for x and z similtaneously. If one tries to eliminate y we get

$$2x - z = u - y, \quad -4x + 2z = v - 3y + 4$$

and this system has a solution if $v - 3y + 4 = -2(u - y)$ and no solution otherwise. But we cannot eliminate x or z, since $(-4, 2) = -2(2, -1)$. The pullback of $dudv$ is

$$(dudv)^* = (2dx + dy - dz)(-4dx + 3dy + 2dz) = 10dxdy + 5dydz,$$

which is nonzero, and this shows that the rank $r = 2$. The fact that we can solve for x, y, given z is reflected in the fact that in the pullback the coefficient of $dxdy$ is nonzero. The fact that we can solve for z, y, given x is reflected in the fact that in the pullback the coefficient of $dydz$ is nonzero. The fact that we cannot solve for x, z, is reflected in the fact that there is no $dxdz$ in the above pullback. The rank $r = 2$ and Eq. (1) becomes $3 = 1 + 2$.

The exteriors powers of C are

$$C^{(1)} = C^T = \begin{pmatrix} 2 & -4 \\ 1 & 3 \\ -1 & 2 \end{pmatrix}, \quad C^{(2)} = \begin{pmatrix} 10 \\ 0 \\ 5 \end{pmatrix},$$

given that

$$\begin{vmatrix} 2 & 1 \\ -4 & 3 \end{vmatrix} = 6 - (-4) = 10, \quad \begin{vmatrix} 2 & -1 \\ -4 & 2 \end{vmatrix} = 2 \cdot 2 - (-4)(-1) = 0$$

and

$$\begin{vmatrix} 1 & -1 \\ 3 & 2 \end{vmatrix} = 2 - (-3) = 5.$$

2. Prove that if the function $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at a point $\vec{a}$, then it is continuous at $\vec{a}$.

There exists the linear map (derivative) $Df : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\lim_{\vec{h}\to\vec{0}} \frac{||f(\vec{a}+\vec{h}) - f(\vec{a}) - Df(\vec{h})||}{||\vec{h}||} = 0.$$

This implies

$$\lim_{\vec{h}\to\vec{0}} ||f(\vec{a}+\vec{h}) - f(\vec{a}) - Df(\vec{h})|| = \lim_{\vec{h}\to\vec{0}} \frac{||f(\vec{a}+\vec{h}) - f(\vec{a}) - Df(\vec{h})||}{||\vec{h}||}||\vec{h}|| = 0.$$

This implies that

$$\lim_{\vec{h}\to\vec{0}} ||f(\vec{a}+\vec{h}) - f(\vec{a})|| = \lim_{\vec{h}\to\vec{0}} ||f(\vec{a}+\vec{h}) - f(\vec{a}) - Df(\vec{h}) + Df(\vec{h})||$$

$$\leq \lim_{\vec{h}\to\vec{0}} ||f(\vec{a}+\vec{h}) - f(\vec{a}) - Df(\vec{h})|| + \lim_{\vec{h}\to\vec{0}} ||Df(\vec{h})|| = 0 + \lim_{\vec{h}\to\vec{0}} ||Df(\vec{h})||.$$

So it suffices to prove that $\lim_{\vec{h}\to\vec{0}} ||Df(\vec{h})|| = 0$. For every linear map T we have proved that there exists a constant M, such that

$$||T(\vec{h})|| \leq M||\vec{h}||.$$

This implies that $\lim_{\vec{h}\to\vec{0}} T(\vec{h}) = \vec{0}$, i.e. $\lim_{\vec{h}\to\vec{0}} ||T(\vec{h})|| = 0$.

3. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x,y) = \begin{cases} \frac{x^2-y^2}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}.$$

Show that $f(x,y)$ is not continuous at $(0,0)$. Show that the partial derivatives at $(0,0)$ do not exist.

We investigate the behavior of $f(x,y)$ on lines through the origin: $y = kx$, where k is the slope. Then for $x \neq 0$

$$f(x,kx) = \frac{x^2 - k^2x^2}{x^2 + k^2x^2} = \frac{1-k^2}{1+k^2}.$$

By choosing different k's, we get different limits

$$\lim_{x\to 0} f(x,kx),$$

e.g. for $k=1$ we get 0 and for $k=2$ we get $-3/5$. Consequently the function is not continuous at $(0,0)$.

We have

$$\frac{\partial f}{\partial x}(0,0)=\lim_{x\to 0}\frac{f(x,0)-f(0,0)}{x}=\lim_{x\to 0}\frac{(x^2-0^2)/(x^2+0^2)-0}{x}=\lim_{x\to 0}\frac{1}{x}$$

and this limit is $+\infty$, $x\to 0^+$ and $-\infty$, $x\to 0^-$.

We have

$$\frac{\partial f}{\partial y}(0,0)=\lim_{y\to 0}\frac{f(0,y)-f(0,0)}{y}=\lim_{y\to 0}\frac{(0^2-y^2)/(0^2+y^2)-0}{y}=\lim_{y\to 0}\frac{-1}{y}$$

and this limit is $+\infty$, $y\to 0^-$ and $-\infty$, $y\to 0^+$.

4. Consider the function $f:\mathbb{R}^2\to\mathbb{R}$ given by

$$f(x,y)=\begin{cases}\frac{xy}{x^2+y^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}.$$

Show that $f(x,y)$ is not continuous at $(0,0)$. Show that the partial derivatives at $(0,0)$ exist and compute them. Show that the function is not differentiable at $(0,0)$.

We investigate the behavior of $f(x,y)$ on lines through the origin: $y=kx$, where k is the slope. Then for $x\neq 0$

$$f(x,kx)=\frac{xkx}{x^2+k^2x^2}=\frac{k}{1+k^2}.$$

By choosing different k's, we get different limits

$$\lim_{x\to 0} f(x,kx).$$

Consequently the function is not continuous at $(0,0)$.

We have

$$\frac{\partial f}{\partial x}(0,0)=\lim_{x\to 0}\frac{f(x,0)-f(0,0)}{x}=\lim_{x\to 0}\frac{(x\cdot 0)/(x^2+0^2)-0}{x}=\lim_{x\to 0}\frac{0}{x}=0$$

$$\frac{\partial f}{\partial y}(0,0)=\lim_{y\to 0}\frac{f(0,y)-f(0,0)}{y}=\lim_{y\to 0}\frac{(0\cdot y)/(0^2+y^2)-0}{y}=\lim_{y\to 0}\frac{0}{y}=0$$

According to the first exercise, since the function is not continuous at $(0,0)$, it is not differentiable at $(0,0)$.

Second method: Since we have computed the partial derivatives at $(0,0)$ to be 0, if the derivative $Df:\mathbb{R}^2\to\mathbb{R}$ exists, then it is the zero linear map given by the matrix $(0,0)$. This is because we proved that if $Df(\vec{h})=A\vec{h}$, then

$$a_{ij}=\frac{\partial f_i}{\partial x_j}(\vec{a}).$$

According to the definition of the derivative this means that ($\vec{h} = (h_1, h_2)$)

$$\lim_{\vec{h}\to\vec{0}} \frac{||f(\vec{h}) - f(\vec{0})||}{||\vec{h}||} = 0.$$

However, when we approach $\vec{0}$ along the line $h_2 = h_1$ we get

$$\lim_{h_1\to 0} \frac{|h_1 \cdot h_1|}{(h_1^2 + h_1^2)^{3/2}} = \lim_{h_1\to 0} \frac{h_1^2}{2^{3/2} h_1^3} = \lim_{h_1\to 0} \frac{1}{2^{3/2} h_1}$$

and this tend to $\pm\infty$ acording to h_1 being positive or negative. This contradiction implies that f is not differentiable at $(0, 0)$.

5. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x, y) = \begin{cases} \frac{x^3}{x^2+y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}.$$

Show that $f(x, y)$ is continuous at $(0, 0)$. Show that the partial derivatives at $(0, 0)$ exist and compute them. Show that the function is not differentiable at $(0, 0)$.
Since $|x| \leq \sqrt{x^2 + y^2}$ we get

$$\left|\frac{x^3}{x^2 + y^2}\right| \leq \frac{(x^2 + y^2)^{3/2}}{x^2 + y^2} = (x^2 + y^2)^{1/2}.$$

This implies that

$$|\lim_{(x,y)\to(0,0)} f(x, y)| \leq \lim_{(x,y)\to(0,0)} (x^2 + y^2)^{1/2} = 0.$$

This means that $f(x, y)$ is continuous at $(0, 0)$. We have

$$\frac{\partial f}{\partial x}(0, 0) = \lim_{x\to 0} \frac{f(x, 0) - f(0, 0)}{x} = \lim_{x\to 0} \frac{(x^3)/(x^2 + 0^2) - 0}{x} = \lim_{x\to 0} \frac{x}{x} = 1$$

$$\frac{\partial f}{\partial y}(0, 0) = \lim_{y\to 0} \frac{f(0, y) - f(0, 0)}{y} = \lim_{y\to 0} \frac{(0^3)/(0^2 + y^2) - 0}{y} = \lim_{y\to 0} \frac{0}{y} = 0$$

So, if the function is differentiable at $(0, 0)$, its derivative is given by

$$Df(h_1, h_2) = (1, 0) \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = 1 \cdot h_1 + 0 \cdot h_2 = h_1.$$

We know that

$$\lim_{(h_1,h_2)\to(0,0)} \frac{||f(\vec{h}) - f(\vec{0}) - Df(\vec{h})||}{||\vec{h}||} = 0,$$

and we rewrite it using the definition of f and the computation of Df:

$$\lim_{(h_1,h_2)\to(0,0)} \frac{||h_1^3/(h_1^2 + h_2^2) - 0 - h_1||}{\sqrt{h_1^2 + h_2^2}} = 0.$$

We simplify this to get

$$\lim_{(h_1,h_2)\to(0,0)} \frac{||(h_1^3 - h_1^3 - h_1h_2^2)/(h_1^2+h_2^2)||}{\sqrt{h_1^2+h_2^2}} = \lim_{(h_1,h_2)\to(0,0)} \frac{h_2^2|h_1|}{(h_1^2+h_2^2)^{3/2}} = 0.$$

However, if we approach $(0,0)$ along the various lines $h_2 = kh_1$ we get different limits:

$$\lim_{h_1\to 0} \frac{k^2|h_1|^3}{(h_1^2+k^2h_1^2)^{3/2}} = \lim_{h_1\to 0} \frac{k^2|h_1|^3}{(1+k^2)^{3/2}|h_1|^3} = \frac{k^2}{(1+k^2)^{3/2}},$$

e.g. for $k=0$ we get limit 0, while for $k=1$ we get limit $1/2^{3/2}$. This contradiction means that f is not differentiable at $(0,0)$.

6. Show that the gradient

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

of a function $f:\mathbb{R}^2 \to \mathbb{R}$ has length in polar coordinates given by:

$$||\nabla f||^2 = \left(\frac{\partial f}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial f}{\partial \theta}\right)^2.$$

We have $x = r\cos\theta$ and $y = r\sin\theta$. The chain rule gives

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta. \tag{2}$$

$$\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta} = \frac{\partial f}{\partial x}(-r\sin\theta) + \frac{\partial f}{\partial y}r\cos\theta.$$

The last equation gives:

$$\frac{1}{r}\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x}(-\sin\theta) + \frac{\partial f}{\partial y}\cos\theta. \tag{3}$$

We square (2) and (3) and add to get

$$\left(\frac{\partial f}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial f}{\partial \theta}\right)^2 = \left(\frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta\right)^2 + \left(\frac{\partial f}{\partial x}(-\sin\theta) + \frac{\partial f}{\partial y}\cos\theta\right)^2$$

$$= \left(\frac{\partial f}{\partial x}\right)^2\cos^2\theta + 2\frac{\partial f}{\partial x}\frac{\partial f}{\partial y}\cos\theta\sin\theta + \left(\frac{\partial f}{\partial y}\right)^2\sin^2\theta + \left(\frac{\partial f}{\partial x}\right)^2\sin^2\theta - 2\frac{\partial f}{\partial x}\frac{\partial f}{\partial y}\cos\theta\sin\theta + \left(\frac{\partial f}{\partial y}\right)^2\cos^2\theta$$

$$= \left(\frac{\partial f}{\partial x}\right)^2(\cos^2\theta + \sin^2\theta) + \left(\frac{\partial f}{\partial y}\right)^2(\sin^2\theta + \cos^2\theta) = \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 = ||\nabla f||^2.$$