Math 434/734

Homework 5

1. Assume that the mixed partial derivatives of the given function are equal.

   (a) Show that div (curl $\mathbf{F}$) = 0, which in alternative notation is $\nabla \cdot (\nabla \times \mathbf{F}) = 0$.

   (b) Let ω be the one form corresponding to $\mathbf{F}$. Show that $d(d\omega) = 0$ and that this is the same calculation as in (a).

   (a) Let $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$. Then

$$\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{vmatrix} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathbf{k}.$$

$$\begin{aligned} \text{div curl } \mathbf{F} &= \frac{\partial}{\partial x}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) \\ &= \frac{\partial^2 F_3}{\partial x \partial y} - \frac{\partial^2 F_2}{\partial x \partial z} + \frac{\partial^2 F_1}{\partial y \partial z} - \frac{\partial^2 F_3}{\partial y \partial x} + \frac{\partial^2 F_2}{\partial z \partial x} - \frac{\partial^2 F_1}{\partial z \partial y} = 0. \end{aligned}$$

   (b) The one-form corresponding to $\mathbf{F}$ is

$$\omega = F_1 dx + F_2 dy + F_3 dz$$

   and its exterior derivative is (notice that it corresponds to curl $\mathbf{F}$)

$$d\omega = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) dydz + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) dzdx + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dxdy.$$

$$\begin{aligned} d(d\omega) &= \frac{\partial}{\partial x}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) dxdydz + \frac{\partial}{\partial y}\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) dydzdx + \frac{\partial}{\partial z}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dzdxdy \\ &= \left(\frac{\partial^2 F_3}{\partial x \partial y} - \frac{\partial^2 F_2}{\partial x \partial z} + \frac{\partial^2 F_1}{\partial y \partial z} - \frac{\partial^2 F_3}{\partial y \partial x} + \frac{\partial^2 F_2}{\partial z \partial x} - \frac{\partial^2 F_1}{\partial z \partial y}\right) dxdydz = 0 dxdydz \end{aligned}$$

   interchanging the positions of dx, dy and dz so that the 3-form is in $dxdydz$ order.

2. Use Green's theorem to evaluate

$$\int_C x^3 y dx + xy dy$$

   where C is the square with vertices at $(0,0)$, $(2,0)$, $(2,2)$ and $(0,2)$. The curve C bounds the square region S, so that, according to Green's theorem,

$$\int_C \omega = \int_S d\omega = \int_S \frac{\partial(x^3 y)}{\partial y} dydx + \frac{\partial(xy)}{\partial x} dxdy = \int_S -x^3 dxdy + ydxdy = \int_S (-x^3 + y) dxdy.$$

This is now an ordinary double integral on the region and we get

$$\int_S d\omega = \int_0^2 \int_0^2 (-x^3+y)dxdy = \int_0^2 \left[-\frac{x^4}{4}+yx\right]_0^2 dy = \int_0^2 -\frac{16}{4}+2ydy$$

$$= \left[-4y+y^2\right]_0^2 = -8+4=-4.$$

Remark: If you want to avoid Green's theorem, you have to compute 4 line integrals. Although 2 of them are 0, the ones along the axes, this is still work.

3. Show that the area of a region S with boundary C can be computed as the (line) integral

$$\text{Area } (S) = \int_C \frac{1}{2}(xdy - ydx)$$

Use this formula to compute the area of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

The answer is πab.

We use Green's theorem:

$$\int_C \frac{1}{2}(xdy-ydx) = \int_S \frac{1}{2}\left(\frac{\partial x}{\partial x}dxdy - \frac{\partial y}{\partial y}dydx\right) = \int_S \frac{1}{2}(dxdy-dydx) = \int_S \frac{1}{2}2dxdy$$

$$= \int_S dxdy = \text{Area } (S).$$

For the ellipse we use the parametrization $x = a\cos t$ and $y = b\sin t$, $0 \le t \le 2\pi$. We calculate the pullback of the form $(xdy - ydx)/2$ to be

$$\frac{1}{2}(a\cos t b\cos t - b\sin t(-a\sin t))dt = \frac{1}{2}(ab\cos^2 t + ab\sin^2 t)dt = \frac{1}{2}abdt.$$

Then

$$\text{Area } (S) = \int_0^{2\pi} \frac{1}{2}abdt = \frac{1}{2}ab2\pi = \pi ab.$$

4. Use Green's theorem to evaluate

$$\int_C (x^2+y)dx - (3x+y^3)dy$$

where C is the ellipse $x^2+4y^2=4$.

Calling S the elliptical region bounded by C we get

$$\int_C \omega = \int_S d\omega = \int_S \frac{\partial(x^2+y)}{\partial y}dydx - \frac{\partial(3x+y^3)}{\partial x}dxdy = \int_S 1dydx - 3dxdy = \int_S -4dxdy$$

$$= -4\text{Area } (S) = -4\pi 2\cdot 1 = -8\pi$$

since the ellipse can be written as $x^2/4+y^2=1$, i.e. it has semiaxes of length $a=2$ and $b=1$.

Remark: If you want to avoid Green's theorem, you have to parametrize the ellipse as $x = 2\cos t$ and $y = \sin t$, $0 \le y \le 2\pi$. Then we compute the pullback of the one form to be

$$(x^2 + y)dx - (3x + y^3)dy = ((4\cos^2 t + \sin t)(-2\sin t) - (6\cos t + \sin^3 t)\cos t)dt$$
$$= (-8\cos^2 t \sin t - 2\sin^2 t - 6\cos^2 t - \sin^3 t \cos t)dt$$

The integral can be split and the parts computed using various substitutions $u = \cos t$ or $u = \sin t$, but we already see the advantage of Green's theorem.

5. Compute the (line) integral (integral of the 1-form)

$$\int_C y\,dx + (2x - z)dy + (z - x)dz$$

where C is the intersection of the sphere $x^2+y^2+z^2 = 4$ and the plane $z = 1$ oriented so that the disc cut on the sphere by the plane has unit normal vector $\mathbf{k}$. Do the calculation by parametrizing the curve AND by using Stokes' theorem.

The disc has radius $\sqrt{3}$ by Pythagoras' theorem. Its boundary is the curve C. (If you do not see this, draw the graph!) The parametrization is $x = \sqrt{3}\cos t$, $y = \sqrt{3}\sin t$, $z = 1$, $0 \le t \le 2\pi$. Then $dx = -\sqrt{3}\sin t$, $dy = \sqrt{3}\cos t$, $dz = 0$. The pullback gives

$$\int_C \omega = \int_0^{2\pi} = \sqrt{3}\sin t\sqrt{3}(-\sin t)dt + (2\sqrt{3}\cos t - 1)\sqrt{3}\cos t dt + 0dt$$

$$= \int_0^{2\pi}(-3\sin^2 t + 2\cdot 3\cos^2 t - \sqrt{3}\cos t)dt = \int_0^{2\pi} -3\frac{1-\cos 2t}{2} + 6\frac{1+\cos 2t}{2} - \sqrt{3}\cos t dt$$
$$= (-3/2 + 6/2)2\pi = 3\pi$$

since the integrals of $\cos t$ and $\cos 2t$ over a period 2π are 0.

Using Stokes' theorem: Since the curve is the boundary of the disc S centered at $(0, 0, 1)$ of radius $\sqrt{3}$ and area 3π (notice that the orientations match)

$$\int_C \omega = \int_S d\omega = \int_S 1dydx + 2dxdy - dzdy + dzdz - dxdz = \int_S -dxdy + 2dxdy$$

$$= \int_S dxdy = \text{Area } (S) = 3\pi$$

since $dz = 0$ on the disc which is on a horizontal plane.

6. Compute the (line) integral (integral of the 1-form)

$$\int_C (y - z)dx + (3x + z)dy + (x + 2y)dz$$

where C is the intersection of the paraboloid $z = 4-x^2-y^2$ and the plane $x+y+z = 0$. Use a direct calculation AND Stokes' theorem.

What is not obvious is what shape the curve C has. We solve the two given equations to get

$$x^2+y^2 = 4-z = 4+x+y \Longrightarrow x^2-x+y^2-y = 4 \Longrightarrow (x-1/2)^2+(y-1/2)^2 = 4+1/4+1/4$$

by completing the square. This equation $(x-1/2)^2+(y-1/2)^2=9/2$ describes a cylinder with base the circle with the same equation in the xy-plane. It is a cylinder, as the z variable does not show up. For every point $(x,y,0)$ on the circle in the xy-plane, all points (x,y,z) with the same x, y coordinates and any z are points on the cylinder. The curve C lies on this cylinder and on the plane $x+y+z=0$, so it is their intersection i.e. en ellipse. Actually we do not need to know this but it helps with the parametrization:

$$x = 1/2 + \frac{3}{\sqrt{2}}\cos t, \quad y = 1/2 + \frac{3}{\sqrt{2}}\sin t, \quad 0 \leq y \leq 2\pi$$

parametrize the base circle ($z=0$) and in our case C is parametrized by these equations and

$$z = -x - y = -1 - \frac{3}{\sqrt{2}}(\cos t + \sin t).$$

These imply

$$dx = -\frac{3}{\sqrt{2}}\sin t, \quad dy = \frac{3}{\sqrt{2}}\cos t, \quad dz = -\frac{3}{\sqrt{2}}(\cos t - \sin t).$$

We compute the pullback and the integral:

$$\int_C \omega = \int_0^{2\pi} \left(\frac{1}{2} + \frac{3}{\sqrt{2}}\sin t + 1 + \frac{3}{\sqrt{2}}(\sin t + \cos t)\right)\left(-\frac{3}{\sqrt{2}}\sin t\right)$$
$$+ \left(\frac{3}{2} + \frac{9}{\sqrt{2}}\cos t - 1 - \frac{3}{\sqrt{2}}\sin t - \frac{3}{\sqrt{2}}\cos t\right)\left(\frac{3}{\sqrt{2}}\cos t\right)$$
$$+ \left(\frac{1}{2} + \frac{3}{\sqrt{2}}\cos t + 1 + \frac{2\cdot 3}{\sqrt{2}}\sin t\right)\left(-\frac{3}{\sqrt{2}}(\cos t - \sin t)\right) dt$$

Since the integrals $\int_0^{2\pi}\sin t dt = \int_0^{2\pi}\cos t dt = \int_0^{2\pi}\cos t \sin t dt = 0$ while

$$\int_0^{2\pi}\sin^2 t dt = \int_0^{2\pi}\frac{1-\cos 2t}{2}dt = \pi, \quad \int_0^{2\pi}\cos^2 t dt = \int_0^{2\pi}\frac{1+\cos 2t}{2}dt = \pi,$$

we get

$$\int_C \omega = -\frac{9}{2}\pi - \frac{9}{2}\pi + \frac{27}{2}\pi - \frac{9}{2}\pi - \frac{9}{2}\pi + \frac{18}{2}\pi = \frac{9}{2}\pi.$$

Using Stokes' theorem: We calculate $d\omega$ and then its pullback to the xy-plane.

$$d\omega = 1dydx - dzdx + 3dxdy + dzdy + dxdz + 2dydz = 2dxdy + dydz - 2dzdx.$$

According to Stokes' theorem

$$\int_C \omega = \int_S d\omega$$

where S is the elliptical region enclosed by C on the plane $x+y+z=0$. We parametrize this region by using the disc in the xy-plane which is the projection. See the Figure of the cylinder and the curve C. Since we work on the plane $z=-x-y$, we have $dz=-dx-dy$ and we plug this to get the pullback of $d\omega$ on the xy-plane:

$$(d\omega)^* = 2dxdy + dy(-dx-dy) - 2(-dx-dy)dx = 2dxdy - dydx - 2(-dy)dx = dxdy.$$

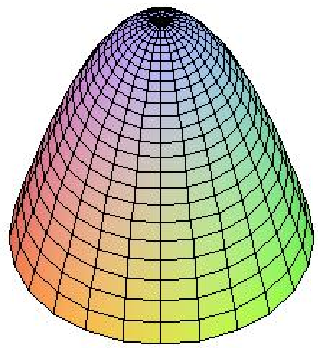

Figure 1: The paraboloid

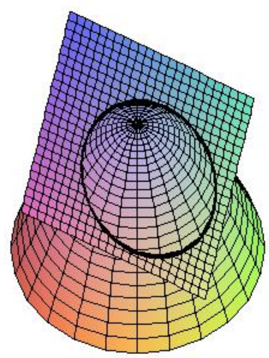

Figure 2: The contour as the intersection of the paraboloid and the plane

Integrating over the projection the form $dxdy$ means just calculating the area of the projection: this is the area of the circle of radius $3/\sqrt{2}$. This area is

$$\pi\frac{9}{2} = \int_S d\omega = \int_C \omega.$$

7. A central force can be written in the form

$$\mathbf{F}(r) = f(r)\mathbf{e}_r,$$

where $\mathbf{e}_r$ is a unit vector in the radial direction and $f(r)$ is a scalar function (it shows that the magnitude of the force depends only on the distance $r = (x^2 + y^2 + z^2)^{1/2}$. Show by direct calculation of the curl of this force that it is irrotational.

We have $\mathbf{e}_r = \mathbf{r}/r = (1/r)(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$. So we set $g(r) = f(r)/r$ and write

$$\mathbf{F}(r) = g(r)(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}).$$

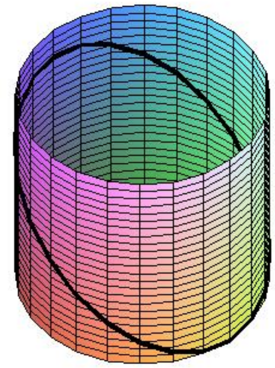

Figure 3: The contour on the cylinder

$$\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xg(r) & yg(r) & zg(r) \end{vmatrix} = \left(z\frac{\partial g}{\partial y} - y\frac{\partial g}{\partial z} \right) \mathbf{i} - \left(z\frac{\partial g}{\partial x} - x\frac{\partial g}{\partial z} \right) \mathbf{j} + \left(y\frac{\partial g}{\partial x} - x\frac{\partial g}{\partial y} \right) \mathbf{k}$$

since when we differentiate in y or z we treat x as a constant and similarly for the other variables. We now recall the derivatives of r in x, y and z (if you do not remember them, just differentiate $r = (x^2 + y^2 + z^2)^{1/2}$):

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}.$$

The chain rule for the function $g(r) = g(x, y, z)$ now gives

$$\frac{\partial g}{\partial x} = \frac{dg}{dr}\frac{\partial r}{\partial x} = g'(r)x/r$$

and similarly

$$\frac{\partial g}{\partial y} = g'(r)y/r, \quad \frac{\partial g}{\partial z} = g'(r)z/r.$$

As a result

$$\text{curl } \mathbf{F} = (g'yz/r - g'zy/r)\mathbf{i} - (g'xz/r - g'zx/r)\mathbf{j} + (g'xy/r - g'yx/r)\mathbf{k} = \vec{0}.$$

8. Verify the following identities in which f and g are arbitrary differentiable scalar functions of (x, y, z), and $\mathbf{F}$ and $\mathbf{G}$ are arbitrary differentiable vector functions. Write and prove the corresponding statement for forms.

(a) $\nabla(fg) = f\nabla g + g\nabla f$, where ∇ is the gradient of the function i.e.

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}.$$

This identity can also be written as

$$\text{grad } (fg) = f\text{grad } g + g\text{grad } f.$$

(b) $\nabla \cdot (f\mathbf{F}) = f\nabla \cdot \mathbf{F} + \mathbf{F} \cdot \nabla f$. This identity can also be written as

$$\text{div } (f\mathbf{F}) = f\text{div } \mathbf{F} + \mathbf{F} \cdot \text{grad} f.$$

(c) $\nabla \times (f\mathbf{F}) = f\nabla \times \mathbf{F} + (\nabla f) \times \mathbf{F}$. This identity can also be written as

$$\text{curl } (f\mathbf{F}) = f\text{curl } \mathbf{F} + (\text{grad } f) \times \mathbf{F}.$$

All these identities are essentially the product rule in multidimensional settings.

(a) We have by the standard product rule:

$$\frac{\partial(fg)}{\partial x} = f\frac{\partial g}{\partial x} + g\frac{\partial f}{\partial x}, \quad \frac{\partial(fg)}{\partial y} = f\frac{\partial g}{\partial y} + g\frac{\partial f}{\partial y}, \quad \frac{\partial(fg)}{\partial z} = f\frac{\partial g}{\partial z} + g\frac{\partial f}{\partial z}.$$

$$\nabla(fg) = \frac{\partial(fg)}{\partial x}\mathbf{i} + \frac{\partial(fg)}{\partial y}\mathbf{j} + \frac{\partial(fg)}{\partial z}\mathbf{k} = \left(f\frac{\partial g}{\partial x} + g\frac{\partial f}{\partial x}\right)\mathbf{i} + \left(f\frac{\partial g}{\partial y} + g\frac{\partial f}{\partial y}\right)\mathbf{j} + \left(f\frac{\partial g}{\partial z} + g\frac{\partial f}{\partial z}\right)\mathbf{k}$$

$$= f\left(\frac{\partial g}{\partial x}\mathbf{i} + \frac{\partial g}{\partial y}\mathbf{j} + \frac{\partial g}{\partial z}\mathbf{k}\right) + g\left(\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}\right) = f\nabla g + g\nabla f.$$

The corresponding statement for 1-forms is:

$$d(fg) = f(dg) + g(df).$$

Recall ∇f corresponds to the 1-form df and ∇g corresponds to the 1-form dg. Also $\nabla(fg)$ corresponds to the one form $d(fg)$. The proof is rather obvious:

$$d(fg) = \frac{\partial(fg)}{\partial x}dx + \frac{\partial(fg)}{\partial y}dy + \frac{\partial(fg)}{\partial z}dz = \left(f\frac{\partial g}{\partial x} + g\frac{\partial f}{\partial x}\right)dx + \left(f\frac{\partial g}{\partial y} + g\frac{\partial f}{\partial y}\right)dy + \left(f\frac{\partial g}{\partial z} + g\frac{\partial f}{\partial z}\right)dz$$

$$= f\left(\frac{\partial g}{\partial x}dx + \frac{\partial g}{\partial y}dy + \frac{\partial g}{\partial z}dz\right) + g\left(\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz\right) = fdg + gdf.$$

(b) We have $f\mathbf{F} = fF_1\mathbf{i} + fF_2\mathbf{j} + fF_3\mathbf{k}$, if $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$. Then

$$\nabla\cdot(f\mathbf{F}) = \text{div } (f\mathbf{F}) = \frac{\partial(fF_1)}{\partial x} + \frac{\partial(fF_2)}{\partial y} + \frac{\partial(fF_3)}{\partial z} = \left(\frac{\partial f}{\partial x}F_1 + f\frac{\partial F_1}{\partial x}\right) + \left(\frac{\partial f}{\partial y}F_2 + f\frac{\partial F_2}{\partial y}\right)$$

$$+ \left(\frac{\partial f}{\partial z}F_3 + f\frac{\partial F_3}{\partial z}\right) = f\left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) + \left(\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}\right)\cdot(F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k})$$

$$= f\text{div } \mathbf{F} + \text{grad } f \cdot \mathbf{F}.$$

For the corresponding statement with forms we realize that the divergence of a vector field shows up in Stokes' theorem in relation to the flux of a vector field. So we consider the 2-form

$$\omega = F_1 dydz + F_2 dzdx + F_3 dxdy$$

so that

$$f\omega = fF_1 dydz + fF_2 dzdx + fF_3 dxdy.$$

Then $d\omega$ corresponds to the divergence of ω and $d(f\omega)$ corresponds to the divergence of $f\omega$.

$$d(f\omega) = \frac{\partial(fF_1)}{\partial x}dxdydz + \frac{\partial(fF_2)}{\partial y}dydzdx + \frac{\partial(fF_3)}{\partial z}dzdxdy$$

$$= \left(f\left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) + \frac{\partial f}{\partial x}F_1 + \frac{\partial f}{\partial y}F_2 + \frac{\partial f}{\partial z}F_3 \right) dxdydz$$
$$= (f\text{div }\mathbf{F} + \nabla f \cdot \mathbf{F})\, dxdydz.$$

(c) We have

$$\text{curl }(f\mathbf{F}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ fF_1 & fF_2 & fF_3 \end{vmatrix} = \left(\frac{\partial(fF_3)}{\partial y} - \frac{\partial(fF_2)}{\partial z}\right)\mathbf{i} + \left(\frac{\partial(fF_1)}{\partial z} - \frac{\partial(fF_3)}{\partial x}\right)\mathbf{j}$$

$$+ \left(\frac{\partial(fF_2)}{\partial x} - \frac{\partial(fF_1)}{\partial y}\right)\mathbf{k} = f\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\mathbf{i} + f\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\mathbf{j} + f\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathbf{k}$$
$$+ \left(\frac{\partial f}{\partial y}F_3 - \frac{\partial f}{\partial z}F_2\right)\mathbf{i} + \left(\frac{\partial f}{\partial z}F_1 - \frac{\partial f}{\partial x}F_3\right)\mathbf{j} + \left(\frac{\partial f}{\partial x}F_2 - \frac{\partial f}{\partial y}F_1\right)\mathbf{k}$$
$$= f\text{curl }\mathbf{F} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial f/\partial x & \partial f/\partial y & \partial f/\partial z \\ F_1 & F_2 & F_3 \end{vmatrix} = f\text{curl }\mathbf{F} + \nabla f \times \mathbf{F}.$$

The corresponding statement for one forms is: Let ω be a 1-form (corresponding to $\mathbf{F}$)

$$\omega = F_1dx + F_2dy + F_3dz.$$

This implies $f\omega = fF_1dx + fF_2dy + fF_3dz$. Then

$$d(f\omega) = fd\omega + (df)\omega,$$

where in this case $d\omega$ corresponds to curl $\mathbf{F}$, df corresponds to ∇f, $d(f\omega)$ corresponds to curl $(f\omega)$. The proof is as follows:

$$d(f\omega) = \left(\frac{\partial(fF_1)}{\partial y}dy + \frac{\partial(fF_1)}{\partial z}dz\right)dx + \left(\frac{\partial(fF_2)}{\partial x}dx + \frac{\partial(fF_2)}{\partial z}dz\right)dy + \left(\frac{\partial(fF_3)}{\partial x}dx + \frac{\partial(fF_3)}{\partial y}dy\right)dz$$

$$= f\left(\frac{\partial F_1}{\partial y}dydx + \frac{\partial F_1}{\partial z}dzdx + \frac{\partial F_2}{\partial x}dxdy + \frac{\partial F_2}{\partial z}dzdy + \frac{\partial F_3}{\partial x}dxdz + \frac{\partial F_3}{\partial y}dydz\right)$$
$$+ \left(\frac{\partial f}{\partial y}dydx + \frac{\partial f}{\partial z}dzdx\right)F_1 + \left(\frac{\partial f}{\partial x}dxdy + \frac{\partial f}{\partial z}dzdy\right)F_2 + \left(\frac{\partial f}{\partial x}dxdz + \frac{\partial f}{\partial y}dydz\right)F_3$$
$$= fd\omega + \left(\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz\right)(F_1dx + F_2dy + F_3dz) = fd\omega + (df)\omega$$