Math 434/734

Homework 4

1. (a) Calculate the divergence of the following vector fields:

(i) $x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$, (ii) $yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$, (iii) $(-xy\mathbf{i} + x^2\mathbf{j})/(x^2+y^2), \quad (x,y) \neq (0,0)$.

(b) Calculate the exterior derivative $d\omega$ for the following differential 2-forms ω:

$$\text{(i) } x^2\,dydz + y^2\,dzdx + z^2\,dxdy, \text{(ii) } yz\,dydz + xz\,dzdx + xy\,dxdy,$$

$$\text{(iii) } (-xy\,dydz + x^2\,dzdx)/(x^2+y^2), \quad (x,y) \neq (0,0).$$

What do you notice?

(a) We have

$$\text{div }(x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}) = \frac{\partial x^2}{\partial x} + \frac{\partial y^2}{\partial y} + \frac{\partial z^2}{\partial z} = 2x + 2y + 2z.$$

$$\text{div }(yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}) = \frac{\partial yz}{\partial x} + \frac{\partial xz}{\partial y} + \frac{\partial xy}{\partial z} = 0 + 0 + 0 = 0.$$

$$\text{div }(-xy\mathbf{i} + x^2\mathbf{j})/(x^2+y^2) = \frac{\partial}{\partial x}\frac{-xy}{x^2+y^2} + \frac{\partial}{\partial y}\frac{x^2}{x^2+y^2} = \frac{-y(x^2+y^2) - (-xy)(2x)}{(x^2+y^2)^2}$$

$$+\frac{-x^2\cdot 2y}{(x^2+y^2)^2} = \frac{-yx^2 - y^3 + 2x^2y - 2yx^2}{(x^2+y^2)^2} = \frac{-x^2y - y^3}{(x^2+y^2)^2} = -\frac{y}{x^2+y^2}$$

(b) We follow the standard rules $dxdx = dydy = dzdz = 0$ and the interchange relations $dxdy = -dydx$, $dxdz = -dzdx$, $dydz = -dzdy$. We calculate:

$$d(x^2\,dydz + y^2\,dzdx + z^2\,dxdy) = 2x\,dxdydz + 2y\,dydzdx + 2z\,dzdxdy = (2x+2y+2z)\,dxdydz,$$

$$d(yz\,dydz + xz\,dzdx + xy\,dxdy) = (zdy+ydz)dydz + (zdx+xdz)dzdx + (ydx+xdy)dxdy$$

$$= z\,dydy\,dz - ydy\,dzdz - zdz\,dxdx + x\,dzdz\,dx + y\,dxdx\,dy = xdx\,dydy = 0+0+0+0+0+0 = 0,$$

$$d\left(\frac{-xy\,dydz + x^2\,dzdx}{(x^2+y^2)}\right) = \frac{\partial}{\partial x}\frac{-xy}{x^2+y^2}dxdydz + \frac{\partial}{\partial y}\frac{-xy}{x^2+y^2}dydydz + \frac{\partial}{\partial z}\frac{-xy}{x^2+y^2}dzdydz$$

$$+\frac{\partial}{\partial x}\frac{x^2}{x^2+y^2}dxdzdx + \frac{\partial}{\partial y}\frac{x^2}{x^2+y^2}dydzdx + \frac{\partial}{\partial z}\frac{x^2}{x^2+y^2}dzdzdx$$

$$= \frac{\partial}{\partial x}\frac{-xy}{x^2+y^2}dydzdx + \frac{\partial}{\partial y}\frac{x^2}{x^2+y^2}dxdydz = \left(\frac{\partial}{\partial x}\frac{-xy}{x^2+y^2} + \frac{\partial}{\partial y}\frac{x^2}{x^2+y^2}\right)dxdydz$$

$$= \text{div}(-xy\mathbf{i} + x^2\mathbf{j})/(x^2+y^2)dxdydz$$

We notice that the forms ω in (b) correspond to the vector fields in (a). The divergence of the vector fields are the factors in front of $dxdydz$ in $d\omega$. This is a general feature and applies to all vector fields.

2. An electrostatic field is given by

$$\mathbf{E} = \lambda(yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}),$$

where λ is a constant. Use Gauss' law to find the total charge enclosed by the surface consisting of the hemisphere $z = (R^2 - x^2 - y^2)^{1/2}$ and its circular base on the xy-plane.

According to Gauss' law the total charge ρ divided by the dielectric constant ϵ_0 is the flux of the field over the surface. So we need to calculate the flux.

On the circular base R the unit normal is $-\mathbf{k}$ so that

$$\mathbf{E} \cdot \mathbf{n} = -\lambda xy.$$

We compute $1/\lambda$ of the flux through this base with $dS = dxdy$ and using polar coordinates to get

$$\int_R -xydxdy = \int_0^{2\pi} \int_0^R -r\cos\theta\, r\sin\theta\, rdrd\theta = \int_0^R -r^3 dr \int_0^{2\pi} \cos\theta \sin\theta\, d\theta$$

$$= \int_0^R -r^3 dr \left[\frac{\sin^2\theta}{2}\right]_0^{2\pi} = \int_0^R -r^3 dr \cdot 0 = 0.$$

On the hemisphere the unit normal vector is $\mathbf{r}/R = (x\mathbf{i} + y\mathbf{j} + z\mathbf{k})/R$. So

$$\mathbf{E} \cdot \mathbf{n} = \frac{\lambda}{R}(yzx + xzy + xyz) = \frac{3xyz\lambda}{R}.$$

To calculate dS we notice that as in Homework 3

$$\frac{\partial z}{\partial x} = -\frac{x}{z}, \quad \frac{\partial z}{\partial y} = -\frac{y}{z}$$

on the hemisphere. Since

$$dS = \sqrt{1 + (z)_x^2 + (z)_y^2} = \sqrt{1 + (-x/z)^2 + (-y/z)^2} = \sqrt{\frac{z^2 + x^2 + y^2}{z^2}} = R/z,$$

$$\mathbf{E} \cdot \mathbf{n} dS = \frac{3xyz\lambda}{R}\frac{R}{z} = 3xy\lambda.$$

This gives the integral (up to the factor of λ)

$$\int_R 3xy\, dxdy = 0$$

by the previous calculation on the base. The result is that we include 0 charge inside the hemisphere.

Second method: Use the divergence theorem: by the previous exercise div $\mathbf{E} = 0$. So

$$\int_{\partial S} \mathbf{E} \cdot \mathbf{n}\, dS = \int_S \text{div } \mathbf{E}\, dxdydz = \int_S 0\, dxdydz = 0.$$

Third method: For the surface integral on the hemisphere we can use the 2-form

$$yz\, dydz + xz\, dzdx + xy\, dxdy$$

and its pullback on the xy-plane. We have as above

$$\frac{\partial z}{\partial x} = -\frac{x}{z}, \quad \frac{\partial z}{\partial y} = -\frac{y}{z}$$

and this gives for the pullback

$$yz\,dy(-x/zdx - y/zdy) + xz\,(-x/zdx - y/zdy)dx + xy\,dxdy = xy\,dxdy + xy\,dxdy + xy\,dxdy$$

$$= 3xy\,dxdy.$$

The region of integration is the disc R and we end up with the same integral as before.

3. (a) Calculate the divergence of the function

$$\mathbf{F}(x, y, z) = f(x)\mathbf{i} + f(y)\mathbf{j} + f(-2z)\mathbf{k}$$

and show that it is zero at the point $(c, c, -c/2)$. Here $f(x)$ is a function of a single variable x.

(b) Calculate the divergence of the

$$\mathbf{G}(x, y, z) = f(y, z)\mathbf{i} + g(x, z)\mathbf{j} + h(x, y)\mathbf{k},$$

where f, g and h are differentiable functions of two variables.

(a) We have

$$\frac{\partial f(x)}{\partial x} = f'(x), \quad \frac{\partial f(y)}{\partial y} = f'(y),$$

while by the chain rule in one-variable calculus:

$$\frac{\partial f(-2z)}{\partial z} = f'(-2z)(-2).$$

This gives:

$$\text{div } \mathbf{F}(x, y, z) = f'(x) + f'(y) - 2f'(-2z).$$

We plug $(c, c, -c/2)$ to get

$$\text{div } \mathbf{F}(c, c, -c/2) = f'(c) + f'(c) - 2f'(-2(-c/2)) = 2f'(c) - 2f'(c) = 0.$$

(b) We clearly have

$$\frac{\partial f(y, z)}{\partial x} = 0, \quad \frac{\partial g(x, z)}{\partial y} = 0, \quad \frac{\partial h(x, y)}{\partial z} = 0.$$

The result is that div $\mathbf{G}(x, y, z) = 0 + 0 + 0 = 0$.

4. (a) Calculate the curl of the following vector fields

(i) $z^2\mathbf{i} + x^2\mathbf{j} - y^2\mathbf{k}$, (ii) $x\mathbf{i} + y\mathbf{j} + (x^2 + y^2)\mathbf{k}$, (iii) $(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})/(x^2 + y^2 + z^2)^{3/2}$, $(x, y, z) \neq (0, 0, 0)$.

(b) Calculate the exterior derivative of the one forms:

$$\text{(i) } z^2dx + x^2dy - y^2dz, \text{(ii) } x\,dx + y\,dy + (x^2 + y^2)dz,$$

(iii) $(x\,dx + y\,dy + z\,dz)/(x^2+y^2+z^2)^{3/2}$, $(x,y,z) \neq (0,0,0)$.

What do you notice?

(a) Using the determinant formula for the curl we get for (i)

$$\text{curl}\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ z^2 & x^2 & -y^2 \end{vmatrix} = \left(\frac{\partial(-y^2)}{\partial y} - \frac{\partial x^2}{\partial z}\right)\mathbf{i} - \left(\frac{\partial(-y^2)}{\partial x} - \frac{\partial z^2}{\partial z}\right)\mathbf{j} + \left(\frac{\partial x^2}{\partial x} - \frac{\partial z^2}{\partial y}\right)\mathbf{k}$$

$$= -2y\mathbf{i} + 2z\mathbf{j} + 2x\mathbf{k}.$$

For (ii) we get

$$\text{curl}\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x & y & x^2+y^2 \end{vmatrix} = \left(\frac{\partial(x^2+y^2)}{\partial y} - \frac{\partial y}{\partial z}\right)\mathbf{i} - \left(\frac{\partial(x^2+y^2)}{\partial x} - \frac{\partial x}{\partial z}\right)\mathbf{j} + \left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}\right)\mathbf{k}$$

$$= 2y\mathbf{i} - 2x\mathbf{j}.$$

For (iii) we calculate first the partial derivatives of x/r^3 with respect to y and z. Here $r^2 = x^2+y^2+z^2$. We have

$$\frac{\partial(x/r^3)}{\partial y} = \frac{\partial}{\partial y}\frac{x}{(x^2+y^2+z^2)^{3/2}} = -\frac{3}{2}x(x^2+y^2+z^2)^{-5/2}2y = -3xyr^{-5}.$$

$$\frac{\partial(x/r^3)}{\partial z} = \frac{\partial}{\partial z}\frac{x}{(x^2+y^2+z^2)^{3/2}} = -\frac{3}{2}x(x^2+y^2+z^2)^{-5/2}2z = -3xzr^{-5}.$$

Similarly we get for the other partial derivatives:

$$\frac{\partial(y/r^3)}{\partial x} = \frac{\partial}{\partial x}\frac{y}{(x^2+y^2+z^2)^{3/2}} = -\frac{3}{2}y(x^2+y^2+z^2)^{-5/2}2x = -3xyr^{-5}.$$

$$\frac{\partial(y/r^3)}{\partial z} = \frac{\partial}{\partial z}\frac{y}{(x^2+y^2+z^2)^{3/2}} = -\frac{3}{2}y(x^2+y^2+z^2)^{-5/2}2z = -3xzr^{-5}.$$

$$\frac{\partial(z/r^3)}{\partial x} = \frac{\partial}{\partial x}\frac{z}{(x^2+y^2+z^2)^{3/2}} = -\frac{3}{2}z(x^2+y^2+z^2)^{-5/2}2x = -3xzr^{-5}.$$

$$\frac{\partial(z/r^3)}{\partial y} = \frac{\partial}{\partial y}\frac{z}{(x^2+y^2+z^2)^{3/2}} = -\frac{3}{2}z(x^2+y^2+z^2)^{-5/2}2y = -3zyr^{-5}.$$

In particular we see that

$$\frac{\partial(x/r^3)}{\partial y} = \frac{\partial(y/r^3)}{\partial x}, \quad \frac{\partial(x/r^3)}{\partial z} = \frac{\partial(z/r^3)}{\partial x}, \quad \frac{\partial(y/r^3)}{\partial z} = \frac{\partial(z/r^3)}{\partial y}.$$

By expanding the determinant for the curl we get that this time it is $\vec{0}$.

For (b) we calculate the exterior derivative using the standard rules. We get for (i):

$$d(z^2dx+x^2dy-y^2dz) = \left(\frac{\partial z^2}{\partial z}\right)dzdx + \left(\frac{\partial x^2}{\partial x}\right)dxdy - \left(\frac{\partial y^2}{\partial y}\right)dydz = 2zdzdx+2xdxdy-2ydydz$$

We see that this two form corresponds to the curl of the given vector field, calculated in (a) to be $-2y\mathbf{i} + 2z\mathbf{j} + 2x\mathbf{k}$.

For (ii) we have no contribution from $F_1 = x$, which has only derivative in x and gives a $dx\,dx$. Similarly we have no contribution from $F_2 = y$, since it has derivative only in y and this gives $dy\,dy$.. The result is:

$$d(x\,dx + y\,dy + (x^2+y^2)dz) = \left(\frac{\partial(x^2+y^2)}{\partial x}dx + \frac{\partial(x^2+y^2)}{\partial y}dy\right) dz$$

$$= 2x\,dxdz + 2y\,dydz = -2x\,dzdx + 2y\,dydz$$

and this 2-form corresponds to the curl of the given vector field calculated in (a): $2y\mathbf{i} - 2x\mathbf{j}$.

For (iii) we write the 1-form as $\omega = F_1 dx + F_2 dy + F_3 dz$, where $F_1 = x/r^3$, $F_2 = y/r^3$, $F_3 = z/r^3$. We calculated all the necessary partial derivatives above and got:

$$(F_1)_y = (F_2)_x, \quad (F_2)_z = (F_3)_y, \quad (F_1)_z = (F_3)_x.$$

Now we have, denoting with the subscript x, y, z the partial derivatives,

$$d\omega = ((F_1)_y dy + (F_1)_z dz)\,dx + ((F_2)_x dx + (F_2)_z dz)dy + ((F_3)_x dx + (F_3)_y dy)dz$$

$$= ((F_3)_y - (F_2)_z)dydz + ((F_1)_z - (F_3)_x)dzdx + ((F_2)_x - (F_1)_y)dxdy = 0dydz + 0dzdx + 0dxdy.$$

This form correspond to the curl computed above to be the vector $\vec{0}$.