1. (a) Calculate the divergence of the following vector fields:

(i) \( x^2i + y^2j + z^2k \), (ii) \( yzi + xzj + xyk \), (iii) \( (-xyi + x^2j)/(x^2 + y^2) \), \((x, y) \neq (0, 0)\).

(b) Calculate the exterior derivative \( d\omega \) for the following differential 2-forms \( \omega \):

(i) \( x^2 dydz + y^2 dzdx + z^2 dxdy \), (ii) \( yzdydz + xz dzdx + xy dxdy \),

(iii) \( (-xy dydz + x^2 dzdx)/(x^2 + y^2) \), \((x, y) \neq (0, 0)\).

What do you notice?

(a) We have

\[
\text{div} \left( x^2i + y^2j + z^2k \right) = \frac{\partial x^2}{\partial x} + \frac{\partial y^2}{\partial y} + \frac{\partial z^2}{\partial z} = 2x + 2y + 2z.
\]

\[
\text{div} \left( yzi + xzj + xyk \right) = \frac{\partial yz}{\partial x} + \frac{\partial xz}{\partial y} + \frac{\partial xy}{\partial z} = 0 + 0 + 0 = 0.
\]

\[
\text{div} \left( -xyi + x^2j \right)/(x^2 + y^2) = \frac{\partial}{\partial y} \left( -xy \right) + \frac{\partial}{\partial x} \left( x^2 \right) = -y(x^2 + y^2) - (-xy)(2x) + \frac{-x^2 \cdot 2y}{(x^2 + y^2)^2} = \frac{-y(x^2 + y^2)}{(x^2 + y^2)^2} = -\frac{y}{x^2 + y^2}.
\]

(b) We follow the standard rules \( dx dx = dy dy = dz dz = 0 \) and the interchange relations \( dx dy = -dy dx, \; dx dz = -dz dx, \; dy dz = -dz dy \). We calculate:

\[
d(x^2 dydz + y^2 dzdx + z^2 dxdy) = 2x dx dy dz + 2y dy dz dx + 2z dz dx dy = (2x + 2y + 2z) dx dy dz,
\]

\[
d(yz dydz + xz dzdx + xy dxdy) = (z dy + y dz) dy dz + (z dx + x dz) dz dx + (y dx + x dy) dxdy
\]

\[= z dy dz - y dz dy dz - z dz dx + x dz dz dx + y dx dx dy = x dx dy dz = 0 + 0 + 0 + 0 + 0 = 0,
\]

\[
d \left( \frac{-xy dydz + x^2 dzdx}{(x^2 + y^2)} \right) = \frac{\partial}{\partial x} \left( -xy \right) dx dy dz dx + \frac{\partial}{\partial y} \left( -xy \right) dy dz dx dy dz dx
\]

\[+ \frac{\partial}{\partial x} \frac{x^2}{x^2 + y^2} dx dz dx dz dx + \frac{\partial}{\partial y} \frac{x^2}{x^2 + y^2} dy dz dx dy dz dx
\]

\[= \frac{\partial}{\partial x} \left( -xy \right) dy dz dx + \frac{\partial}{\partial y} \left( -xy \right) dy dz dx
\]

\[= \text{div}(-xyi + x^2j)/(x^2 + y^2) dxdy dz.
\]

We notice that the forms \( \omega \) in (b) correspond to the vector fields in (a). The divergence of the vector fields are the factors in front of \( dxdy dz \) in \( d\omega \). This is a general feature and applies to all vector fields.
2. An electrostatic field is given by
\[ \mathbf{E} = \lambda (yz \mathbf{i} + xz \mathbf{j} + xy \mathbf{k}) , \]
where \( \lambda \) is a constant. Use Gauss’ law to find the total charge enclosed by the surface consisting of the hemisphere \( z = (R^2 - x^2 - y^2)^{1/2} \) and its circular base on the \( xy \)-plane. According to Gauss’ law the total charge \( \rho \) divided by the dielectric constant \( \epsilon_0 \) is the flux of the field over the surface. So we need to calculate the flux.

On the circular base \( R \) the unit normal is \(-\mathbf{k}\) so that
\[ \mathbf{E} \cdot \mathbf{n} = -\lambda xy. \]

We compute \( 1/\lambda \) of the flux through this base with \( dS = dxdy \) and using polar coordinates to get
\[
\int_{-R}^{R} -xydxdy = \int_{0}^{2\pi} \int_{0}^{R} -r \cos \theta r \sin \theta r dr d\theta = \int_{0}^{2\pi} -r^3 dr \int_{0}^{R} \cos \theta \sin \theta d\theta \\
= \int_{0}^{R} -r^3 dr \left[ \frac{\sin^2 \theta}{2} \right]_{0}^{2\pi} = \int_{0}^{R} -r^3 dr \cdot 0 = 0.
\]

On the hemisphere the unit normal vector is \( \mathbf{r}/R = (x \mathbf{i} + y \mathbf{j} + z \mathbf{k})/R \). So
\[ \mathbf{E} \cdot \mathbf{n} = \frac{\lambda}{R} (yzx + xzy + xyz) = \frac{3xyz\lambda}{R} . \]

To calculate \( dS \) we notice that as in Homework 3
\[ \frac{\partial z}{\partial x} = -\frac{x}{z}, \quad \frac{\partial z}{\partial y} = -\frac{y}{z} \]
on the hemisphere. Since
\[ dS = \sqrt{1 + (z^2_x)^2 + (z^2_y)^2} = \sqrt{1 + (-x/z)^2 + (-y/z)^2} = \sqrt{\frac{z^2 + x^2 + y^2}{z^2}} = R/z, \]
\[ \mathbf{E} \cdot \mathbf{n} dS = \frac{3xyz\lambda R}{z} = 3xy\lambda . \]
This gives the integral (up to the factor of \( \lambda \))
\[ \int_{R} 3xy dxdy = 0 \]
by the previous calculation on the base. The result is that we include 0 charge inside the hemisphere.

Second method: Use the divergence theorem: by the previous exercise \( \text{div } \mathbf{E} = 0 \).
\[ \int_{\partial S} \mathbf{E} \cdot \mathbf{n} dS = \int_{S} \text{div } \mathbf{E} dxdydz = \int_{S} 0 dxdydz = 0. \]

Third method: For the surface integral on the hemisphere we can use the 2-form
\[ yz \, dydz + xz \, dzdx + xy \, dxdy \]
and its pullback on the $xy$-plane. We have as above
\[
\frac{\partial z}{\partial x} = -\frac{x}{z}, \quad \frac{\partial z}{\partial y} = -\frac{y}{z}
\]
and this gives for the pullback
\[
yz \, dy(-x/z \, dx - y/z \, dy) + xz \, (-x/z \, dx - y/z \, dy) \, dx + xy \, dx \, dy = xy \, dx \, dy + xy \, dx \, dy + xy \, dx \, dy
\]
\[
= 3xy \, dx \, dy.
\]
The region of integration is the disc $R$ and we end up with the same integral as before.

3. (a) Calculate the divergence of the function
\[
\mathbf{F}(x, y, z) = f(x)\mathbf{i} + f(y)\mathbf{j} + f(-2z)\mathbf{k}
\]
and show that it is zero at the point $(c, c, -c/2)$. Here $f(x)$ is a function of a single variable $x$.

(b) Calculate the divergence of the
\[
\mathbf{G}(x, y, z) = f(y, z)\mathbf{i} + g(x, z)\mathbf{j} + h(x, y)\mathbf{k},
\]
where $f$, $g$ and $h$ are differentiable functions of two variables.

(a) We have
\[
\frac{\partial f(x)}{\partial x} = f'(x), \quad \frac{\partial f(y)}{\partial y} = f'(y),
\]
while by the chain rule in one-variable calculus:
\[
\frac{\partial f(-2z)}{\partial z} = f'(-2z)(-2).
\]
This gives:
\[
\text{div } \mathbf{F}(x, y, z) = f'(x) + f'(y) - 2f'(-2z).
\]
We plug $(c, c, -c/2)$ to get
\[
\text{div } \mathbf{F}(c, c, -c/2) = f'(c) + f'(c) - 2f'(-c/2)) = 2f'(c) - 2f'(c) = 0.
\]

(b) We clearly have
\[
\frac{\partial f(y, z)}{\partial x} = 0, \quad \frac{\partial g(x, z)}{\partial y} = 0, \quad \frac{\partial h(x, y)}{\partial z} = 0.
\]
The result is that $\text{div } \mathbf{G}(x, y, z) = 0 + 0 + 0 = 0$.

4. (a) Calculate the curl of the following vector fields
\[
(i) \ z^2\mathbf{i} + x^2\mathbf{j} - y^2\mathbf{k}, (ii) \ x\mathbf{i} + y\mathbf{j} + (x^2+y^2)\mathbf{k}, (iii) \ (x\mathbf{i} + y\mathbf{j} + z\mathbf{k})/(x^2+y^2+z^2)^{3/2}, (x, y, z) \neq (0, 0, 0).
\]

(b) Calculate the exterior derivative of the one forms:
\[
(i) \ z^2 \, dx + x^2 \, dy - y^2 \, dz, (ii) \ x \, dx + y \, dy + (x^2 + y^2) \, dz,
\]
(iii) \((x \, dx + y \, dy + z \, dz)/(x^2 + y^2 + z^2)^{3/2}, (x, y, z) \neq (0, 0, 0)\).

What do you notice?

(a) Using the determinant formula for the curl we get for \(i\)

\[
\text{curl} \mathbf{F} = \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
x^2 & \frac{1}{y^2} & \frac{1}{z^2}
\end{vmatrix} = \left( \frac{\partial(-y^2)}{\partial y} - \frac{\partial x^2}{\partial z} \right) \mathbf{i} - \left( \frac{\partial(-y^2)}{\partial x} - \frac{\partial z^2}{\partial z} \right) \mathbf{j} + \left( \frac{\partial x^2}{\partial x} - \frac{\partial z^2}{\partial y} \right) \mathbf{k}
\]

\[-\mathbf{2y} \mathbf{i} + 2z \mathbf{j} + 2x \mathbf{k}.
\]

For \(ii\) we get

\[
\text{curl} \mathbf{F} = \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
x & \frac{1}{y} & \frac{1}{z}
\end{vmatrix} = \left( \frac{\partial(x^2 + y^2)}{\partial y} - \frac{\partial y}{\partial z} \right) \mathbf{i} - \left( \frac{\partial(x^2 + y^2)}{\partial x} - \frac{\partial x}{\partial z} \right) \mathbf{j} + \left( \frac{\partial y}{\partial x} - \frac{\partial x}{\partial y} \right) \mathbf{k}
\]

\[-2y \mathbf{i} - 2x \mathbf{j}.
\]

For \(iii\) we calculate first the partial derivatives of \(x/r^3\) with respect to \(y\) and \(z\). Here \(r^2 = x^2 + y^2 + z^2\). We have

\[
\frac{\partial(x/r^3)}{\partial y} = \frac{\partial}{\partial y} \left( \frac{x}{x^2 + y^2 + z^2}^{3/2} \right) = -\frac{3}{2} \frac{x(x^2 + y^2 + z^2)^{-5/2} \cdot 2y}{x^2 + y^2 + z^2} = -3xyr^{-5}.
\]

\[
\frac{\partial(x/r^3)}{\partial z} = \frac{\partial}{\partial z} \left( \frac{x}{x^2 + y^2 + z^2}^{3/2} \right) = -\frac{3}{2} \frac{x(x^2 + y^2 + z^2)^{-5/2} \cdot 2z}{x^2 + y^2 + z^2} = -3xzr^{-5}.
\]

Similarly we get for the other partial derivatives:

\[
\frac{\partial(y/r^3)}{\partial x} = \frac{\partial}{\partial x} \left( \frac{y}{x^2 + y^2 + z^2}^{3/2} \right) = -\frac{3}{2} \frac{y(x^2 + y^2 + z^2)^{-5/2} \cdot 2x}{x^2 + y^2 + z^2} = -3xyr^{-5}.
\]

\[
\frac{\partial(y/r^3)}{\partial z} = \frac{\partial}{\partial z} \left( \frac{y}{x^2 + y^2 + z^2}^{3/2} \right) = -\frac{3}{2} \frac{y(x^2 + y^2 + z^2)^{-5/2} \cdot 2z}{x^2 + y^2 + z^2} = -3xzr^{-5}.
\]

\[
\frac{\partial(z/r^3)}{\partial x} = \frac{\partial}{\partial x} \left( \frac{z}{x^2 + y^2 + z^2}^{3/2} \right) = -\frac{3}{2} \frac{z(x^2 + y^2 + z^2)^{-5/2} \cdot 2x}{x^2 + y^2 + z^2} = -3xzr^{-5}.
\]

\[
\frac{\partial(z/r^3)}{\partial y} = \frac{\partial}{\partial y} \left( \frac{z}{x^2 + y^2 + z^2}^{3/2} \right) = -\frac{3}{2} \frac{z(x^2 + y^2 + z^2)^{-5/2} \cdot 2y}{x^2 + y^2 + z^2} = -3yzr^{-5}.
\]

In particular we see that

\[
\frac{\partial(x/r^3)}{\partial y} = \frac{\partial(y/r^3)}{\partial x}, \quad \frac{\partial(x/r^3)}{\partial z} = \frac{\partial(z/r^3)}{\partial x}, \quad \frac{\partial(y/r^3)}{\partial z} = \frac{\partial(z/r^3)}{\partial y}.
\]

By expanding the determinant for the curl we get that this time it is \(\mathbf{0}\).

For \(b\) we calculate the exterior derivative using the standard rules. We get for \(i\):

\[
d(z^2dx + x^2dy - dz^2) = \left( \frac{\partial z^2}{\partial z} \right) dzdx + \left( \frac{\partial x^2}{\partial x} \right) dx - \left( \frac{\partial y^2}{\partial y} \right) dydz = 2zdzdx + 2xdxdy - 2ydydz
\]

We see that this two form corresponds to the curl of the given vector field, calculated in \(a\) to be \(-2y \mathbf{i} + 2z \mathbf{j} + 2x \mathbf{k}\).
For (ii) we have no contribution from $F_1 = x$, which has only derivative in $x$ and gives a $dx \, dx$. Similarly we have no contribution from $F_2 = y$, since it has derivative only in $y$ and this gives $dy \, dy$. The result is:

$$d(x \, dx + y \, dy + (x^2 + y^2)dz) = \left( \frac{\partial(x^2 + y^2)}{\partial x} \, dx + \frac{\partial(x^2 + y^2)}{\partial y} \, dy \right) dz$$

$$= 2x \, dx \, dz + 2y \, dy \, dz = -2x \, dz \, dx + 2y \, dy \, dz$$

and this 2-form corresponds to the curl of the given vector field calculated in (a); $2yi - 2xj$.

For (iii) we write the 1-form as $\omega = F_1 dx + F_2 dy + F_3 dz$, where $F_1 = x/r^3$, $F_2 = y/r^3$, $F_3 = z/r^3$. We calculated all the necessary partial derivatives above and got:

$$(F_1)_y = (F_2)_x, \quad (F_2)_z = (F_3)_y, \quad (F_1)_z = (F_3)_x.$$

Now we have, denoting with the subscript $x, y, z$ the partial derivatives,

$$d\omega = ((F_1)_y dy + (F_1)_z dz) \, dx + ((F_2)_x dx + (F_2)_z dz) \, dy + ((F_3)_x dx + (F_3)_y dy) dz$$

$$= ((F_3)_y - (F_2)_z) dy dz + ((F_1)_z - (F_3)_x) dz dx + ((F_2)_x - (F_1)_y) dx dy = 0 dy dz + 0 dz dx + 0 dx dy.$$

This form correspond to the curl computed above to be the vector $\vec{0}$. 
