Math 434/734

Homework 3

1. Find a unit normal vector \( \mathbf{n} \) to each of the following surfaces

   \[ (a) \quad z = 2 - x - y, \quad (b) \quad z = (x^2 + y^2)^{1/2}. \]

(a) A tangent vector to the surface on a plane \( y = c \), is given by

\[
\mathbf{u} = \mathbf{i} + \frac{\partial z}{\partial x} \mathbf{k} = \mathbf{i} - \mathbf{k}.
\]

A tangent vector to the surface on the plane \( x = c \) is given by

\[
\mathbf{v} = \mathbf{j} + \frac{\partial z}{\partial y} \mathbf{k} = \mathbf{j} - \mathbf{k},
\]

as the partial derivatives gives the slopes of the tangent lines on the intersection of the surface with the planes \( y = c \) and \( x = c \) respectively. A normal vector is \( \mathbf{u} \times \mathbf{v} \), which is

\[
\begin{vmatrix}
  \mathbf{i} & \mathbf{j} & \mathbf{k} \\
  1 & 0 & -1 \\
  0 & 1 & -1
\end{vmatrix} = \mathbf{k} + \mathbf{i} + \mathbf{j}.
\]

The unit normal vector is

\[
\mathbf{n} = \frac{\mathbf{k} + \mathbf{i} + \mathbf{j}}{|\mathbf{k} + \mathbf{i} + \mathbf{j}|} = \frac{1}{\sqrt{3}} (\mathbf{k} + \mathbf{i} + \mathbf{j}).
\]

(b) We calculate the partial derivatives:

\[
\frac{\partial z}{\partial x} = \frac{2x}{2(x^2 + y^2)^{1/2}} = \frac{x}{z}, \quad \frac{\partial z}{\partial y} = \frac{2y}{2(x^2 + y^2)^{1/2}} = \frac{y}{z}.
\]

With

\[
\mathbf{u} = \mathbf{i} + \frac{x}{z} \mathbf{k}, \quad \mathbf{v} = \mathbf{j} + \frac{y}{z} \mathbf{k}
\]

\[
\mathbf{u} \times \mathbf{v} = \begin{vmatrix}
  \mathbf{i} & \mathbf{j} & \mathbf{k} \\
  1 & 0 & \frac{x}{z} \\
  0 & 1 & \frac{y}{z}
\end{vmatrix} = \mathbf{k} - \frac{x}{z} \mathbf{i} - \frac{y}{z} \mathbf{j}.
\]

The unit normal vector is

\[
\mathbf{n} = \frac{\mathbf{k} - \frac{x}{z} \mathbf{i} - \frac{y}{z} \mathbf{j}}{|\mathbf{k} - \frac{x}{z} \mathbf{i} - \frac{y}{z} \mathbf{j}|} = \frac{\mathbf{k} - (x/z)\mathbf{i} - (y/z)\mathbf{j}}{\sqrt{1 + x^2/z^2 + y^2/z^2}} = \frac{\mathbf{k} - (x/z)\mathbf{i} - (y/z)\mathbf{j}}{\sqrt{z^2 + x^2 + y^2/z}} = \frac{z\mathbf{k} - x\mathbf{i} - y\mathbf{j}}{\sqrt{2z}}.
\]
2. Calculate the integral of the 1-form \( z^2dx + x^2dy - y^2dz \) along the square of size \( s \) with vertices \((x_0 - s/2, x_0 - s/2, 0), (x_0 + s/2, x_0 - s/2, 0), (x_0 + s/2, x_0 + s/2, 0), (x_0 - s/2, x_0 + s/2, 0)\) oriented counterclockwise.

Along the lower horizontal side \( C_1 \) we use the parametrization \((x, y, z) = (t, x_0 - s/2, 0), x_0 - s/2 \leq t \leq x_0 + s/2\). We have \(dx/dt = 1, dy/dt = 0, dz/dt = 0\). The (line) integral becomes

\[
\int_{C_1} z^2dx + x^2dy - y^2dz = \int_{x_0-s/2}^{x_0+s/2} 0 \cdot 1 + t^2 \cdot 0 - (x_0 - s/2)^2 \cdot 0 dt = \int_{x_0-s/2}^{x_0+s/2} 0 dt = 0.
\]

Along the upper horizontal side \( C_2 \) we calculate the integral over \(-C_2\), with the opposite orientation and we use the parametrization \((x, y, z) = (t, x_0 + s/2, 0), x_0 - s/2 \leq t \leq x_0 + s/2\). We have \(dx/dt = 1, dy/dt = 0, dz/dt = 0\). The (line) integral becomes

\[
\int_{-C_2} z^2dx + x^2dy - y^2dz = \int_{x_0-s/2}^{x_0+s/2} 0 \cdot 1 + t^2 \cdot 0 - (x_0 + s/2)^2 \cdot 0 dt = \int_{x_0-s/2}^{x_0+s/2} 0 dt = 0.
\]

Along the right side of the square \( C_3 \) we use the parametrization \((x, y, z) = (x_0 + s/2, t, 0), x_0 - s/2 \leq t \leq x_0 + s/2\). We have \(dx/dt = 0, dy/dt = 1, dz/dt = 0\). The (line) integral becomes

\[
\int_{C_3} z^2dx + x^2dy - y^2dz = \int_{x_0-s/2}^{x_0+s/2} 0 \cdot 0 + (x_0+s/2)^2 \cdot 1 - t^2 \cdot 0 dt = \int_{x_0-s/2}^{x_0+s/2} (x_0+s/2)^2 dt = s \cdot (x_0+s/2)^2,
\]

since we are integrating a constant function over an interval of length \( s \).

Along the left side of the square \( C_4 \) we calculate the integral over \(-C_4\), i.e. with the opposite orientation and use the parametrization \((x, y, z) = (x_0 - s/2, t, 0), x_0 - s/2 \leq t \leq x_0 + s/2\). We have \(dx/dt = 0, dy/dt = 1, dz/dt = 0\). The (line) integral becomes

\[
\int_{-C_4} z^2dx + x^2dy - y^2dz = \int_{x_0-s/2}^{x_0+s/2} 0 \cdot 0 + (x_0-s/2)^2 \cdot 1 - t^2 \cdot 0 dt = \int_{x_0-s/2}^{x_0+s/2} (x_0-s/2)^2 dt = s \cdot (x_0-s/2)^2,
\]

since we are integrating a constant function over an interval of length \( s \).

The final integral is the difference of the above results, as

\[
\int_C z^2dx + x^2dy - y^2dz = \int_{C_1} - \int_{-C_2} + \int_{C_3} - \int_{-C_4} z^2dx + x^2dy - y^2dz = s \cdot (x_0+s/2)^2 - s \cdot (x_0-s/2)^2 = s(x_0+s/2+x_0-s/2)(x_0+s/2-x_0+s/2) = s2x_0s = 2s^2x_0.
\]

3. Evaluate the integral \( \int_S \mathbf{F} \cdot n dS \) (surface integral or flow) of the vector field

\[
\mathbf{F}(x, y, z) = xi - zk,
\]

where \( S \) is the portion of the plane \( x + y + 2z = 2 \) in the first octant and the unit normal is in the positive \( x, y \) and \( z \) direction.

The 2-form corresponding to \( \mathbf{F} \) is

\[
xdydz - zdxdy
\]
Figure 1: The square contour.

and we calculate its pullback under the parametrization of the portion of the plane in the first octant

\[ z = 1 - \frac{x}{2} - \frac{y}{2}, \quad 0 \leq x \leq 2, \quad 0 \leq y \leq 2 - x. \]

What we have done is project the plane to the \( xy \)-plane and parametrized the triangle \( T \), which is the projection in the first quadrant, as a vertically simple region. The triangle \( T \) is oriented in the counterclockwise direction. We have \( dz = -(1/2)dx - (1/2)dy \) and

\[
xdydz - zdxdy = xdy(-1/2dx - 1/2dy) - (1-y/2-x/2)dxdy = -(x/2)dydx - (1-y/2-x/2)dxdy
\]

\[
= (x/2 - 1 + y/2 + x/2)dxdy = (x - 1 + y/2)dxdy.
\]

The integral of the 2-form is calculated as a double integral over the region parametrized as vertically simple region.

\[
\int \mathbf{F} \cdot \mathbf{n} dS = \int_T (x-1+y/2)dydx = \int_0^2 \int_0^{2-x} (x-1+y/2)dydx = \int_0^2 [(x - 1)y + y^2/4]_{y=0}^{y=2-x} dx
\]

\[
= \int_0^2 (x-1)(2-x) + (2-x)^2/4 dx = \int_0^2 -x^2 + 3x - 2 + x^2/4 - x + 1 dx = \int_0^2 -3x^2/4 + 2x - 1 dx
\]

\[
= [-(x^3/4 + x^2 - x)]_0^2 = -8/4 + 2^2 - 2 = 0.
\]

4. Evaluate the integral \( \int_S \mathbf{F} \cdot \mathbf{n} dS \) (surface integral or flow) of the vector field

\[
\mathbf{F}(x, y, z) = xi + yj + zk,
\]

where \( S \) is the hemisphere \( z = \sqrt{a^2 - x^2 - y^2} \) oriented counterclockwise as seen from the outside. A simple calculation of partial derivatives gives
Figure 2: Projection in the first quadrant is lightly-shaded.

Figure 3: The portion of the plane in the first octant.
\[ \frac{\partial z}{\partial x} = -\frac{x}{z}, \quad \frac{\partial z}{\partial y} = -\frac{y}{z}. \]

This gives
\[ dz = -\frac{x}{z} dx - \frac{y}{z} dy. \]

The projection of the hemisphere on the xy-plane is the disc
\[ T = \{(x, y), x^2 + y^2 \leq a^2 \} \]
oriented counterclockwise. Then
\[
\int_S \mathbf{F} \cdot \mathbf{n} \, dS = \int_S x \, dydz + y \, dzdx + z \, dxdy = \int_T xdy \left( -\frac{x}{z} dx - \frac{y}{z} dy \right) + y \left( -\frac{x}{z} dx - \frac{y}{z} dy \right) \, dx + z \, dxdy
\]
\[
= \int_T -\frac{x^2}{z} dydx - \frac{y^2}{z} dydx + z \, dxdy = \int_T \frac{x^2 + y^2 + z^2}{z} \, dxdy = \int_T \frac{a^2}{\sqrt{a^2 - x^2 - y^2}} \, dxdy
\]
We switch to polar coordinates \( x = r \cos \theta, \ y = r \sin \theta, \ x^2 + y^2 = r^2, \ dx \, dy = r \, dr \, d\theta \) to get
\[
\int_T \frac{a^2}{\sqrt{a^2 - x^2 - y^2}} \, dx \, dy = \int_0^{2\pi} \int_0^a \frac{a^2 r}{\sqrt{a^2 - r^2}} \, dr \, d\theta = \int_0^{2\pi} \left[ -a^2 \sqrt{a^2 - r^2} \right]_0^a \, d\theta
\]
\[
= \int_0^{2\pi} a^2 \sqrt{a^2} \, d\theta = 2\pi a^3.
\]

Remark: In this example the interpretation as a flow gives a much easier way of calculation: The vector field is radial i.e. in the direction of the vector \( \mathbf{r} \) from the origin to every point on the hemisphere. So \( \mathbf{F} \cdot \mathbf{n} = |\mathbf{F}| = |\mathbf{r}| = a \). We are integrating the constant 2-form \( adS \) on the hemisphere, so we get the area of the hemisphere, which is \( 2\pi a^2 \) times \( a \), i.e. \( 2\pi a^3 \).
5. Evaluate the integral \( \int_S F \cdot n \, dS \) (surface integral or flow) of the vector field

\[
F(x, y, z) = y\, j + k,
\]

where \( S \) is the portion of the paraboloid \( z = 1 - x^2 - y^2 \) above the \( xy \)-plane oriented counterclockwise as seen from above.

We have

\[
\frac{\partial z}{\partial x} = -2x, \quad \frac{\partial z}{\partial y} = -2y, \quad dz = -2x \, dx = 2y \, dy.
\]

The pullback of the form \( y \, dz \, dx + dx \, dy \) is

\[
y(-2x \, dx - 2y \, dy) \, dx + dx \, dy = 2y^2 \, dx \, dy + dx \, dy = (1 + 2y^2) \, dx \, dy.
\]

The projection of the surface on the \( xy \)-plane is the disc of radius 1 centered at \((0, 0)\) oriented counterclockwise. We evaluate the resulting integral in polar coordinates

\[
\int_0^{2\pi} \int_0^1 (1 + 2r^2 \sin^2 \theta) \, r \, dr \, d\theta = \int_0^{2\pi} \int_0^1 r \, dr \, d\theta + \int_0^{2\pi} \int_0^1 2r^3 \sin^2 \theta \, dr \, d\theta = \pi + \int_0^{2\pi} \left[ \frac{r^4}{2} \right]_0^1 \sin^2 \theta \, d\theta = \pi + \frac{1}{2} \int_0^{2\pi} \sin^2 \theta \, d\theta = \pi + \frac{\pi}{2} = \frac{3\pi}{2}.
\]

6. (a) Assume a surface \( S \) is parametrized as a function \( y = h(x, z) \), where \((x, z) \in D\).

Show that

\[
\int_S Adydz + Bdzdx + Cdxdy = \int_D \left\{ -A(x, h(x, z), z) \frac{\partial h}{\partial x} + B(x, h(x, z), z) - C(x, h(x, z), z) \frac{\partial h}{\partial z} \right\} dz \, dx.
\]

(b) Assume a surface \( S \) is parametrized as a function \( x = g(y, z) \), where \((y, z) \in D\).

Show that

\[
\int_S Adydz + Bdzdx + Cdxdy = \int_D \left\{ A(g(y, z), y, z) - B(g(y, z), y, z) \frac{\partial g}{\partial y} - C(g(y, z), y, z) \frac{\partial g}{\partial z} \right\} dy \, dz.
\]
We calculate the pullbacks of $Adydz + Bdzdx + Cdxdy$ under the maps $x = x$, $y = h(x, z)$ and $z = z$ (resp. $x = g(y, z)$, $y = y$, $z = z$). We get

$$dy = \frac{\partial h}{\partial x} dx + \frac{\partial h}{\partial z} dz$$

in (a) and

$$dx = \frac{\partial g}{\partial y} dy + \frac{\partial g}{\partial z} dz$$

in (b). This gives

$$Adydz + Bdzdx + Bdxdy = A(x, h(x, z), z) \left( \frac{\partial h}{\partial x} dx + \frac{\partial h}{\partial z} dz \right) dz + B(x, h(x, z), z) dz dx$$

$$+ C(x, h(x, z), z) dx \left( \frac{\partial h}{\partial x} dx + \frac{\partial h}{\partial z} dz \right) = A \frac{\partial h}{\partial x} dxdz + Bdzdx + C \frac{\partial h}{\partial z} dxdz$$

$$= -A \frac{\partial h}{\partial x} dxdz + Bdzdx - C \frac{\partial h}{\partial z} dxdz, \quad \text{as } dzdz = 0, \quad dxdx = 0, \quad dxdz = -dzdx$$

for (a) and

$$Adydz + Bdzdx + Bdxdy = A(g(y, z), y, z) dydz + B(g(y, z), y, z) dz \left( \frac{\partial g}{\partial y} dy + \frac{\partial g}{\partial z} dz \right)$$

$$+ C(g(y, z), y, z) \left( \frac{\partial g}{\partial y} dy + \frac{\partial g}{\partial z} dz \right) dy = Adydz + B \frac{\partial g}{\partial y} dzdy + C \frac{\partial g}{\partial z} dzdy$$

$$= Adydz - B \frac{\partial g}{\partial y} dydz - C \frac{\partial g}{\partial z} dydz, \quad \text{as } dzdz = 0, \quad dydy = 0, \quad dydz = -dzdy$$

for (b).