Math 434/734

Homework 3

1. Find a unit normal vector $\mathbf{n}$ to each of the following surfaces

$$\text{(a) } z = 2 - x - y, \quad \text{(b) } z = (x^2 + y^2)^{1/2}.$$

(a) A tangent vector to the surface on a plane $y = c$, is given by

$$\mathbf{u} = \mathbf{i} + \frac{\partial z}{\partial x}\mathbf{k} = \mathbf{i} - \mathbf{k}.$$

A tangent vector to the surface on the plane $x = c$ is given by

$$\mathbf{v} = \mathbf{j} + \frac{\partial z}{\partial y}\mathbf{k} = \mathbf{j} - \mathbf{k},$$

as the partial derivatives gives the slopes of the tangent lines on the intersection of the surface with the planes $y = c$ and $x = c$ respectively. A normal vector is $\mathbf{u} \times \mathbf{v}$, which is

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} = \mathbf{k} + \mathbf{i} + \mathbf{j}.$$

The unit normal vector is

$$\mathbf{n} = \frac{\mathbf{k} + \mathbf{i} + \mathbf{j}}{|\mathbf{k} + \mathbf{i} + \mathbf{j}|} = \frac{1}{\sqrt{3}}\left(\mathbf{k} + \mathbf{i} + \mathbf{j}\right).$$

(b) We calculate the partial derivatives:

$$\frac{\partial z}{\partial x} = \frac{2x}{2(x^2 + y^2)^{1/2}} = \frac{x}{z}, \quad \frac{\partial z}{\partial y} = \frac{2y}{2(x^2 + y^2)^{1/2}} = \frac{y}{z}.$$

With

$$\mathbf{u} = \mathbf{i} + \frac{x}{z}\mathbf{k}, \quad \mathbf{v} = \mathbf{j} + \frac{y}{z}\mathbf{k}$$

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & x/z \\ 0 & 1 & y/z \end{vmatrix} = \mathbf{k} - \frac{x}{z}\mathbf{i} - \frac{y}{z}\mathbf{j}.$$

The unit normal vector is

$$\mathbf{n} = \frac{\mathbf{k} - (x/z)\mathbf{i} - (y/z)\mathbf{j}}{|\mathbf{k} - (x/z)\mathbf{i} - (y/z)\mathbf{j}|} = \frac{\mathbf{k} - (x/z)\mathbf{i} - (y/z)\mathbf{j}}{\sqrt{1 + x^2/z^2 + y^2/z^2}} = \frac{\mathbf{k} - (x/z)\mathbf{i} - (y/z)\mathbf{j}}{\sqrt{z^2 + x^2 + y^2}/z} = \frac{z\mathbf{k} - x\mathbf{i} - y\mathbf{j}}{\sqrt{2}z}.$$

2. Calculate the integral of the 1-form $z^2dx + x^2dy - y^2dz$ along the square of size s with vertices $(x_0 - s/2, x_0 - s/2, 0)$, $(x_0 + s/2, x_0 - s/2, 0)$, $(x_0 + s/2, x_0 + s/2, 0)$, $(x_0 - s/2, x_0 + s/2, 0)$ oriented counterclockwise.

   Along the lower horizontal side C_1 we use the parametrization $(x, y, z) = (t, x_0 - s/2, 0)$, $x_0 - s/2 \le t \le x_0 + s/2$. We have $dx/dt = 1$, $dy/dt = 0$, $dz/dt = 0$. The (line) integral becomes

$$\int_{C_1} z^2dx + x^2dy - y^2dz = \int_{x_0-s/2}^{x_0+s/2} 0 \cdot 1 + t^2 \cdot 0 - (x_0 - s/2)^2 \cdot 0dt = \int_{x_0-s/2}^{x_0+s/2} 0dt = 0.$$

   Along the upper horizontal side C_2 we calculate the integral over $-C_2$, with the opposite orientation and we use the parametrization $(x, y, z) = (t, x_0 + s/2, 0)$, $x_0 - s/2 \le t \le x_0 + s/2$. We have $dx/dt = 1$, $dy/dt = 0$, $dz/dt = 0$. The (line) integral becomes

$$\int_{-C_2} z^2dx + x^2dy - y^2dz = \int_{x_0-s/2}^{x_0+s/2} 0 \cdot 1 + t^2 \cdot 0 - (x_0 + s/2)^2 \cdot 0dt = \int_{x_0-s/2}^{x_0+s/2} 0dt = 0.$$

   Along the right side of the square C_3 we use the parametrization $(x, y, z) = (x_0 + s/2, t, 0)$, $x_0 - s/2 \le t \le x_0 + s/2$. We have $dx/dt = 0$, $dy/dt = 1$, $dz/dt = 0$. The (line) integral becomes

$$\int_{C_3} z^2dx+x^2dy-y^2dz = \int_{x_0-s/2}^{x_0+s/2} 0\cdot0+(x_0+s/2)^2\cdot1-t^2\cdot0dt = \int_{x_0-s/2}^{x_0+s/2} (x_0+s/2)^2dt = s\cdot(x_0+s/2)^2,$$

   since we are integrating a constant function over an interval of length s.

   Along the left side of the square C_4 we calculate the integral over $-C_4$, i.e. with the opposite orientation and use the parametrization $(x, y, z) = (x_0 - s/2, t, 0)$, $x_0 - s/2 \le t \le x_0 + s/2$. We have $dx/dt = 0$, $dy/dt = 1$, $dz/dt = 0$. The (line) integral becomes

$$\int_{-C_4} z^2dx+x^2dy-y^2dz = \int_{x_0-s/2}^{x_0+s/2} 0\cdot0+(x_0-s/2)^2\cdot1-t^2\cdot0dt = \int_{x_0-s/2}^{x_0+s/2} (x_0-s/2)^2dt = s\cdot(x_0-s/2)^2,$$

   since we are integrating a constant function over an interval of length s.

   The final integral is the difference of the above results, as

$$\int_C z^2dx + x^2dy - y^2dz = \int_{C_1} - \int_{-C_2} + \int_{C_3} - \int_{-C_4} z^2dx + x^2dy - y^2dz$$

$$= s\cdot(x_0+s/2)^2 - s\cdot(x_0-s/2)^2 = s(x_0+s/2+x_0-s/2)(x_0+s/2-x_0+s/2) = s2x_0s = 2s^2x_0.$$

3. Evaluate the integral $\int_S \mathbf{F} \cdot \mathbf{n}dS$ (surface integral or flow) of the vector field

$$\mathbf{F}(x, y, z) = x\mathbf{i} - z\mathbf{k},$$

   where S is the portion of the plane $x + y + 2z = 2$ in the first octant and the unit normal is in the positive x, y and z direction.

   The 2-form corresponding to $\mathbf{F}$ is

$$xdydz - zdxdy$$

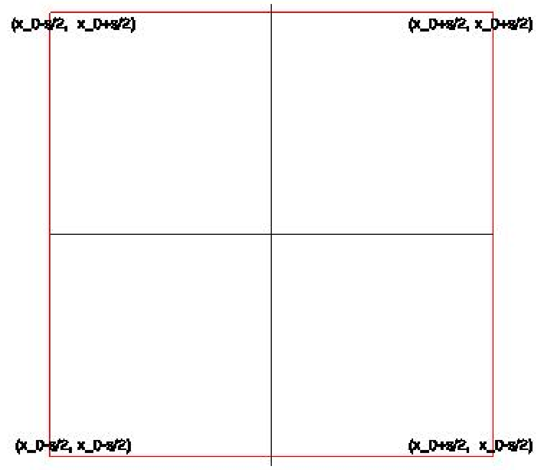

Figure 1: The square contour.

and we calculate its pullback under the parametrization of the portion of the plane in the first octact

$$z = 1 - x/2 - y/2, \quad 0 \le x \le 2, \quad 0 \le y \le 2 - x.$$

What we have done is project the plane to the xy-plane and parametrized the triangle T, which is the projection in the first quadrant, as a vertically simple region. The triangle T is oriented in the counterclockwise direction. We have $dz = -(1/2)dx - (1/2)dy$ and

$$xdydz - zdxdy = xdy(-1/2dx - 1/2dy) - (1 - y/2 - x/2)dxdy = -(x/2)dydx - (1 - y/2 - x/2)dxdy$$

$$= (x/2 - 1 + y/2 + x/2)dxdy = (x - 1 + y/2)dxdy.$$

The integral of the 2-form is calculated as a double integral over the region parametrized as vertically simple region.

$$\int \mathbf{F}\cdot\mathbf{n}\, dS = \int_T (x-1+y/2)dydx = \int_0^2 \int_0^{2-x} (x-1+y/2)dydx = \int_0^2 \left[(x-1)y + y^2/4\right]_{y=0}^{y=2-x} dx$$

$$= \int_0^2 (x-1)(2-x)+(2-x)^2/4\, dx = \int_0^2 -x^2+3x-2+x^2/4-x+1\, dx = \int_0^2 -3x^2/4+2x-1\, dx$$

$$= \left[-x^3/4 + x^2 - x\right]_0^2 = -\frac{8}{4} + 2^2 - 2 = 0.$$

4. Evaluate the integral $\int_S \mathbf{F} \cdot \mathbf{n} dS$ (surface integral or flow) of the vector field

$$\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k},$$

where S is the hemisphere $z = \sqrt{a^2 - x^2 - y^2}$ oriented counterclockwise as seen from the outside. A simple calculation of partial derivatives gives

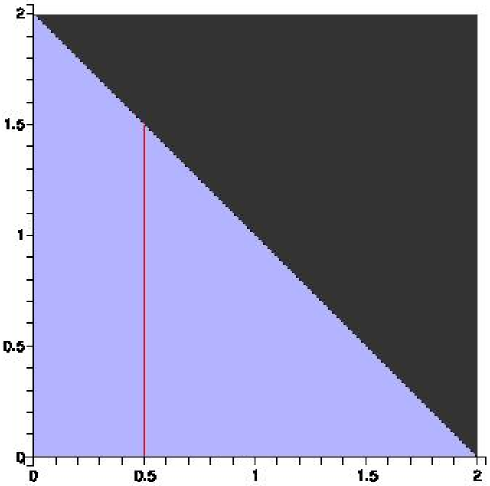

Figure 2: Projection in the first quadrant is lightly-shaded.

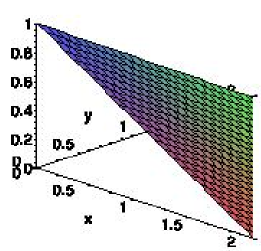

Figure 3: The portion of the plane in the first octant.

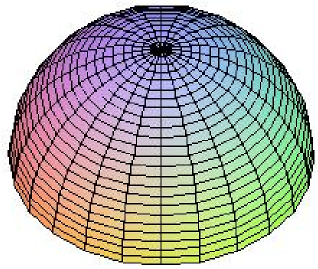

Figure 4: The hemisphere.

$$\frac{\partial z}{\partial x} = -\frac{x}{z}, \quad \frac{\partial z}{\partial y} = -\frac{y}{z}.$$

This gives

$$dz = -\frac{x}{z}dx - \frac{y}{z}dy.$$

The projection of the hemisphere on the xy-plane is the disc

$$T = \{(x, y), x^2 + y^2 \leq a^2\}$$

oriented counterclockwise. Then

$$\int_S \mathbf{F}\cdot\mathbf{n}\, dS = \int_S x\, dydz + y\, dzdx + z\, dxdy = \int_T xdy\left(-\frac{x}{z}dx - \frac{y}{z}dy\right) + y\left(-\frac{x}{z}dx - \frac{y}{z}dy\right)dx + zdxdy$$

$$= \int_T -\frac{x^2}{z}dydx - \frac{y^2}{z}dydx + zdxdy = \int_T \frac{x^2 + y^2 + z^2}{z}dxdy = \int_T \frac{a^2}{\sqrt{a^2 - x^2 - y^2}}dxdy$$

We switch to polar coordinates $x = r\cos\theta$, $y = r\sin\theta$, $x^2 + y^2 = r^2$, $dxdy = rdrd\theta$ to get

$$\int_T \frac{a^2}{\sqrt{a^2 - x^2 - y^2}}dxdy = \int_0^{2\pi}\int_0^a \frac{a^2 r}{\sqrt{a^2 - r^2}}drd\theta = \int_0^{2\pi}\left[-a^2\sqrt{a^2 - r^2}\right]_0^a d\theta$$

$$= \int_0^{2\pi} a^2\sqrt{a^2}d\theta = 2\pi a^3.$$

Remark: In this example the interpretation as a flow gives a much easier way of calculation: The vector field is radial i.e. in the direction of the vector $\mathbf{r}$ from the origin to every point on the hemisphere. So $\mathbf{F}\cdot\mathbf{n} = |\mathbf{F}| = |\mathbf{r}| = a$. We are integrating the constant 2-form adS on the hemisphere, so we get the area of the hemisphere, which is $2\pi a^2$ times a, i.e. $2\pi a^3$.

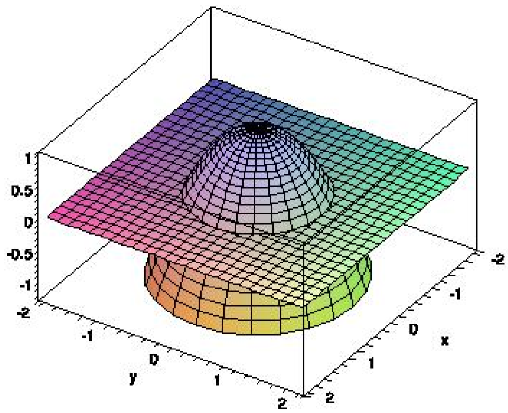

Figure 5: The paraboloid and the xy-plane.

5. Evaluate the integral $\int_S \mathbf{F}\cdot\mathbf{n}dS$ (surface integral or flow) of the vector field

$$\mathbf{F}(x,y,z) = y\mathbf{j} + \mathbf{k},$$

where S is the portion of the paraboloid $z = 1 - x^2 - y^2$ above the xy-plane oriented counterclockwise as seen from above.

We have

$$\frac{\partial z}{\partial x} = -2x, \quad \frac{\partial z}{\partial y} = -2y, \quad dz = -2x\,dx = 2y\,dy.$$

The pullback of the form $ydzdx + dxdy$ is

$$y(-2xdx - 2ydy)dx + dxdy = 2y^2dxdy + dxdy = (1 + 2y^2)dxdy.$$

The projection of the surface on the xy-plane is the disc of radius 1 centered at $(0,0)$ oriented counterclockwise. We evaluate the resulting integral in polar coordinates

$$\int_0^{2\pi}\int_0^1 (1+2r^2\sin^2\theta)rdrd\theta = \int_0^{2\pi}\int_0^1 rdrd\theta + \int_0^{2\pi}\int_0^2 2r^3\sin^2\theta drd\theta = \pi + \int_0^{2\pi}\left[\frac{r^4}{2}\right]_0^1 \sin^2\theta d\theta$$

$$= \pi + \int_0^{2\pi}\frac{1}{2}\sin^2\theta d\theta = \pi + \int_0^{2\pi}\frac{1}{4} - \frac{\cos(2\theta)}{4}d\theta = \pi + \frac{1}{4}2\pi - \left[\frac{\sin(2\theta)}{8}\right]_0^{2\pi} = \pi + \frac{\pi}{2} = \frac{3\pi}{2}.$$

6. (a) Assume a surface S is parametrized as a function $y = h(x,z)$, where $(x,z)\in D$. Show that

$$\int_S Adydz + Bdzdx + Cdxdy = \int_D \left\{-A(x,h(x,z),z)\frac{\partial h}{\partial x} + B(x,h(x,z),z) - C(x,h(x,z),z)\frac{\partial h}{\partial z}\right\} dzdx$$

(b) Assume a surface S is parametrized as a function $x = g(y,z)$, where $(y,z)\in D$. Show that

$$\int_S Adydz + Bdzdx + Cdxdy = \int_D \left\{A(g(y,z),y,z) - B(g(y,z),y,z)\frac{\partial g}{\partial y} - C(g(y,z),y,z)\frac{\partial g}{\partial z}\right\} dydz.$$

We calculate the pullbacks of $Adydz + Bdzdx + Cdxdy$ under the maps $x = x$, $y = h(x,z)$ and $z = z$ (resp. $x = g(y,z)$, $y = y$, $z = z$). We get

$$dy = \frac{\partial h}{\partial x}dx + \frac{\partial h}{\partial z}dz$$

in (a) and

$$dx = \frac{\partial g}{\partial y}dy + \frac{\partial g}{\partial z}dz$$

in (b). This gives

$$Adydz+Bdzdx+Bdxdy = A(x,h(x,z),z)\left(\frac{\partial h}{\partial x}dx + \frac{\partial h}{\partial z}dz\right)dz+B(x,h(x,z),z)dzdx$$

$$+C(x,h(x,z),z)dx\left(\frac{\partial h}{\partial x}dx + \frac{\partial h}{\partial z}dz\right) = A\frac{\partial h}{\partial x}dxdz + Bdzdx + C\frac{\partial h}{\partial z}dxdz$$

$$= -A\frac{\partial h}{\partial x}dzdx + Bdzdx - C\frac{\partial h}{\partial z}dzdx, \quad \text{as } dzdz = 0, \quad dxdx = 0, \quad dxdz = -dzdx$$

for (a) and

$$Adydz + Bdzdx + Bdxdy = A(g(y,z),y,z)dydz + B(g(y,z),y,z)dz\left(\frac{\partial g}{\partial y}dy + \frac{\partial g}{\partial z}dz\right)$$

$$+C(g(y,z),y,z)\left(\frac{\partial g}{\partial y}dy + \frac{\partial g}{\partial z}dz\right)dy = Adydz + B\frac{\partial g}{\partial y}dzdy + C\frac{\partial g}{\partial z}dzdy$$

$$= Adydz - B\frac{\partial g}{\partial y}dydz - C\frac{\partial g}{\partial z}dydz, \quad \text{as } dzdz = 0, \quad dydy = 0, \quad dydz = -dzdy$$

for (b).