Math 434/734

Homework 2

1. Section 1.1, 1(b) We calculate the vector $\vec{PQ}$ by subtracting the coordinates of P from the coordinates of Q.

$$\vec{PQ} = (11-3)\mathbf{i} + (14-12)\mathbf{j} + (-7-4)\mathbf{k} = 8\mathbf{i} + 2\mathbf{j} - 11\mathbf{k}$$

so that $dx = 8$, $dy = 2$ and $dz = -11$ and the one form $2dx + 3dy + 5dz$ is evaluated to give

$$2 \cdot 8 + 3 \cdot 2 + 5 \cdot (-11) = 16 + 6 - 55 = -33.$$

2. Section 1.1, 2(b)

Let the differential one form is $Adx + bdy$. For $P(4,2)$ and $Q(6,3)$, we plug for $\vec{PQ} = 2\mathbf{i} + \mathbf{j}$ the values $dx = 2$, $dy = 1$ to get

$$5 = A \cdot 2 + B \cdot 1$$

and for $P(-2,1)$ and $Q(1,3)$ we get $\vec{PQ} = 3\mathbf{i} + 2\mathbf{j}$ the values $dx = 3$ and $dy = 2$ to get

$$2 = A \cdot 3 + B \cdot 2.$$

We solve the system. A quick way is Cramer's rule: the determinant of the coefficients is

$$\begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} = 2 \cdot 2 - 3 \cdot 1 = 1.$$

$$A = \frac{\begin{vmatrix} 5 & 1 \\ 2 & 2 \end{vmatrix}}{1} = \frac{10-2}{1} = 8,$$

$$B = \frac{\begin{vmatrix} 2 & 5 \\ 3 & 2 \end{vmatrix}}{1} = \frac{4-15}{1} = -11.$$

So the differential form is $8dx - 11dy$. We computed it, without reference to any vectors fields.

3. Section 1.2, 1 (explained in class)

The projection of the triangle on the xy-plane is the oriented triangle from $(0,0)$ towards $(1,2)$ and then towards $(1,4)$. This triangle is oriented counterclockwise, so we just measure its area. The base is from $(1,2)$ to $(1,4)$ and has length 2. The height is 1. So the signed area is $1 \cdot 2/2 = 1$.

The projection of the triangle on the xz plane is the oriented triangle from $(0,0)$ towards $(1,3)$ and then towards $(1,0)$. This is oriented counterclockwise as seen from the positive y-axis. So the signed area is negative. We compute it: the height is 3 and the base is 1. The signed area is $3/2$. However in the 2-form we have $3dxdy$ and this is oriented the opposite direction and the contribution is $-3 \cdot 3/3 = -9/2$.

The total is $-9/2 + 1 = -7/2$.

4. Section 1.2, 3

   The vector from $(0, 0)$ towards (x_1, y_1) is $\vec{u} = x_1\mathbf{i} + y_1\mathbf{j}$ and the vector from $(0, 0)$ towards (x_2, y_2) is $\vec{v} = x_2\mathbf{i} + y_2\mathbf{j}$. The triangle is positively oriented if the two vectors in this order define the same orientation as the standard vectors $\mathbf{i}$ and $\mathbf{j}$ (in this order). Said differently, if $\vec{u} \times \vec{v}$ is in the direction of the vector $\mathbf{k} = \mathbf{i} \times \mathbf{j}$. We compute

$$\vec{u} \times \vec{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_1 & y_1 & 0 \\ x_2 & y_2 & 0 \end{vmatrix} = (x_1y_2 - x_2y_2)\mathbf{k}.$$

   This vector is in the direction of $\mathbf{k}$ iff the constant in front is positive. The condition for positive orientation is $x_1y_2 - x_2y_1 > 0$.

5. Section 1.2, 4

   We translate the triangle so that the first vertex is at $(0, 0)$ and apply the previous exercise. So we look at the triangle with vertices $(0, 0)$, $(x_2 - x_1, y_2 - y_1)$ and $(x_3 - x_1, y_3 - y_1)$ in this order. The condition for positive orientation is

$$(x_2 - x_1)(y_3 - y_1) - (y_2 - y_1)(x_3 - x_1) > 0.$$

   We actually can understand this through a determinant:

$$\begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ x_1 & x_2 - x_1 & x_3 - x_1 \\ y_1 & y_2 - y_1 & y_3 - y_1 \end{vmatrix} = \begin{vmatrix} x_2 - x_1 & x_3 - x_1 \\ y_2 - y_1 & y_3 - y_1 \end{vmatrix},$$

   where we subtracted the first column from the second and third and expanded along the first row. The result is exactly the expression above, when we expand the 2×2 determinant.

6. Section 1.3, 2 (b)

   The points are $P(1, 1)$, $Q(3, 1)$ and $R(2, 3)$.

   Using geometry: the base is PQ, which has length 2 and the height is 2. The area is $2 \cdot 2/2 = 2$ and the oriented area is the same, as the orientation is counterclockwise.

   Using pullbacks: We map the vertices of the standard triangle T from $(0, 0)$ towards $(1, 0)$ towards $(0, 1)$ to the vertices of PQR in this order, using an affine map.

   This we will do in a systematic way. We look at the vectors $\vec{PQ} = 2\mathbf{i}$ and $\vec{PR} = \mathbf{i} + 2\mathbf{j}$. This gives the affine map

$$\begin{aligned} x &= 1 + 2u + v \\ y &= 1 + 2v. \end{aligned}$$

   So we compute the pullback

$$dxdy = (2du + dv)(2dv) = 4dudv.$$

   The area of the triangle T is $1/2$ and using the pullback we get that the area of the oriented triangle PQR is $4 \cdot 1/2 = 2$.

7. Section 1.3, 3 (a) Since $\vec{PQ} = -5\mathbf{i}$ and $\vec{PR} = -2\mathbf{i} + 3\mathbf{j} - 3\mathbf{k}$, we define the affine map

$$\begin{aligned} x &= 3 - 5u - 2v \\ y &= 1 + 3v \\ z &= 4 - 3v \end{aligned}$$

(check that when $u = v = 0$ we get $(x, y, z) = (3, 1, 4)$ i.e. P, when $u = 1, v = 0$ we get $(x, y, z) = (-2, 1, 4)$, i.e. Q and, finally, when $u = 0, v = 1$ we get $(x, y, z) = (1, 4, 1)$, i.e. R). We compute the pullaback:

$$3dydz + 2dxdy = 3(3dv)(-3dv) + 2(-5du - 2dv)(3dv) = -27dvdv - 30dudv - 12dvdv = -30dudv.$$

Since the oriented area of T is $1/2$ we get that the value of the two form is $-30 \cdot 1/2 = -15$.

8. Section 1.5, 1 (solution in the back of the book)

9. Section 2.1, 1 (explained in class)

The force of attraction towards the sum is is the direction of the unit vector $\mathbf{r}/|\mathbf{r}|$. Since the magnitude is $|\mathbf{F}(\mathbf{r})| = k/|\mathbf{r}|^2$, we get (with the notation $|\mathbf{r}| = r$, $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + y\mathbf{k}$)

$$\mathbf{F}(x, y, z) = \frac{k}{r^2}\frac{\mathbf{r}}{r} = \frac{k\mathbf{r}}{r^3} = \frac{kx}{r^3}\mathbf{i} + \frac{ky}{r^3}\mathbf{j} + \frac{ky}{r^3}\mathbf{k}.$$

To this vector field we associate the 1-form

$$\frac{kx}{r^3}dx + \frac{ky}{r^3}dy + \frac{kz}{r^3}dz.$$

10. Section 2.1, 2 (explained in class)

If the velocity field is $v_1\mathbf{i} + v_2\mathbf{j}$, then the 1-form describing the flow is $-v_2dx + v_1dy$. Since a circle of radius r has length $2\pi r$ and the same amount flows through any circle, which is equal to $2\pi r|\mathbf{v}| = 2\pi k$ for some constant k, we get that $|\mathbf{v}| = k/r$, when $|\mathbf{r}| = r$. The vector in the radial direction at $\mathbf{r}$ is $\mathbf{r}/r$, so

$$\mathbf{v} = \frac{k}{r}\frac{\mathbf{r}}{r} = \frac{k\mathbf{r}}{r^2} = \frac{kx}{r^2}\mathbf{i} + \frac{ky}{r^2}\mathbf{j}.$$

Then the one-form is

$$-v_2dx + v_1dy = \frac{-ky}{x^2 + y^2}dx + \frac{kx}{x^2 + y^2}dy.$$

11. Section 2.1, 3 (explained in class) The flow through the surface of a ball of radius r is constant, and this area is $4\pi r^2$. If the speed of the flow is constant on the sphere of radius r, then

$$4\pi r^2|\mathbf{v}| = 4\pi k,$$

for an appropriate constant k. This gives $|\mathbf{v}| = k/r^2$ and

$$\mathbf{v} = \frac{k}{r^2}\frac{\mathbf{r}}{r} = \frac{k}{r^3}\mathbf{r} = \frac{kx}{r^3}\mathbf{i} + \frac{ky}{r^3}\mathbf{j} + \frac{kz}{r^3}\mathbf{k}.$$

The correspondence between the velocity $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ and the 2-form defining the flow is $v_1dydz + v_2dzdx + v_3dxdy$ so we get the 2-form

$$\frac{kx}{r^3}dydz + \frac{ky}{r^3}dzdx + \frac{kz}{r^3}dxdy.$$

12. Section 2.4, 1(a)

If $y = \sin(uv)$, then $dy = \cos(uv)vdu + \cos(uv)udv$, by using the standard chain rule in one variable calculus to get

$$\frac{d\sin(uv)}{du} = \cos(uv)v \quad \frac{d\sin(uv)}{dv} = \cos(uv)u.$$

Similarly for $z = uv^2$ we get $dz = v^2du + 2uvdv$ and these give together

$$dydz = (\cos(uv)vdu+\cos(uv)udv)(v^2du+2uvdv) = \cos(uv)2uv^2\,dudv+\cos(uv)uv^2\,dvdu$$
$$= \cos(uv)(2uv^2 - uv^2)dudv = \cos(uv)uv^2\,dudv$$

and finally

$$xdydz = \cos(uv)\cos(uv)(uv^2)dudv = \cos^2(uv)(uv^2)dudv.$$

13. Section 2.4, 2

Since $x = \sqrt{t}$ and $y = t$, we get

$$\frac{dx}{dt} = \frac{1}{2\sqrt{t}}, \quad \frac{dy}{dt} = 1.$$

The infinitesimal rate of work at time t is

$$x(t)\frac{dy}{dt} + y(t)\frac{dx}{dt} = \sqrt{t}\cdot 1 + t\frac{1}{2\sqrt{t}} = \frac{3}{2}\sqrt{t}.$$

For the whole work in the interval $t = 1$ to $t = 4$ we get

$$W = \int_1^4 \frac{3}{2}\sqrt{t}\,dt.$$

14. Section 2.4, 4 (explained in class)

With

$$\begin{aligned} x &= \cos\theta\cos\phi \\ y &= \sin\theta\cos\phi \\ z &= \sin\phi \end{aligned}$$

we get

$$\begin{aligned} dx &= \sin\theta\cos\phi d\theta - \cos\theta\sin\phi d\phi \\ dy &= \cos\theta\cos\phi d\theta - \sin\theta\sin\phi d\phi \\ dz &= \cos\phi d\phi \end{aligned}$$

Now

$$\begin{aligned} xdydz &= \cos\theta\cos\phi(\cos\theta\cos^2\phi d\theta d\phi) \\ ydzdx &= \sin\theta\cos\phi(\cos\phi(-\sin\theta\cos\phi)d\phi d\theta \\ &= \sin^2\theta\cos^3\phi d\theta d\phi \\ zdxdy &= \sin\phi(\sin^2\theta\cos\phi\sin\phi d\theta d\phi + \cos^2\theta\sin\phi\cos\phi d\theta d\phi) \\ &= \sin^2\phi\cos\phi d\theta d\phi \end{aligned}$$

Finally

$$xdydz+ydzdx+zdxdy = (\cos^2\theta\cos^3\phi+\sin^2\theta\cos^3\phi+\sin^2\phi\cos\phi)d\theta d\phi = (\cos^3\phi+\sin^2\phi\cos\phi)d\theta d\phi$$
$$= (\cos^2\phi + \sin^2\phi)\cos\phi d\theta d\phi = \cos\phi d\theta d\phi$$