Math 434/734

Homework 2

1. Section 1.1, 1(b) We calculate the vector \( \vec{PQ} \) by subtracting the coordinates of \( P \) from the coordinates of \( Q \).

\[
\vec{PQ} = (11 - 3)i + (14 - 12)j + (-7 - 4)k = 8i + 2j - 11k
\]

so that \( dx = 8 \), \( dy = 2 \) and \( dz = -11 \) and the one form \( 2dx + 3dy + 5dz \) is evaluated to give

\[
2 \cdot 8 + 3 \cdot 2 + 5 \cdot (-11) = 16 + 6 - 55 = -33.
\]

2. Section 1.1, 2(b)

Let the differential one form is \( Adx + bdy \). For \( P(4,2) \) and \( Q(6,3) \), we plug for \( \vec{PQ} = 2i + j \) the values \( dx = 2 \), \( dy = 1 \) to get

\[
5 = A \cdot 2 + B \cdot 1
\]

and for \( P(-2,1) \) and \( Q(1,3) \) we get \( \vec{PQ} = 3i + 2j \) the values \( dx = 3 \) and \( dy = 2 \) to get

\[
2 = A \cdot 3 + B \cdot 2.
\]

We solve the system. A quick way is Cramer’s rule: the determinant of the coefficients is

\[
A = \begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} = 2 \cdot 2 - 3 \cdot 1 = 1.
\]

\[
B = \begin{vmatrix} 5 & 1 \\ 2 & 2 \end{vmatrix} = \frac{10 - 2}{1} = 8,
\]

\[
B = \begin{vmatrix} 2 & 5 \\ 3 & 2 \end{vmatrix} = \frac{4 - 15}{1} = -11.
\]

So the differential form is \( 8dx - 11dy \). We computed it, without reference to any vectors fields.

3. Section 1.2, 1 (explained in class)

The projection of the triangle on the \( xy \)-plane is the oriented triangle from \( (0,0) \) towards \( (1,2) \) and then towards \( (1,4) \). This triangle is oriented counterclockwise, so we just measure its area. The base is from \( (1,2) \) to \( (1,4) \) and has length 2. The height is 1. So the signed area is \( 1 \cdot 2/2 = 1 \).

The projection of the triangle on the \( xz \) plane is the oriented triangle from \( (0,0) \) towards \( (1,3) \) and then towards \( (1,0) \). This is oriented counterclockwise as seen from the positive \( y \)-axis. So the signed area is negative. We compute it: the height is 3 and the base is 1. The signed area is \( 3/2 \). However in the 2-form we have \( 3dxdy \) and this is oriented the opposite direction and the contribution is \( -3 \cdot 3/3 = -9/2 \).

The total is \( -9/2 + 1 = -7/2 \).
4. Section 1.2, 3

The vector from \((0,0)\) towards \((x_1,y_1)\) is \(\vec{u} = x_1 \mathbf{i} + y_1 \mathbf{j}\) and the vector from \((0,0)\) towards \((x_2,y_2)\) is \(\vec{v} = x_2 \mathbf{i} + y_2 \mathbf{j}\). The triangle is positively oriented if the two vectors in this order define the same orientation as the standard vectors \(\mathbf{i}\) and \(\mathbf{j}\) (in this order). Said differently, if \(\vec{u} \times \vec{v}\) is in the direction of the vector \(\mathbf{k} = \mathbf{i} \times \mathbf{j}\). We compute

\[
\vec{u} \times \vec{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_1 & y_1 & 0 \\ x_2 & y_2 & 0 \end{vmatrix} = (x_1 y_2 - x_2 y_1) \mathbf{k}.
\]

This vector is in the direction of \(\mathbf{k}\) iff the constant in front is positive. The condition for positive orientation is \(x_1 y_2 - x_2 y_1 > 0\).

5. Section 1.2, 4

We translate the triangle so that the first vertex is at \((0,0)\) and apply the previous exercise. So we look at the triangle with vertices \((0,0)\), \((x_2 - x_1, y_2 - y_1)\) and \((x_3 - x_1, y_3 - y_1)\) in this order. The condition for positive orientation is

\[
(x_2 - x_1)(y_3 - y_1) - (y_2 - y_1)(x_3 - x_1) > 0.
\]

We actually can understand this through a determinant:

\[
\begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ x_1 & x_2 - x_1 & x_3 - x_1 \\ y_1 & y_2 - y_1 & y_3 - y_1 \end{vmatrix} = \begin{vmatrix} x_2 - x_1 & x_3 - x_1 \\ y_2 - y_1 & y_3 - y_1 \end{vmatrix},
\]

where we subtracted the first column from the second and third and expanded along the first row. The result is exactly the expression above, when we expand the \(2 \times 2\) determinant.

6. Section 1.3, 2 (b)

The points are \(P(1,1)\), \(Q(3,1)\) and \(R(2,3)\).

Using geometry: the base is \(PQ\), which has length 2 and the height is 2. The area is \(2 \cdot 2/2 = 2\) and the oriented area is the same, as the orientation is counterclockwise.

Using pullbacks: We map the vertices of the standard triangle \(T\) from \((0,0)\) towards \((1,0)\) towards \((0,1)\) to the vertices of \(PQR\) in this order, using an affine map. This we will do in a systematic way. We look at the vectors \(\vec{PQ} = 2 \mathbf{i}\) and \(\vec{PR} = \mathbf{i} + 2 \mathbf{j}\). This gives the affine map

\[
x = 1 + 2u + v \\
y = 1 + 2v.
\]

So we compute the pullback

\[
dxdy = (2du + dv)(2dv) = 4dudv.
\]

The area of the triangle \(T\) is \(1/2\) and using the pullback we get that the area of the oriented triangle \(PQR\) is \(4 \cdot 1/2 = 2\).
7. Section 1.3, 3 (a) Since \( \vec{PQ} = -5\hat{i} \) and \( \vec{PR} = -2\hat{i} + 3\hat{j} - 3\hat{k} \), we define the affine map

\[
\begin{align*}
x &= 3 - 5u - 2v \\
y &= 1 + 3v \\
z &= 4 - 3v
\end{align*}
\]

(check that when \( u = v = 0 \) we get \((x, y, z) = (3, 1, 4)\) i.e. \( P \), when \( u = 1, v = 0 \) we get \((x, y, z) = (-2, 1, 4)\) i.e. \( Q \) and, finally, when \( u = 0, v = 1 \) we get \((x, y, z) = (1, 4, 1)\), i.e. \( R \)). We compute the pullback:

\[
3dydz + 2dxdy = 3(3dv)(-3dv) + 2(-5du - 2dv)(3dv) = -27dvdv - 30dudv - 12dvdv = -30dudv.
\]

Since the oriented area of \( T \) is \( 1/2 \) we get that the value of the two form is \(-30 \cdot 1/2 = -15\).

8. Section 1.5, 1 (solution in the back of the book)

9. Section 2.1, 1 (explained in class)

The force of attraction towards the sum is the direction of the unit vector \( \hat{r} \).

Since the magnitude is \( |\vec{F}(\hat{r})| = k/|\hat{r}|2 \), we get (with the notation \( |\hat{r}| = r \), \( \vec{r} = xi + yj + zk \))

\[
\vec{F}(x, y, z) = \frac{k}{r^2} \hat{r} = \frac{k}{r^3} \hat{r}^3 = \frac{kx}{r^3} \hat{i} + \frac{ky}{r^3} \hat{j} + \frac{kz}{r^3} \hat{k}.
\]

To this vector field we associate the 1-form

\[
\frac{kx}{r^3} dx + \frac{ky}{r^3} dy + \frac{kz}{r^3} dz.
\]

10. Section 2.1, 2 (explained in class)

If the velocity field is \( v_1 \hat{i} + v_2 \hat{j} \), then the 1-form describing the flow is \(-v_2 dx + v_1 dy\). Since a circle of radius \( r \) has length \( 2\pi r \) and the same amount flows through any circle, which is equal to \( 2\pi r |\hat{v}| = 2\pi k \) for some constant \( k \), we get that \( |\vec{v}| = k/r \), when \( |\hat{r}| = r \). The vector in the radial direction at \( \hat{r} \) is \( \hat{r}/r \), so

\[
\vec{v} = \frac{k}{r^2} \hat{r} = \frac{k}{r^3} \hat{r}^3 = \frac{kx}{r^3} \hat{i} + \frac{ky}{r^3} \hat{j}.
\]

Then the one-form is

\[
-v_2 dx + v_1 dy = -\frac{k y}{x^2 + y^2} dx + \frac{kx}{x^2 + y^2} dy.
\]

11. Section 2.1, 3 (explained in class) The flow through the surface of a ball of radius \( r \) is constant, and this area is \( 4\pi r^2 \). If the speed of the flow is constant on the sphere of radius \( r \), then

\[
4\pi r^2 |\vec{v}| = 4\pi k,
\]

for an appropriate constant \( k \). This gives \( |\vec{v}| = k/r^2 \) and

\[
\vec{v} = \frac{k}{r^2} \hat{r} = \frac{k}{r^3} \hat{r}^3 = \frac{kx}{r^3} \hat{i} + \frac{ky}{r^3} \hat{j} + \frac{kz}{r^3} \hat{k}.
\]

The correspondence between the velocity \( \vec{v} = v_1 \hat{i} + v_2 \hat{j} + v_3 \hat{k} \) and the 2-form defining the flow is \( v_1 dydz + v_2 dzdx + v_3 dxdy \) so we get the 2-form

\[
\frac{kx}{r^3} dydz + \frac{ky}{r^3} dzdx + \frac{kz}{r^3} dxdy.
\]
12. Section 2.4, 1(a)
If \( y = \sin(uv) \), then \( dy = \cos(uv)vdu + \cos(uv)udv \), by using the standard chain rule in one variable calculus to get
\[
\frac{d\sin(uv)}{du} = \cos(uv)v \quad \frac{d\sin(uv)}{dv} = \cos(uv)u.
\]
Similarly for \( z = uv^2 \) we get \( dz = v^2du + 2uvdv \) and these give together
\[
dydz = (\cos(uv)vdu + \cos(uv)udv)(v^2du + 2uvdv) = \cos(uv)2uv^2dudv + \cos(uv)uv^2dvdv
\]
and finally
\[
xdydz = \cos(uv)\cos(uv)(uv^2)dudv = \cos^2(uv)(uv^2)dudv.
\]

13. Section 2.4, 2
Since \( x = \sqrt{t} \) and \( y = t \), we get
\[
\frac{dx}{dt} = \frac{1}{2\sqrt{t}}, \quad \frac{dy}{dt} = 1.
\]
The infinitesimal rate of work at time \( t \) is
\[
x(t)\frac{dy}{dt} + y(t)\frac{dx}{dt} = \sqrt{t} \cdot 1 + t \cdot \frac{1}{2\sqrt{t}} = \frac{3}{2}\sqrt{t}.
\]
For the whole work in the interval \( t = 1 \) to \( t = 4 \) we get
\[
W = \int_{1}^{4} \frac{3}{2}\sqrt{t} \, dt.
\]

14. Section 2.4, 4 (explained in class)
With
\[
x = \cos \theta \cos \phi \\
y = \sin \theta \cos \phi \\
z = \sin \phi
\]
we get
\[
dx = \sin \theta \cos \phi \, d\theta - \cos \theta \sin \phi \, d\phi \\
dy = \cos \theta \cos \phi \, d\theta - \sin \theta \sin \phi \, d\phi \\
dz = \cos \phi \, d\phi
\]
Now
\[
xdydz = \cos \theta \cos \phi (\cos \theta \cos^2 \phi \, d\theta d\phi) \\
ydzdx = \sin \theta \cos \phi (\cos \phi (-\sin \theta \cos \phi) \, d\phi d\theta) \\
\quad = \sin^2 \theta \cos^3 \phi \, d\theta d\phi \\
zxdy = \sin \phi (\sin^2 \theta \cos \phi \sin \phi \, d\theta d\phi + \cos^2 \theta \sin \phi \cos \phi \, d\phi d\theta) \\
\quad = \sin^2 \phi \cos \phi \, d\theta d\phi
\]
Finally
\[
xdydz + ydzdx + zxdy = (\cos^2 \theta \cos^3 \phi + \sin^2 \theta \cos^3 \phi + \sin^2 \phi \cos \phi \, d\theta d\phi) = (\cos^3 \phi + \sin^2 \phi \cos \phi) \, d\theta d\phi \\
\quad = (\cos^2 \phi + \sin^2 \phi) \cos \phi \, d\theta d\phi = \cos \phi \, d\theta d\phi
\]