Math 434/734

Homework 1

1. For $\mathbf{a} = \langle 1, -2 \rangle$, and $\mathbf{b} = \langle -3, 4 \rangle$ compute $\mathbf{a} + \mathbf{b}$, $3\mathbf{a} - 2\mathbf{b}$, $\mathbf{a} \cdot \mathbf{b}$, $||\mathbf{a}||$, $||\mathbf{b}||$, $||\mathbf{a} + \mathbf{b}||$. Find a unit vector $\mathbf{u}$ in the same direction as $\mathbf{b}$.

2. Consider the triangle with vertices at $A(2, -1, 0)$, $B(5, -4, 3)$, $C(1, -3, 2)$. Find the angles of the triangle and show it is a right triangle. Find the projection of $\vec{AB}$ onto $\vec{AC}$.

3. (a) Write the equation of a sphere with radius 5 and center $(3, -1, 2)$.

   (b) Write the equation of a sphere with center $(-1, 0, 2)$, if the point $(4, 12, 2)$ is on the sphere.

   (c) Find the center and the radius of the sphere with equation $x^2+y^2+z^2+4x-6y = 0$. *Hint:* Complete the square.

4. Find the equation of a plane with normal vector $\langle 1, 3, 5 \rangle$ that passes through the point $(2, -3, -2)$.

5. Compute the cross product $\vec{u} \times \vec{v}$, where $\vec{u} = \langle 1, 2, 3 \rangle$, and $\vec{v} = \mathbf{j} + \mathbf{k}$. Show that it is perpendicular to both $\vec{u}$ and $\vec{v}$.

6. Consider the points $P(1, -1, 0)$, $Q(2, 1, -1)$ and $R(3, -2, 0)$.

   (a) Find a vector normal to the plane that passes through them.

   (b) Find the equation of the plane that passes through them.

   (c) Find the area of the triangle that has these points as vertices.

   (d) Show that it is a right triangle.

7. Write the equations in polar coordinates:
$$x^2 + y^2 = 16, \quad x + y = 3, \quad y = 4, \quad x^2 + y^2 - 2x = 0.$$

8. Write the equations in rectangular coordinates:
$$r = 5 \sec \theta \quad \theta = \pi/2, \quad r^2 = 8 \csc(2\theta).$$

9. Consider polar coordinates in the xy-plane given by the equations
$$r = \sqrt{x^2 + y^2}, \quad \theta = \tan^{-1}(y/x),$$
   which can be solved for x, y to give
$$x = r \cos \theta, \quad y = r \sin \theta.$$
   Find the partial derivatives
$$\frac{\partial r}{\partial x}, \quad \frac{\partial r}{\partial y}, \quad \frac{\partial \theta}{\partial x}, \quad \frac{\partial \theta}{\partial y}, \quad \frac{\partial x}{\partial r}, \quad \frac{\partial y}{\partial r}, \quad \frac{\partial x}{\partial \theta}, \quad \frac{\partial y}{\partial \theta}.$$

Figure 1: Alice meets Humpty Dumpty (Illustration to Chapter 6 of Through the Looking-Glass) by John Tenniel. Wood-engraving by the Dalziels.

10. Find the area of the triangle with vertices $(0,0)$, $(1,1)$, $(-2,1)$ using a double integral.

11. Evaluate the integrals.

$$\int_0^{\pi}\int_x^{\pi}\frac{\sin y}{y}\,dy\,dx, \quad \int_0^{\sqrt{\pi}}\int_y^{\sqrt{\pi}}\sin(x^2)\,dx\,dy, \quad \int_0^1\int_{\tan^{-1}x}^{\pi/4}\sec y\,dy\,dx,$$

$$\int\int_R e^{x^2+y^2}\,dA, \quad R=\left\{(x,y), x^2+y^2\le 4\right\}.$$

12. Humpty Dumpty looks like an egg. When he is standing up his shape can be described by the equations:

$$z=\frac{5}{4}\sqrt{1600-(x^2+y^2)}$$

for the upper part of his body above the xy-plane (it is hard to say whether the xy-plane is his waist or his neck) and

$$z=-\sqrt{1600-(x^2+y^2)}$$

for the lower part of his body (below the xy-plane). All numbers in this problem are in centimeters.

> Humpty Dumpty sat on a wall:
> Humpty Dumpty had a great fall.
> All the King's horses and all the King's men
> Couldn't put Humpty Dumpty in his place again."

Find the volume of Humpty Dumpty.

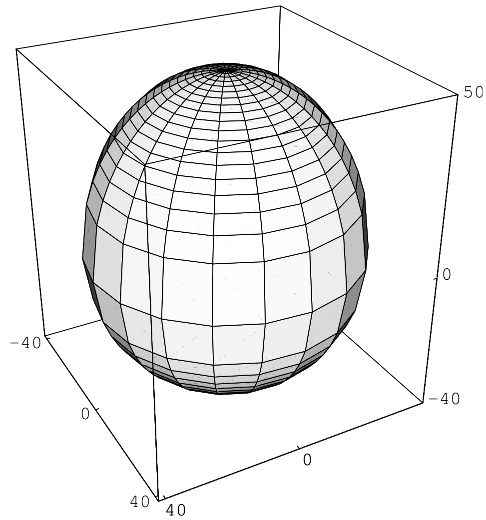

Figure 2: Humpty Dumpty

Solutions

1. $\mathbf{a}+\mathbf{b} = \langle 1-3, -2+4\rangle = \langle -2, 2\rangle$.

   $3\mathbf{a}-2\mathbf{b} = \langle 3, -6\rangle - \langle -6, 8\rangle = \langle 9, -14\rangle$.

   $\mathbf{a}\cdot\mathbf{b} = 1(-3) + (-2)4 = -3-8 = -11$.

   $||\mathbf{a}|| = \sqrt{1^2+(-2)^2} = \sqrt{5}$, $||\mathbf{b}|| = \sqrt{(-3)^2+4^2} = \sqrt{9+16} = 5$, $||\mathbf{a}+\mathbf{b}|| = \sqrt{(-2)^2+2^2} = \sqrt{8}$.

   $\mathbf{u} = \frac{1}{||\mathbf{b}||}\mathbf{b} = \langle -3/5, 4/5\rangle$.

2. $\vec{AB} = \langle 5-2, -4-(-1), 3-0\rangle = \langle 3, -3, 3\rangle$, $\vec{AC} = \langle 1-2, -3-(-1), 2-0\rangle = \langle -1, -2, 2\rangle$, $\vec{BC} = \langle 1-5, -3-(-4), 2-3\rangle = \langle -4, 1, -1\rangle$.

   $||\vec{AB}|| = \sqrt{3^3+(-3)^2+3^2} = \sqrt{27}$, $||\vec{AC}|| = \sqrt{(-1)^2+(-2)^2+2^2} = \sqrt{9} = 3$, $||\vec{BC}|| = \sqrt{(-4)^2+1^2+(-1)^2} = \sqrt{18}$.

   $$\cos B\hat{A}C = \frac{\vec{AB}\cdot\vec{AC}}{||\vec{AB}||||\vec{AC}||} = \frac{3(-1)+(-3)(-2)+3\cdot 2}{\sqrt{27}\cdot 3} = \frac{9}{3\sqrt{27}} = \frac{1}{\sqrt{3}}$$

   so $B\hat{A}C = \arccos 1/\sqrt{3} = 0.9553$.

   $$\cos A\hat{B}C = \frac{\vec{BA}\cdot\vec{BC}}{||\vec{BA}||||\vec{BC}||} = \frac{\langle -3, 3, -3\rangle\cdot\langle -4, 1, -1\rangle}{\sqrt{27}\sqrt{18}} = \frac{18}{\sqrt{27}\sqrt{18}} = \sqrt{\frac{2}{3}}$$

   so $A\hat{B}C = \arccos\sqrt{\frac{2}{3}} = 0.615$.

   $$\cos A\hat{C}B = \frac{\vec{CA}\cdot\vec{CB}}{||\vec{CA}||||\vec{CB}||} = 0$$

   since $\vec{CA}\cdot\vec{CB} = \langle 1, 2, -2\rangle\cdot\langle 4, -1, 1\rangle = 4-2-2 = 0$. This means that the angle is $\pi/2$ and this is the right angle of the triangle.

$$\text{proj}_{\vec{AC}}\vec{AB} = \frac{\vec{AB}\cdot\vec{AC}}{||\vec{AC}||^2}\cdot\vec{AC} = \frac{9}{9}\vec{AC} = \vec{AC}.$$

This is obvious because we project the hypotenuse on a vertical side.

3. (a) $(x-3)^2+(y-(-1))^2+(z-2)^2=5^2$, $(x-3)^2+(y+1)^2+(z-2)^2=25$

   (b) We need to find the radius. This is the distance from the center to the point on the sphere and is equal to

   $$\sqrt{(4-(-1))^2+(12-0)^2+(2-2)^2} = \sqrt{5^2+12^2} = \sqrt{25+144} = \sqrt{169} = 13.$$

   The equation of the sphere is $(x+1)^2+(y-0)^2+(z-2)^2=13^2$.

   (c) $x^2+y^2+z^2+4x-6y = x^2+4x+4-4+y^2-6y+9-9+z^2=0$, $(x+2)^2-4+(y-3)^2-9+z^2=0$, $(x+2)^2+(y-3)^2+z^2=13$. The sphere has radius $\sqrt{13}$ and center $(-2,3,0)$.

4. The equation has the form $1x+3y+5z=k$. We find k so that the point lies on the plane. $2+3(-3)+5(-2)=k$, which gives $k=-17$. The equation of the plane is $x+3y+5z=-17$.

5. $$\vec{u}\times\vec{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ 0 & 1 & 1 \end{vmatrix} = -\mathbf{i}-\mathbf{j}+\mathbf{k}$$

   $$(\vec{u}\times\vec{v})\cdot\vec{u} = \langle -1,-1,1\rangle\cdot\langle 1,2,3\rangle = -1-2+3=0.$$

   $$(\vec{u}\times\vec{v})\cdot\vec{v} = \langle -1,-1,1\rangle\cdot\langle 0,1,1\rangle = -1+1=0.$$

6. (a) $\vec{PQ}=\langle 1,2,-1\rangle$, $\vec{PR}=\langle 2,-1,0\rangle$

   $$\vec{n} = \vec{PQ}\times\vec{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -1 \\ 2 & -1 & 0 \end{vmatrix} = -\mathbf{i}-2\mathbf{j}-5\mathbf{k}.$$

   (b) $-1(x-1)-2(y+1)-5(z-0)=0$ which gives the equation

   $-x-2y-5z+1-2=0$, i.e $x+2y+5z=-1$

   (c)

   $$\text{Area} = \frac{1}{2}||\vec{PQ}\times\vec{PR}|| = \frac{1}{2}\sqrt{1^2+2^2+5^2} = \frac{\sqrt{30}}{2}.$$

   (d) $\vec{PQ}\cdot\vec{PR} = 1\cdot 2+2\cdot(-1)+(-1)\cdot 0=0$, so $\vec{PQ}$ is perpendicular to $\vec{PR}$ and the angle at P is $\pi/2$.

7. $x^2+y^2=16$ gives $r^2=16$, which gives $r=4$.

   $x+y=3$ gives $r\cos\theta+r\sin\theta=3$, i.e. $r=3/(\cos\theta+\sin\theta)$.

   $y=4$ gives $r\sin\theta=4$ which is $r=4/\sin\theta=4\csc\theta$.

   $x^2+y^2-2x=0$ gives $r^2-2r\cos\theta=0$, $r-2\cos\theta=0$, $r=2\cos\theta$.

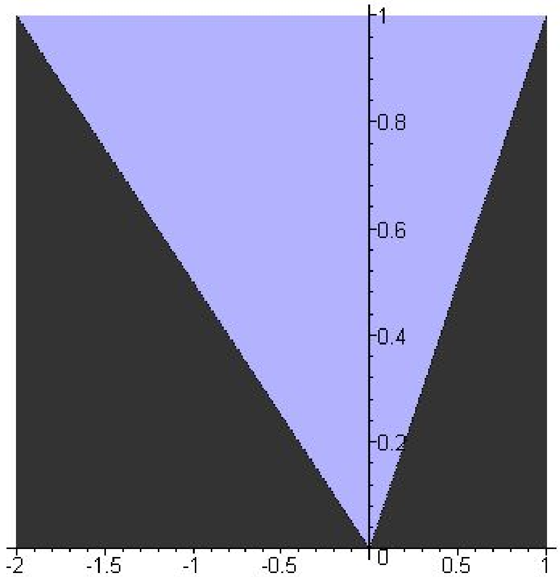

Figure 3: Region for the triangle is lightly-shaded.

8. $r = 5 \sec \theta$ gives $r \cos \theta = 5$, i.e. $x = 5$.

   $\theta = \pi/2$ represents the positive y axis, which has equation $x = 0$.

   $r^2 = 8 \csc(2\theta)$ gives $r^2 \sin(2\theta) = 8$, where we now use the identity from trigonometry $\sin(2x) = 2 \sin(x) \cos(x)$. We get $r^2 2 \sin \theta \cos \theta = 8$, which gives $xy = 4$.

9.
$$\frac{\partial r}{\partial x} = \frac{1}{2\sqrt{x^2+y^2}} \cdot 2x = \frac{x}{\sqrt{x^2+y^2}}, \quad \frac{\partial r}{\partial y} = \frac{1}{2\sqrt{x^2+y^2}} \cdot 2y = \frac{y}{\sqrt{x^2+y^2}}.$$
$$\frac{\partial \theta}{\partial x} = \frac{1}{1+(y/x)^2} \frac{\partial (y/x)}{\partial x} = \frac{1}{1+(y/x)^2} \frac{-y}{x^2} = -\frac{y}{x^2+y^2}.$$
$$\frac{\partial \theta}{\partial y} = \frac{1}{1+(y/x)^2} \frac{\partial (y/x)}{\partial y} = \frac{1}{1+(y/x)^2} \frac{1}{x} = \frac{x}{x^2+y^2}.$$
$$\frac{\partial x}{\partial r} = \cos\theta, \quad \frac{\partial y}{\partial r} = \sin\theta, \quad \frac{\partial x}{\partial \theta} = -r\sin\theta, \quad \frac{\partial y}{\partial \theta} = r\cos\theta.$$

10. The equations of the sides of the triangle are: $y = 1$ for the top, $y = x$, $y = -x/2 \Leftrightarrow x = -2y$. We consider the region as horizontally simple. The area is

$$\int\int_R 1\,dA = \int_0^1 \int_{-2y}^{y} 1\,dx\,dy = \int_0^1 [x]_{-2y}^{y}\,dy = \int_0^1 y-(-2y)\,dy = \int_0^1 3y\,dy = \left[\frac{3y^2}{2}\right]_0^1 = 3/2.$$

We could have computed the area as a vertically simple region as well. But then we need to split the integral into two parts, since there is a different formula for the side of the triangle in the second quadrant and the side in the first quadrant.

$$\int\int_R 1\,dA = \int_{-2}^0 \int_{-x/2}^1 1\,dy\,dx + \int_0^1 \int_x^1 1\,dy\,dx = \int_{-2}^0 [y]_{-x/2}^1\,dx + \int_0^1 [y]_x^1\,dx$$

$$= \int_{-2}^0 1-(-x/2)\,dx + \int_0^1 1-x\,dx = \int_{-2}^0 1+x/2\,dx + [x-x^2/2]_0^1 = [x+x^2/4]_{-2}^0 + [x-x^2/2]_0^1$$

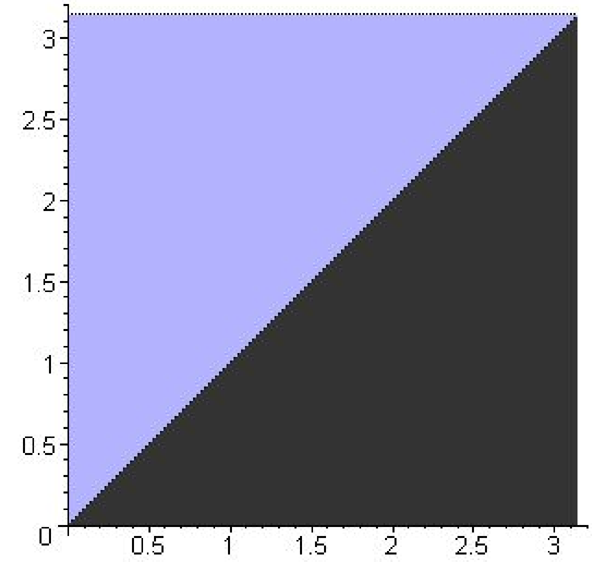

Figure 4: Region for (a) is lightly shaded

$$= -(-2+1) + (1 - 1^2/2) = 1 + 1/2 = 3/2.$$

11. In the first two integrals it is impossible to perform the integration in the order given. In the last one, one knows the antiderivative of $\sec(y)$ to be $\ln|\sec(y)+\tan(y)|$ but it is difficult to work with. So in all cases we reverse the order of integration.

$$\text{(a)} \int_0^\pi \int_x^\pi \frac{\sin y}{y}\, dy\, dx = \int_0^\pi \int_0^y \frac{\sin y}{y}\, dx\, dy = \int_0^\pi \left[\frac{\sin y}{y} x\right]_0^y dy$$

$$= \int_0^\pi \frac{\sin y}{y} y\, dy = \int_0^\pi \sin y\, dy = [-\cos y]_0^\pi = -\cos\pi + \cos 0 = 1 + 1 = 2.$$

$$\text{(b)} \int_0^{\sqrt{\pi}} \int_y^{\sqrt{\pi}} \sin(x^2)\, dx\, dy = \int_0^{\sqrt{\pi}} \int_0^x \sin(x^2)\, dy\, dx = \int_0^{\sqrt{\pi}} [\sin(x^2) y]_0^x\, dx$$

$$= \int_0^{\sqrt{\pi}} \sin(x^2) x\, dx = \int_0^\pi \sin(u)\, du/2 = \frac{1}{2}[-\cos(u)]_0^\pi = \frac{1}{2}(1+1) = 1,$$

where we have used the substitution $u = x^2 \Longrightarrow du = 2x dx$.

(c) We solve $y = \tan^{-1} x$ to get $x = \tan(y)$.

$$\int_0^1 \int_{\tan^{-1} x}^{\pi/4} \sec(y)\, dy\, dx = \int_0^{\pi/4} \int_0^{\tan(y)} \sec(y)\, dx\, dy = \int_0^{\pi/4} [\sec(y) x]_0^{\tan(y)}\, dy$$

$$= \int_0^{\pi/4} \sec(y)\tan(y)\, dy = \int_0^{\pi/4} \frac{\sin(y)}{\cos^2(y)}\, dy = \int_1^{1/\sqrt{2}} \frac{-du}{u^2} = [1/u]_1^{1/\sqrt{2}} = \sqrt{2} - 1.$$

In the integral we also used the substitution $\cos(y) = u$, which gives $-\sin(y)dy = du$.

(d) We switch to polar coordinates. The region of integration is a disc of radius 2 centered at the origin.

$$\int\int_R e^{x^2+y^2}\, dA = \int_0^{2\pi} \int_0^2 e^{r^2} r\, dr\, d\theta = \int_0^{2\pi} \left[\frac{e^{r^2}}{2}\right]_0^2 d\theta = \int_0^{2\pi} (e^4 - e^0)/2\, d\theta = \pi(e^4 - 1).$$

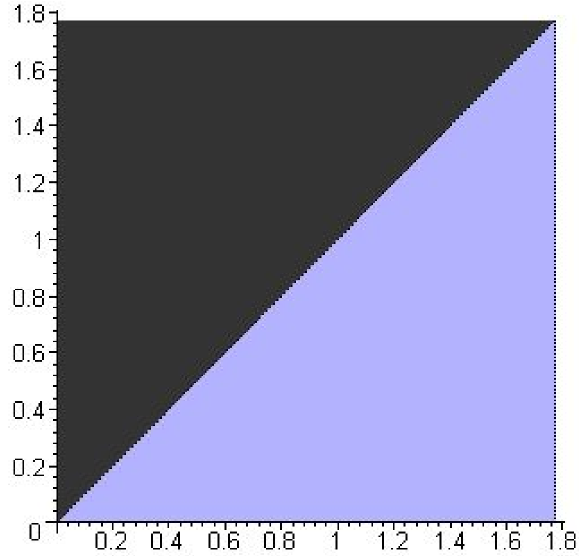

Figure 5: Region for (b) is lightly shaded

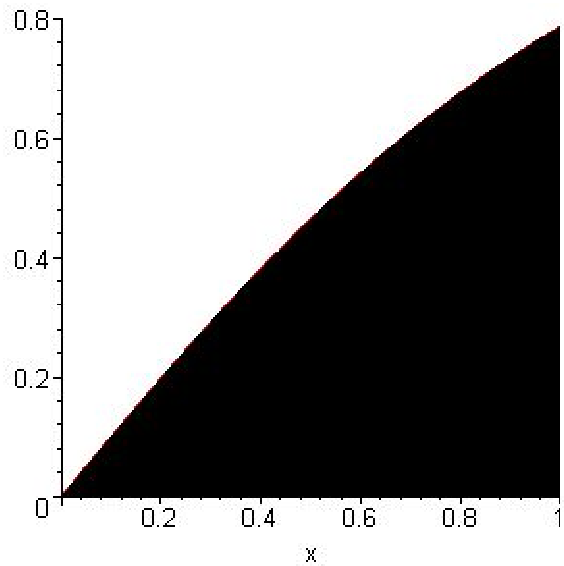

Figure 6: Region for (c) is darkly shaded

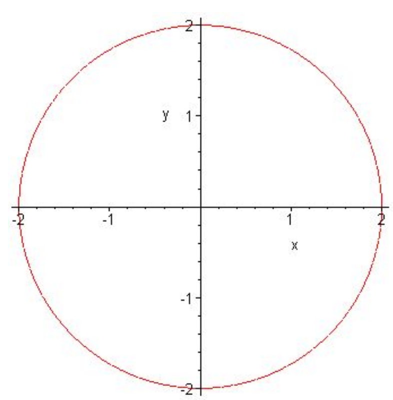

Figure 7: Region for (d)

12. We switch the integral in polar coordinates: $z_u = (5/4)\sqrt{1600 - r^2}$, $z_l = -\sqrt{1600 - r^2}$. The region of integration in the xy-plane can be understood if we set either $z_u = 0$ or $z_l = 0$. Both equations give

$$\sqrt{1600 - r^2} = 0 \Longrightarrow r^2 = 1600 \Longrightarrow r = 40.$$

This is the equation in polar coordinates of a circle of radius 40 centered at the origin. The volume of Humpty Dumpty is

$$V = \int_0^{2\pi} \int_0^{40} (z_u - z_l)\, r\, dr\, d\theta = \int_0^{2\pi} \int_0^{40} (5/4 + 1)\sqrt{1600 - r^2} r\, dr\, d\theta$$

$$= \frac{9}{4} 2\pi \int_0^{40} \sqrt{1600 - r^2} r\, dr = \frac{18\pi}{4} \int_{1600}^{0} \sqrt{u}(-du)/2 = \frac{9\pi}{4} \int_0^{1600} u^{1/2} du$$

$$= \frac{9\pi}{4} \frac{2}{3} [u^{3/2}]_0^{1600} = \frac{3\pi}{2} 1600^{3/2} = \frac{3\pi 64000}{2} = 96000\pi \text{ cm}^3.$$

where we have substituted $u = 1600 - r^2$, which gives $-2rdr = du$ and the limits of integration changed as follows:

$$r = 0 \Longrightarrow u = 1600, \quad r = 40 \Longrightarrow u = 0.$$