Syllabus

The textbook referred to is Mathematical Analysis, a straightforward approach by K. G. Binmore, Second Edition, Cambridge University Press.

Week 1

Reading Assignment: Sections 1.1, 1.2, 1.3, 1.4, Examples 1.5, 1.6, 1.7, Sections 1.9, 1.10, Example 1.11, Sections 1.13, 1.14, Theorems 1.15, 1.16, 1.17, 1.18, Example 1.19, Section 2.1.

Homework (not to be submitted): 1.8 (1), (2), (3), (4), (5), (6), 1.12 (1), (2), (4), (5), (6), 1.20 (1), (2), (4), (6)

Week 2

Reading Assignment: Section 2.2, Examples 2.4, 2.5, Sections 2.6, 2.7, Examples 2.8, Sections 2.9, 2.11, 3.1, 3.2, Theorem 3.3, Example 3.4

Homework (not to be submitted): 2.10 (1), 2.10 (2), 2.10 (3), 2.10 (4), 2.10 (6), 2.13 (1), 2.13 (3), 2.13 (4), 2.13 (5) (i) and (ii) only.

Written Assignment: Problem Set 1, p. 345.

Week 3

Reading Assignment: Theorem 3.5, Section 3.7, Theorem 3.8, Example 3.9, Example 3.10 (optional), Section 4.1, Section 4.2, Example 4.3, Section 4.4, Example 4.5.

Homework (not to be submitted): 3.6 (1), (2), (3), (4), (5), 3.11 (1), (2), (3), (4). Optional: 3.11 (5).

Week 4

Reading Assignment: Section 4.7, Proposition 4.8, Example 4.9, Theorem 4.10, Corollary 4.11, Examples 4.12, 4.13, 4.14, Section 4.15, Example 4.16, Theorem 4.17, Example 4.18, Example 4.19 (Optional).

Homework (not to be submitted): 4.6 (1), (2), (3), 4.20 (1), (2), (3).

Written Assignment: Problem Set 2: 1, 2 (p. 346).

Week 5

Reading Assignment: Section 4.21, Theorems 4.22, 4.23, 4.25, Example 4.24, Section 4.26, Examples 4.27, 4.28.

Homework (not to be submitted): 4.20 (6), 4.29 (1), (2), (3), (4).

Week 6

Reading Assignment: Section 5.1, Theorem 5.2, Example 5.3. Example 5.4 (optional), Examples 5.5, 5.6.

Homework (not to be submitted): 4.20 (4), 5.7 (1), (2). Moreover, investigate the sequence \( \langle x_n \rangle \) defined by: \( x_1 = 1 \) and \( x_{n+1} = (1/5)x_n + 2/5 \). Is in increasing, decreasing, bounded, what is its limit?
Week 7

Reading Assignment: Section 5.8, Theorems 5.9, 5.10, Example 5.11, Section 5.16, Propositions 5.17, 5.18, Theorem 5.19, Example 5.20 and its conclusion in 5.21 (3).

Homework (not to be submitted): 5.21 (1), (2).

Week 8

Reading Assignment: Section 7.1, Example 7.2, Example 7.3, 7.4, 7.5, Section 7.6 Examples 7.7, 7.8, Section 7.9, Example 7.10, Section 7.11, Example 7.12, Section 7.13 Example 7.14.

Written Assignment: Problem Set 2: (3), (4) (p. 346) and Problem Set 4: (1), (2) (p. 348).

Homework (not to be submitted): 7.16 (1), (2), (3), (4), (5), (6).

Week 9

Reading Assignment: Sections 8.1, 8.2, 8.3, Proposition 8.4, Example 8.5, Section 8.6, Example 8.7.


Written Assignment: Problem set 4: 3, 4 (page 348).

Week 10


Homework (not to be submitted): 9.17 (1), (3), (5), (6)

Week 11

Reading Assignment: Sections 10.1, 10.2, Example 10.3, Section 10.5, Theorem 10.6, Examples 10.7, 10.8, Theorem 10.9, Example 10.10, Section 10.12, Theorem 10.13, Example 10.14, Section 11.1, Theorem 11.2, Section 11.3, Theorem 11.4, Section 11.5, Theorems 11.6, 11.7

Homework (not to be submitted): 10.11 (1), (2), (3), (5), 10.15 (3), (4), (5), 11.8 (1), (2), (3), (4)

Written Assignment: Problem Set 5: 2, 3 (ignore the last sentence), 4 (ignore the last sentence), 6.
Projects

The purpose of the projects is to study other sections of Analysis that we have no time to cover in the class. The project requires that you carefully study and understand the parts from Binmore’s book required, or the handouts on the project. You are allowed to consult any book on the project. You have to solve the exercises you are asked. You have to type the project (Maple Worksheet is acceptable). Exposition counts. Among the other references you may use are the following books:

Spivak: Calculus (670 pages) Publisher: Publish or Perish; 3rd edition (September 1994) Language: English ISBN: 0914098896


Project 1: The exponential and logarithmic functions.

Read Chapter 14 from Binmore. Solve exercises 14.3 (2), (3), (4), (5), Exercise 15.6 (4), (5) and exercise 1 from Problem set 7, p. 351.

Project 2: Stirling’s formula.

From Binmore: Learn about Stirling’s formula. Explain Proposition 17.3 and use Maple to compare the actual values of $n!$ with the Stirling approximations. What percentage of error do you get? Work exercise 17.5.3.


From Binmore: Study sections 17.4, 17.6 and work exercises 17.5. (4), (5), (6).

Project 4: Machin’s arctan formula.

Prove the formulas $\arctan(x) + \arctan(y) = \arctan((x + y)/(1 - xy))$ and $\tan(2x) = 2\tan(x)/(1 - \tan(x)^2)$. If $x = \arctan(1/5)$, how much is $\tan(2x)$ and $\tan(4x)$? What about $\tan(4x - \pi/4)$? Prove that $\arctan 1/239 = 4 \arctan(1/5) - \pi/4$. Use this formula and the Taylor series of $\arctan(x)$ (bottom of p. 155) to calculate $\pi$ with 10 decimals.

Project 5: The Wallis’ product formula.

This project requires integration by parts from Calculus II. The theorem is also in the book: 13.21. Prove the reduction formulas on p. 316 of Spivak, and work all parts of problem 26 in p. 328 of the handout. Investigate with Maple the convergence of the product to $\pi$. How many terms do you need to multiply to compute $\pi$ with 3 decimals?

Project 6: Trigonometric functions in terms of area.

Explain in your own words the definition of $\pi$, $A(x)$ and $\sin x$ on p. 259 of Spivak, prove theorem 2 in the same chapter, solve exercises 5 and 6 on p. 269 of the same chapter. How much is $\cos(\pi/5)$? Use the formulas for $\sin(2x)$ and $\sin(3x)$. You are asked to find a formula with square roots.

Project 7: The derivative of $\sin(x)$.

Solve Problem 26 in the exercises of the chapter on Trigonometric functions of Spivak. Prove the formula for $\cos'(x)$ based only on this exercise. Solve exercise 27 as well.
Project 8: Bernoulli numbers.

This is an interesting sequence of numbers. In this project you will learn how they are defined and use them to understand the summation of the p-powers of natural number. Solve Problem 16 (a, b) and 17 (a, b, c, d) from the chapter of Complex power series of Spivak.

Project 9: Construction of the real numbers.

Learn how real numbers are defined from the rational numbers. Fill the missing details from the chapter of Spivak (Epilogue) e.g. theorem on $a + b = b + a$, theorem on $ab = ba$.

Project 10: Countable and uncountable sets.

In this project you will try to understand how large sets can be. Even if they are infinite, one can still be larger than the other. Read pages 9-18 from Kolmogorov-Fomin and work exercises Problem 4 and 5 on page 19. This project is more suitable for graduate students.

Project 11: Metric Spaces: Examples.

Learn the definition of a metric space and work through the examples in Kolmogorov and Fomin: p. 37-44. Solve problems 2, 3, 6, 7 on p. 45. This project is more suitable for graduate students.

Project 12: Convergence, Open sets and Closed sets in metric spaces.

Read p. 45-51 from Kolmogorov and Fomin. Explain examples 1, 2, 3, 5, 6, 7, on page 48. This project is more suitable for graduate students.
Homework Assignments

1. Find all real values of $x$ such that
   
   (i) $\frac{x + 1}{x^2 + 3} < \frac{2}{x}$

   We need to assume that $x \neq 0$ to make sense of the denominator. Notice that $x^2 + 3 > 0$, so there is no extra restriction out of the other denominator.

   
   
   $\frac{x + 1}{x^2 + 3} < \frac{2}{x} \Leftrightarrow \frac{x + 1}{x^2 + 3} - \frac{2}{x} < 0 \Leftrightarrow \frac{(x + 1)x - 2(x^2 + 3)}{x(x^2 + 3)} < 0 \Leftrightarrow -x^2 + x - 6 < 0$

   
   
   We now notice that the quadratic $-x^2 + x - 6$ has no (real) roots, as $b^2 - 4ac = 1^2 - 4(-1)(-6) = 1 - 24 < 0$. Since its leading term is $-1 < 0$ the quadratic is always negative (see pictures on p. 7). On the other hand $x^2 + 3 > x^2 \geq 0$. So $(-x^2 + x - 6)(x^2 + 3) < 0$. We multiple both sides of the inequality by $x^2(x^2 + 3)^2$ to get

   $x(-x^2 + x - 6)(x^2 + 3) < 0 \Leftrightarrow x > 0$

   (how could the product of $x$ with a negative number be negative?)

   (ii) $\left|\frac{1}{x + 1} - 1\right| < 2$

   We assume that $x \neq -1$ to make sense of the denominator. Use 1.20 (1) to get

   
   
   
   $-2 < \frac{1}{x + 1} - 1 < 2 \Leftrightarrow 0 < \frac{1}{x + 1} + 1$ and $\frac{1}{x + 1} - 3 < 0$

   
   
   $\Leftrightarrow 0 < \frac{x + 2}{x + 1}$ and $\frac{1 - 3(x + 1)}{x + 1} < 0 \Leftrightarrow 0 < \frac{x + 2}{x + 1}$ and $\frac{-3x - 2}{x + 1} < 0$

   
   
   Now we multiply the inequalities with $(x + 1)^2 > 0$ to get

   $0 < (x+1)(x+2)$ and $-(3x+2)(x+1) < 0 \Leftrightarrow (x > -1$ or $x < -2)$ and $(x < -1$ or $x > -2/3)$

   looking at the graphs of the quadratics on page 7. From the four cases that follow only two are compatible:

   
   
   $x < -1$ and $x < -2 \Leftrightarrow x < -2$

   and

   
   
   $x > -2/3$ and $x > -1 \Leftrightarrow x > -2/3$

   
   
   
   
   So the solution set is

   $(-\infty, -2) \cup (-2/3, \infty)$.

2. Suppose that for any $\epsilon > 0$, $|x| < \epsilon$. Prove that $x = 0$.

   We imitate example 1.7. and use 1.20 (1).

   $|x| < \epsilon \Leftrightarrow -\epsilon < x < \epsilon$.

   Assume that $x > 0$. Choose $\epsilon = x$ to get a contradiction out of the right inequality: $x < x$.

   Assume that $x < 0$. Choose $\epsilon = -x > 0$ to get a contradiction out of the left inequality: $-(x) < x \Leftrightarrow x < x$.

   The only possibility left is $x = 0$. 
3. Prove that, for any \( y \in (4, 5) \) there exists an \( x \in (4, 5) \) such that \( x < y \). Deduce that (4, 5) has no minimum.

Given \( y \in (4, 5) \) we have \( 4 < y < 5 \). We choose \( x = (4 + y)/2 \), the average of 4 and \( y \). As \( x \) is the average it satisfies \( 4 < x < y \). Together with the previous inequality we get:

\[
4 < x < y < 5 \implies x \in (4, 5) \text{ and } x < y.
\]

To deduce that (4, 5) has no minimum, we assume that we found a \( y \) which is a minimum. It has to belong to the set (4, 5). But then the previous argument produces an element \( x \) of the interval (4, 5), which is smaller than \( y \). The contradiction shows that there is no minimum.

4. For each of the following sets, find the sup, inf, max, and min whenever these exist.

(i) \([0, 3)\): sup\([0, 3)\) = 3 but there is no maximum (this is the same as 2.8 (ii)). On the other hand inf\([0, 3)\) = min\([0, 3)\) = 0.

(ii) \([-1, \infty)\). This set is unbounded above as in Example 2.4 (iii), so it has no supremum or maximum. On the other hand min\([-1, \infty]\) = inf\([-1, \infty]\) = -1.

(iii) \(S = \{0, -1, 3, -6\}\). Do not be confused with the order that the elements are written. The smallest is -6 and the largest is 3. This means (as in example 2.5 (i)) that sup\(S\) = max\(S\) = 3, while inf\(S\) = min\(S\) = -6.

(iv) \([1, 2]\). sup\([1, 2]\) = max\([1, 2]\) = 2, while inf\([1, 2]\) = min\([1, 2]\) = 1.

(v) \(S = \{x : x^2 - 2x - 1 < 0\}\). We first find the roots of the quadratic.

\[
\alpha, \beta = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{2 \pm \sqrt{(-2)^2 - 4 \cdot 1(-1)}}{2} = \frac{2 \pm \sqrt{8}}{2} = \frac{2 \pm 2\sqrt{2}}{2} = 1 \pm \sqrt{2}.
\]

Looking at the graphs on p. 7 we see that the quadratic is negative, when we are between the roots:

\(S = (1 - \sqrt{2}, 1 + \sqrt{2})\).

The set \(S\) has no maximum, as 2.8(ii) and no minimum (it is open at the left endpoint) but inf\(S\) = 1 - \(\sqrt{2}\), sup\(S\) = 1 + \(\sqrt{2}\).

(vi) \(T = \{x^2 - 2x - 1 : x \in \mathbb{R}\}\). This set is the set of values of the quadratic in (v). As the picture on p. 7 suggests, the parabola has a minimum at \(-b/(2a) = 1\) and the minimum value is \(1^2 - 2 \cdot 1 - 1 = -2\). This can also be seen by writing \(x^2 - 2x - 1 = (x - 1)^2 - 2 \geq -2\), as the expression \((x - 1)^2\) is nonnegative, and we also see that at \(x = 1\) we get -2. Consequently, inf\(T\) = min\(T\) = -2. On the other hand the set \(T\) is unbounded above: In fact \(T = [-2, \infty)\). The inclusion \(T \subset [-2, \infty)\) follows from the discussion above. We need to show that every number in \([-2, \infty)\) is in \(T\). This can be seen as follows: Let \(y \in [-2, \infty)\). We try to solve \(x^2 - 2x - 1 = y\). We get:

\[
x^2 - 2x - 1 = y \iff (x - 1)^2 - 2 = y \iff (x - 1)^2 = y + 2.
\]

Since \(y \geq -2\), we get \(y + 2 \geq 0\). This allows to compute square roots and get

\[
x - 1 = \pm \sqrt{y + 2} \iff x = 1 \pm \sqrt{y + 2}.
\]

(vii) \(\{x : x = x + 1\}\). There is no number \(x\) with \(x = x + 1\): it leads to \(0 = 1\), which was specifically ruled out at the beginning of the course. So this set is the empty
set $\emptyset$. This has no maximum or minimum, as it has no elements to serve as such. However, it is bounded above by any number $H$ and below by any number $h$: This may seem surprising but is easy to see:

If $H$ is a number we need to check that for all $x \in \emptyset$ we have $x \leq H$. Since $\emptyset$ has no elements, there is nothing to check, the property holds automatically. The same thing applies for any $h$. For all $x \in \emptyset$ we have $h \leq x$. There is no infimum or supremum, as there is no smallest upper bound and no greatest lower bound.

5. Suppose that $a$ and $b$ are real numbers with $a > 0$. If $S$ is nonempty and bounded above, prove that

$$\sup_{x \in S} (ax + b) = a \sup_{x \in S} x + b.$$ 

Call $M = \sup_{x \in S} x$. Its existence is guaranteed by the continuum property. First we prove that $\sup_{x \in S} (ax + b) \leq a \sup_{x \in S} x + b$. To do this it suffices to prove that the right-hand side is an upper bound for the set $\{ax + b : x \in S\}$. Since $S$ is bounded above by $M$ we have

$x \in S \implies x \leq M \implies ax \leq aM \implies ax + b \leq aM + b,$

where for the multiplication by $a$ we use the important assumption that $a > 0$, so that the inequality is preserved.

In particular the set $\{ax + b : x \in S\}$ has a supremum by the continuum property. Call it $P$. Now we prove that $P \geq aM + b$. This is equivalent to

$$P - b \geq aM \iff \frac{P - b}{a} \geq M.$$ 

So it suffices to prove that $(P - b)/a$ is an upper bound of $S$. Take any $x \in S$. Then $ax + b \in \{ax + b : x \in S\}$, which implies that

$$ax + b \leq P \implies ax \leq P - b \implies x \leq \frac{P - b}{a},$$

where we have used $a > 0$ when we divided by $a$.

6. Find the distance between the number $\xi = 2$ and the set $S$ in the following cases (see Exercise 2.12(4).)

(i) $S = \{-1, 0, 3\}$. The set $\{|2 - (-1)|, |2 - 0|, |2 - 3|\} = \{3, 2, 1\}$ has minimum 1. So $d(2, S) = 1$.

(ii) $S = (0, 3)$. Since $2 \in S$, $0 \in \{\xi - x : x \in S\} = T$. Then inf $T = 0$ as $T$ contains only nonnegative numbers. Said differently: Since $2 \in S$ the distance from 2 to $S$ is 0.

(iii) $S = [2, 4]$. Same as in (ii). $d(2, S) = 0$.

(iv) $S = (2, 4)$. The argument above does not apply. Instead we identify the set $T = \{\xi - x : x \in (2, 4)\}$. Since $x \in (2, 4)$, we have $2 < x < 4$. This implies $-4 < x < -2$ and $-4 < 2 - x < 0$. Taking absolute values produces $2 > 2 - x > 0$. So $T = (0, 2)$. The infimum of this set is 0. So $d(2, S) = 0$, even if $2 \notin S$. 
7. Suppose that \( x > -1 \) and \( x \neq 0 \). Prove by induction that, for any natural number \( n \geq 2 \), \((1 + x)^n > 1 + nx\). Deduce that

\[
\left(1 + \frac{1}{n}\right)^n > 2
\]

for all natural numbers \( n \geq 2 \).

\( P(2) \): \((1 + x)^2 > 1 + 2x\), since \((1 + x)^2 = 1 + 2x + x^2 > 1 + 2x\), as \( x^2 > 0 \) for \( x \neq 0 \).

Assume \( P(n) \): \((1 + x)^n > 1 + nx\). We try to prove \( P(n+1) \): \((1 + x)^{n+1} > 1 + (n+1)x\).

We have

\[
(1 + x)^{n+1} = (1 + x)^n(1 + x) > (1 + nx)(1 + x)
\]

using the inductive hypothesis and the fact that \( 1 + x > 0 \). Then

\[
(1 + x)^{n+1} > 1 + nx + x + nx^2 > 1 + nx + x = 1 + (n+1)x,
\]

as \( nx^2 > 0 \).

We apply the result to \( x = 1/n \) to get

\[
\left(1 + \frac{1}{n}\right)^n > 1 + \frac{1}{n} = 2.
\]

8. Find the following limits:

(i) \( \lim_{n \to \infty} \frac{4n^5 + 5n^3 + 6n}{2n^5 + 1} \).

We factor \( n^5 \) from numerator and denominator to get:

\[
\lim_{n \to \infty} \frac{n^5(4 + 5/n^2 + 6/n^4)}{n^5(2 + 1/n^5)} = \lim_{n \to \infty} \frac{4 + 5/n^2 + 6/n^4}{2 + 1/n^5} = \frac{\lim 4 + \lim 5/n^2 + \lim 6/n^4}{\lim 2 + \lim 1/n^5}
\]

using the combination theorem (i) and (iii). Another application of it gives:

\[
\lim_{n \to \infty} \frac{4n^5 + 5n^3 + 6n}{2n^5 + 1} = \frac{4 + 5 \lim 1/n^2 + 6 \lim 1/n^4}{2 + \lim 1/n^5} = \frac{4 + 5 \cdot 0 + 6 \cdot 0}{2 + 0} = 2,
\]

where we used the fact that \( 1/n^r \to 0 \) for \( r > 0 \) (Exercise 4.6 (2)), and the obvious fact that a constant sequence \( x_n = c \) converges to \( c \).

(ii) \( \lim_{n \to \infty} \frac{3^n + (-2)^n}{3^n - 2^n} \).

We factor \( 3^n \) from both numerator and denominator to get:

\[
\lim_{n \to \infty} \frac{3^n + (-2)^n}{3^n - 2^n} = \lim_{n \to \infty} \frac{3^n(1 + (-2/3)^n)}{3^n(1 - (2/3)^n)} = \lim_{n \to \infty} \frac{1 + (-2/3)^n}{1 - (2/3)^n}
\]

\[
= \frac{\lim 1 + \lim(-2/3)^n}{\lim 1 - \lim(2/3)^n} = \frac{1 + 0}{1 - 0} = 1,
\]

where we used Example 4.12 with \( x = -2/3 \) in the numerator and \( x = 2/3 \) in the denominator.
The sequence defined by: $x_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}$.

So we now know that the sequence is bounded below by 1 and bounded above by 2.

(ii) Assume that the sequence is strictly decreasing.

We assume that $P(n)$ is true, i.e. $x_n > x_{n+1}$, we will prove $P(n)$ is true, i.e. $x_{n+1} > x_{n+2}$. We have

$$x_n > x_{n+1} \implies \frac{1}{5}x_n > \frac{1}{5}x_{n+1} \implies \frac{1}{5}x_n + \frac{2}{5} > \frac{1}{5}x_{n+1} + \frac{2}{5}$$

which is the same as

$$x_{n+1} > x_{n+2}$$

by the recursion. But this is $P(n+1)$.

Second idea, that will lead to a proof: Consider the difference $x_{n+1} - x_n$. We try to prove that

$$x_{n+1} - x_n < 0 \iff \frac{1}{5}x_n + \frac{2}{5} - x_n < 0 \iff -\frac{4}{5}x_n + \frac{2}{5} < 0 \iff \frac{2}{5} < \frac{4}{5}x_n \iff \frac{1}{2} < x_n.$$  (1)

But this we do not know yet!

So we decide to prove by induction $P(n)$: $1/2 < x_n$. $P(1)$ is true, as $1/2 < 1 = x_1$. We assume that $P(n)$ is true, i.e., $1/2 < x_n$ and try to prove that $P(n+1)$ is true, i.e., $1/2 < x_{n+1}$. We have:

$$\frac{1}{2} < x_n \implies \frac{11}{5} < \frac{2}{5}x_n \implies \frac{1}{10} + \frac{2}{5} < \frac{1}{5}x_{n+1} + \frac{2}{5} \implies \frac{1}{10} + \frac{4}{10} < x_{n+1} \implies \frac{1}{2} = \frac{5}{10} < x_{n+1}.$$

So we now know that the sequence is bounded below by $1/2$ and this we can use in (1) to see that the sequence is strictly decreasing. So it converges, say $\lim x_n = l$.

This implies also that $\lim x_{n+1} = l$, since $x_{n+1}$ is a subsequence of $x_n$. Plugging into the recurrence formula we get:

$$\lim x_{n+1} = \frac{1}{5}\lim x_n + \frac{2}{5} \implies l = \frac{1}{5}l + \frac{2}{5}. $$
This is an equation that we easily solve to get
\[ l - \frac{1}{5}l = \frac{2}{5} \iff \frac{4}{5}l = \frac{2}{5} \iff 4l = 2 \iff l = 1/2. \]

There is another method that can be useful: We guess a pattern, formula for the sequence \( x_n \) and prove it by induction. From \( x_2 = 3/5, x_3 = 13/25 \), we start guessing that the denominator is a power of 5, in fact for \( x_n \) it should be \( 5^{n-1} \). The numerator is a bit more difficult. We notice that it is a bit larger than half the denominator: \( 5/2 = 2.5 \) and we have 3 for \( x_2 \). Similarly \( 25/2 = 12.5 \) and we have 13 for \( x_3 \). In fact we see exactly how much larger: the numerator is half the denominator plus \( 1/2 \). So we guess the formula:
\[ x_n = \frac{(1/2)5^{n-1} + (1/2)}{5^{n-1}} = \frac{5^{n-1} + 1}{2 \cdot 5^{n-1}}. \]

Let’s prove this formula by induction.
\( P(1) \) is true, as \( x_1 = 1 \) and
\[ \frac{5^0 + 1}{2 \cdot 5^0} = \frac{1 + 1}{2} = 1. \]

Now we assume \( P(n) \) is true and prove \( P(n + 1) \).
\[ x_n = \frac{5^{n-1} + 1}{2 \cdot 5^{n-1}} \implies \frac{1}{5}x_n = \frac{15^{n-1} + 1}{5 \cdot 2 \cdot 5^{n-1}} = \frac{5^{n-1} + 1}{2 \cdot 5^n} \implies \frac{1}{5}x_n + \frac{2}{5} = \frac{5^{n-1} + 1}{2 \cdot 5^n} + \frac{2}{5} \]
\[ \implies x_{n+1} = \frac{5^{n-1} + 1}{2 \cdot 5^n} + \frac{4 \cdot 5^{n-1}}{5^n} = \frac{5^{n-1} + 1 + 4 \cdot 5^{n-1}}{2 \cdot 5^n} = \frac{5 \cdot 5^{n-1} + 1}{2 \cdot 5^n} = \frac{5^n + 1}{2 \cdot 5^n}. \]

Now we can compute the limit as follows:
\[ x_n = \frac{5^{n-1} + 1}{2 \cdot 5^{n-1}} = \frac{1 + (1/5)^{n-1}}{2} - \frac{1 + 0}{2} = \frac{1}{2}, \]
where we used the fact that for \( |x| < 1 \), we have \( \lim x^n = 0 \), applied to \( (1/5)^{n-1} = 5 \cdot (1/5)^n \).

10. Suppose that \( 0 < k < 1 \) and \( \langle x_n \rangle \) satisfies
\[ |x_{n+1}| < k|x_n| \quad (n = 1, 2, 3, \ldots). \]

Prove that \( x_n \to \infty \) as \( n \to \infty \). Explain why the same conclusion holds if it is only known that \( |x_{n+1}| < k|x_n| \) when \( n > N \) for some natural number \( N \).

\textit{Solution}: We claim that all the terms are \( \neq 0 \). Otherwise, there is a subscript \( m \), such that \( x_m = 0 \), which gives \( |x_{m+1}| < k \cdot 0 = 0 \). Since there is no number with negative absolute value, this proves our claim. We write successively
\[ |x_2| < k|x_1| \]
\[ |x_3| < k|x_2| \]
\[ |x_4| < k|x_3| \]
\[ \ldots \]
\[ |x_{n-1}| < k|x_{n-2}| \]
\[ |x_n| < k|x_{n-1}|. \]
We multiply all the above to get

\[ |x_2||x_3| \cdots |x_{n-1}||x_n| < k^{n-1}|x_1||x_2| \cdots |x_{n-1}| \]

and cancel \(|x_2|, |x_3|, \ldots |x_{n-1}|\) to get

\[ |x_n| < k^{n-1}|x_1|. \]

Since \(0 < k < 1\), we know that \(k_n \to 0\). By the sandwich theorem we get \(x_n \to 0\). If we only know the given inequality for \(n > N\), we start our enumerate from \(n = N + 1\), to get for \(n > N + 2\)

\[ |x_{N+2}| < k|x_{N+1}| \]
\[ |x_{N+3}| < k|x_{N+2}| \]
\[ \vdots \]
\[ |x_n| < k|x_{n-1}|. \]

which gives now \(|x_n| < k^{n-N-1}|x_{N+1}| = k^n(k^{N-1}|x_{N+1}|)\). This also implies \(x_n \to 0\), as the first \(N\) terms of the sequence do not matter for the convergence.

11. Suppose that \(y_n \to l\) as \(n \to \infty\). If \(l < k\), prove that there exists an \(N\) such that \(y_n < k\) for any \(n > N\). Suppose that \(0 < l < 1\) and

\[ \left| \frac{a_{n+1}}{a_n} \right| \to l, \quad \text{as } n \to \infty. \] (2)

Use question 3 to show that \(a_n \to 0\) as \(n \to \infty\). Hence show that, for any real number \(\alpha\) and any \(x\) satisfying \(|x| < 1\),

\[ \lim_{n \to \infty} \frac{\alpha(\alpha - 1) \cdots (\alpha - n + 1)}{n!} x^n = 0. \]

**Solution:** For the part involving \(y_n \to l < k\). Since \(k > l, k - l > 0\) and we can take \(\epsilon = k - l\) in the definition of the limit of the sequence \(y_n \to l\). We get that we can find a number \(N\), such that

\[ n > N \implies |y_n - l| < k - l. \]

Now for any number \(x, x \leq |x|\), so we get

\[ y_n - l \leq |y_n - l| < k - l \implies y_n - l < k - l \implies y_n < k. \]

Suppose now that \(0 < l < 1\) and (2) holds. Pick up a number \(k\), larger than \(l\) but still smaller than 1, e.g. \(k = (l + 1)/2 < 1\). Then we can find \(N\), such that \(|a_{n+1}/a_n| < k\) for \(n > N\). This implies that \(|a_{n+1}| < k|a_n|\) for \(n > N\), which, by exercise 3, implies \(a_n \to 0\).

For the last part we set

\[ a_n = \frac{\alpha(\alpha - 1) \cdots (\alpha - n + 1)}{n!} x^n \]
and consider
\[ \left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{\alpha(\alpha - 1) \cdots (\alpha - n + 1)(\alpha - n + 2)x^{n+1}/(n+1)!}{\alpha(\alpha - 1) \cdots (\alpha - n + 1)x^n/n!} \right| = \left| \frac{(\alpha - n + 2)x}{n+1} \right| \to -x = |x|, \]

since \((n + 1)! = n!(n + 1)\) and \((\alpha - n + 2)/(n + 1) \to -1\). According to the previous part, if \( |x| < 1 \), we get that \( a_n \to 0 \).

12. Draw a diagram illustrating the equation \( x^2 + y^4 = 1 \). Explain why:

(i) the equation does not define a function \( f : \mathbb{R} \to \mathbb{R} \)

There is no \( f(x) \) for \( x > 1 \) or \( x < -1 \). In these cases \( x^2 > 1 \) and this gives \( y^4 < 0 \), which is impossible. Moreover, the graph does not pass the vertical line test.

(ii) the equation does not define a function \( f : [-1, 1] \to [-1, 1] \)

The first problem from (i) is solved by looking only on \([-1, 1]\) but the vertical line test is not satisfied, as the figure shows.

(iii) the equation does define a function \( f : [-1, 1] \to [0, 1] \). Here we also restrict the codomain. It passes the vertical line test.

![Figure 1: The graph does not pass the vertical line test.](image1)

![Figure 2: The graph passes the vertical line test.](image2)

13. Suppose that \( f : (0, \infty) \to (0, \infty) \) is defined by \( f(x) = 1/x \) \((x > 0)\) and that \( g : (0, \infty) \to (0, \infty) \) is defined by \( g(x) = x^2 - 2x + 2, \) \((x > 0)\).

(i) Which of these functions has an inverse? Find a formula for the inverse where it exists.
The function \( f \) has an inverse as it is one-to-one:

\[
f(a) = f(b) \iff \frac{1}{a} = \frac{1}{b} \iff a = b
\]

and onto, i.e. the range of \( f \) is \((0, \infty)\). Its inverse can be computed as follows:

\[
y = \frac{1}{x} \iff yx = 1 \iff x = \frac{1}{y}
\]

so \( f^{-1}(y) = 1/y \). If you want to write it in terms of the \( x \) variable, we get \( f^{-1}(x) = 1/x = f(x) \). The function \( f \) is its own inverse.

The function \( g \) does not have an inverse, because it is not one-to-one: \( g(1/2) = 1/4 - 2(1/2) + 2 = 5/4 \) and \( g(3/2) = 9/4 - 2(3/2) + 2 = 5/4 \). So the function does not pass the horizontal line test.

Figure 3: \( 1/x \) is symmetric around the diagonal \( y = x \)

Figure 4: \( g \) does not pass the horizontal line test

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(ii) Compute \( f \circ g \) and \( g \circ f \).

\[
(f \circ g)(x) = f(g(x)) = f(x^2 - 2x + 2) = \frac{1}{x^2 - 2x + 2},
\]

while

\[
(g \circ f)(x) = g(f(x)) = g(1/x) = (1/x)^2 - 2(1/x) + 2 = \frac{1 - 2x + 2x^2}{x^2}.
\]

14. Evaluate the following limits.

(i) \( \lim_{x \to 3} \frac{x^3 + 5x + 7}{x^4 + 6x^2 + 8} \)
Method 1: Use of Theorem 8.12 (iii) and 8.13. We know that the polynomials in the numerator and denominator are continuous. This implies
\[
\lim_{x \to 3}(x^3 + 5x + 7) = 3^3 + 5 \cdot 3 + 7 = 27 + 15 + 7 = 49,
\]
\[
\lim_{x \to 3}(x^4 + 6x^2 + 8) = 3^4 + 6 \cdot 3^2 + 8 = 81 + 54 + 8 = 143 \neq 0.
\]
Then
\[
\lim_{x \to 3} \frac{x^3 + 5x + 7}{x^4 + 6x^2 + 8} = \frac{49}{143}.
\]

Method 2: Use of sequences. Let \(x_n \to 3\) but \(x_n \neq 3\). Then
\[
\lim f(x_n) = \lim \frac{x_n^3 + 5x_n + 7}{x_n^4 + 6x_n^2 + 8} = \frac{\lim(x_n^3 + 5x_n + 7)}{\lim(x_n^4 + 6x_n^2 + 8)} = \frac{\lim(x_n^3) + \lim(5x_n) + \lim 7}{\lim(x_n^4) + \lim(6x_n^2) + \lim 8}
\]
\[
= \frac{\lim(x_n)^3 + 5 \lim x_n + 7}{\lim(x_n)^4 + 6(\lim x_n)^2 + 8} = \frac{3^3 + 5 \cdot 3 + 7}{3^4 + 6 \cdot 3^2 + 8} = \frac{49}{143}.
\]

(ii) \(\lim_{x \to 0^+} x^{1/2}\)

The limit is 0. Use of the definition: Given \(\epsilon > 0\) we must find \(\delta > 0\) such that
\[
0 < x < \delta \implies |\sqrt{x}| < \epsilon.
\]
Since
\[
|\sqrt{x}| < \epsilon \iff |\sqrt{x}|^2 < \epsilon^2 \iff |(\sqrt{x})^2| < \epsilon^2 \iff |x| < \epsilon^2 \iff -\epsilon^2 < x < \epsilon^2,
\]
we can take \(\delta = \epsilon^2\). (The first \(\iff\) follows from Example 1.6. and the second \(\iff\) follows from Theorem 1.16.) Now \(0 < x < \delta\) implies \(-\epsilon^2 < x < \epsilon^2\) which implies \(|\sqrt{x}| < \epsilon\) by the above inequalities.

(iii) \(\lim_{x \to 0} \frac{(1 + x)^{1/2} - (1 - x)^{1/2}}{x}\)

Recall the formula \(a^2 - b^2 = (a - b)(a + b)\).

We cannot use proposition 8.12 because the limit of the denominator is 0. We have
\[
\frac{(1 + x)^{1/2} - (1 - x)^{1/2}}{x} = \frac{(1 + x)^{1/2} - (1 - x)^{1/2}}{x} \frac{(1 + x)^{1/2} + (1 - x)^{1/2}}{(1 + x)^{1/2} + (1 - x)^{1/2}}
\]
\[
= \frac{(1 + x) - (1 - x)}{x[(1 + x)^{1/2} + (1 - x)^{1/2}]} = \frac{2x}{x[(1 + x)^{1/2} + (1 - x)^{1/2}]} = \frac{2}{(1 + x)^{1/2} + (1 - x)^{1/2}}.
\]
If we prove that
\[
\lim_{x \to 0}(1 + x)^{1/2} = 1, \quad \lim_{x \to 0}(1 - x)^{1/2} = 1,
\]
then Proposition 8.12 gives
\[
\lim_{x \to 0} \frac{(1 + x)^{1/2} - (1 - x)^{1/2}}{x} = \lim_{x \to 0} \frac{2}{(1 + x)^{1/2} + (1 - x)^{1/2}} = \frac{2}{1 + 1} = 1.
\]
We have
\[(1 + x)^{1/2} - 1 = \frac{[(1 + x)^{1/2} - 1] \cdot [(1 + x)^{1/2} + 1]}{(1 + x)^{1/2} + 1} = \frac{1 + x - 1}{(1 + x)^{1/2} + 1} = \frac{x}{(1 + x)^{1/2} + 1}.\]

Now we use the sandwich theorem 8.14. First we observe that the denominator is bigger than 1, it is 1 plus a nonnegative quantity (a square root). This gives
\[
\frac{1}{(1 + x)^{1/2} + 1} \leq 1.
\]

\[|(1 + x)^{1/2} - 1| = \left| \frac{x}{(1 + x)^{1/2} + 1} \right| \leq |x|,
\]
which gives \(-x \leq (1 + x)^{1/2} - 1 \leq x\). Since \(\lim_{x \to 0} x = 0 = \lim_{x \to 0} (-x)\), the sandwich theorem gives
\[\lim_{x \to 0} (1 + x)^{1/2} = 1.
\]

We work similarly for the other limit in (3).

15. Let \(f : \mathbb{R} \to \mathbb{R}\) be defined by
\[
f(x) = \begin{cases} 
(x - 1)^2 & (x < 1) \\
1 & (x = 1) \\
3x + 2 & (x > 1).
\end{cases}
\]

Use the definitions to show that \(f(x) \to 0\) as \(x \to 1^-\) and \(f(x) \to 5\) as \(x \to 1^+\). Is this function
(i) continuous
(ii) continuous on the right
(iii) continuous on the left
at the point 1?

**Solution:** For \(\lim_{x \to 1^-} f(x) = 0\): Given \(\epsilon > 0\) we must find a \(\delta > 0\) such that
\[1 - \delta < x < 1 \implies |f(x) - 0| < \epsilon.
\]

We have, for \(x < 1\),
\[|f(x)| < \epsilon \iff (x - 1)^2 < \epsilon \iff |x - 1|^2 < \epsilon \iff |x - 1| < \sqrt{\epsilon}
\[
\iff -\sqrt{\epsilon} < x - 1 < \sqrt{\epsilon} \iff 1 - \sqrt{\epsilon} < x < 1 + \sqrt{\epsilon}.
\]

If we take \(\delta = \sqrt{\epsilon} > 0\), then \(1 - \delta < x < 1\) implies \(1 - \sqrt{\epsilon} < x < 1 + \sqrt{\epsilon}\), which by the above inequalities implies \(|f(x)| < \epsilon\).

For \(\lim_{x \to 1^+} f(x) = 5\): Given \(\epsilon\) we must find \(\delta > 0\) such that
\[1 < x < 1 + \delta \implies |f(x) - 5| < \epsilon.
\]

We have, for \(x > 1\),
\[|f(x) - 5| < \epsilon \iff |3x + 2 - 5| < \epsilon \iff |3x - 3| < \epsilon \iff -\epsilon < 3x - 3 < \epsilon
\]
\[3 - \epsilon < 3x < 3 + \epsilon \Leftrightarrow 1 - \frac{\epsilon}{3} < x < 1 + \frac{\epsilon}{3}.
\]

If we take \(\delta = \frac{\epsilon}{3}\), then \(1 < x < 1 + \delta\) implies \(1 - \frac{\epsilon}{3} < x < 1 + \frac{\epsilon}{3}\), which by the above inequalities implies \(|f(x) - 5| < \epsilon\).

Since \(f(1) = 1\), which is different from both limits of the function from the left and right at 1, we conclude that the function is neither continuous from the left nor from the right at 1. Moreover, it is not continuous at 1.

Figure 5: The box is a point on the graph, the circles are not

16. Show that the equation \(x^{16} + x^7 - 1 = 0\) has a solution \(\xi \in (0, 1)\).

We apply the intermediate value theorem to \(f(x) = x^{16} + x^7 - 1\) on the interval \([0, 1]\) on which it is continuous. We have: \(f(0) = 0^{16} + 0^7 - 1 = -1 < 0\) and \(f(1) = 1^{16} + 1^7 - 1 = 1 > 0\). The intermediate value theorem guarantees a \(\xi \in (0, 1)\) with \(f(\xi) = 0\).

Figure 6: The polynomial \(f(x) = x^{16} + x^7 - 1\)

17. Show that \(x^3 + x - 1 = 0\) has a solution in \([0, 1]\) and find the first decimal digit of it.
We consider the polynomial \( f(x) = x^3 + x - 1 \) and apply the intermediate value theorem repeatedly on intervals on which it changes sign. We call the solution \( \xi \).

\[
f(0) = -1 < 0, \quad f(1) = 1 + 1 - 1 = 1 > 0 \implies \xi \in (0, 1).
\]

We go to the midpoint \( x = 1/2 \).

\[
f(1/2) = \frac{1}{8} + \frac{1}{2} - 1 = -\frac{3}{8} < 0, \quad f(1) > 0 \implies \xi \in (1/2, 1).
\]

We go to the midpoint \( x = 3/4 \).

\[
f(3/4) = \frac{27}{64} + \frac{3}{4} - 1 = \frac{27 + 48 - 64}{64} = \frac{11}{64} > 0, \quad f(1/2) < 0 \implies \xi \in (1/2, 3/4).
\]

We go to the midpoint \( x = 5/8 \).

\[
f(5/8) = \frac{125}{512} + \frac{5}{8} - 1 = \frac{125 + 320 - 512}{512} = -\frac{67}{512} < 0, \quad f(3/4) > 0, \implies \xi \in (5/8, 3/4).
\]

We go to the midpoint \( x = 11/16 \).

\[
f(11/16) = \frac{51}{4096} > 0, \quad f(5/8) < 0 \implies \xi \in (5/8, 11/16).
\]

Since \( 5/8 = 0.625 \), while \( 11/16 = 0.6875 \), the first decimal digit of the solution is 6.

18. Assume \( f \) is continuous on the interval \([0, 1]\) and \( f(0) = f(1) \). Show that we can find a \( c \in [0, 1/2] \) with

\[
f(c) = f(c + 1/2).
\]

We consider the function

\[
g(x) = f(x) - f(x + 1/2)
\]

which is continuous on \([0, 1/2]\). We have

\[
g(0) = f(0) - f(1/2), \quad g(1/2) = f(1/2) - f(1/2 + 1/2) = f(1/2) - f(1) = f(1/2) - f(0).
\]

So \( g \) changes sign at the endpoints. The intermediate value theorem implies that we can find a \( c \in [0, 1/2] \) with

\[
g(c) = 0 \iff f(c) - f(c + 1/2) = 0 \iff f(c) = f(c + 1/2).
\]

19. A function \( f : [a, b] \to \mathbb{R} \) has the property that \( f(x) \geq 0 \) for \( a \leq x \leq b \), while \( f(a) = 0 \) and \( f(b) = 0 \). If, for each \( x \in [a, b] \) there exists exactly one distinct \( y \in [a, b] \) with \( f(x) = f(y) \), prove that \( f \) cannot be continuous on \([a, b]\). [Hint: If \( f \) is continuous on \([a, b]\) there are points on \([a, b]\) at which it achieves its maximum. These can be used to obtain a contradiction using the intermediate value theorem.]

If \( f \) is continuous on the compact interval, by Theorem 9.12, \( f \) achieves its maximum on \([a, b]\). This maximum \( M \) is either 0 or \( M \neq 0 \). If \( M = 0 \), then \( f(x) = 0 \) on the whole interval, i.e. the function is constant and we violate the condition: \( f(x) = f(y) \) has a unique solution in \( y \) for given \( x \).
So we can assume that $M > 0$ and that it is achieved at $\xi \in [a, b]$. However, since $M > 0$, while $f(a) = f(b) = 0$, we must have $a < \xi < b$. Now we apply the intermediate value theorem for any number $\lambda \in (0, M)$ on the two intervals $[a, \xi]$ and $[\xi, b]$. We get two numbers $\xi_1 \in (a, \xi)$ and $\xi_2 \in (\xi, b)$ with $f(\xi_1) = \lambda = f(\xi_2)$. But then we violate the condition $f(x) = f(y)$ holds for only one distinct $y$, given $x \in [a, b]$. This is a contradiction. We arrive at the contradiction by using the continuity of $f$ on the compact interval $[a, b]$. So the function cannot be continuous on the interval.

20. Show that the function $g : \mathbb{R} \to \mathbb{R}$ defined by

$$g(x) = \begin{cases} x^3 & (x < 0) \\ x^{1/2} & (x \geq 0) \end{cases}$$

is continuous at 0 but that $g$ is not differentiable at 0. Show that the function $h : \mathbb{R} \to \mathbb{R}$ defined by $h(x) = xg(x)$ is differentiable at every point including 0.

$$\lim_{x \to 0^+} g(x) = \lim_{x \to 0^+} x^{1/2} = 0^{1/2} = g(0), \quad \lim_{x \to 0^-} g(x) = \lim_{x \to 0^-} x^3 = 0^3 = 0 = g(0)$$

so the function $g$ is continuous at 0.

For the derivative at 0, we examine the limits from the left and the right:

$$\lim_{x \to 0^+} \frac{g(x) - g(0)}{x - 0} = \lim_{x \to 0^+} \frac{x^{1/2} - 0}{x} = \lim_{x \to 0^+} x^{-1/2} = \infty$$

while

$$\lim_{x \to 0^-} \frac{g(x) - g(0)}{x - 0} = \lim_{x \to 0^-} \frac{x^3 - 0}{x} = \lim_{x \to 0^-} x^2 = 0.$$  

The limits do not agree, in fact, the right-hand limit is infinite. Consequently the derivative at 0 for $g$ does not exist.

Figure 7: $g$ does not have a well-defined tangent line at $x = 0$: from the left we get the $x$-axis, from the right we get the $y$-axis.
However, the functions $g$ is differentiable at any point $x \neq 0$, and, in fact:

$$g'(x) = \begin{cases} 
3x^2 & (x < 0) \\
\frac{1}{2}x^{-1/2} & (x > 0)
\end{cases}$$

This is obvious because away from 0 the function $g$ is given by a formula that does not change and which we can differentiate.

Now we consider the function $h$. For $x \neq 0$ the function $h$ is differentiable as the product of two differentiable functions: $x$ and $g(x)$. It is only 0 which is troublesome. We examine the limits from the left and the right:

$$\lim_{x \to 0^+} \frac{h(x) - h(0)}{x - 0} = \lim_{x \to 0^+} \frac{x \cdot x^{1/2} - 0}{x} = \lim_{x \to 0^+} x^{1/2} = 0$$

while

$$\lim_{x \to 0^-} \frac{h(x) - h(0)}{x - 0} = \lim_{x \to 0^-} \frac{x \cdot x^3 - 0}{x} = \lim_{x \to 0^-} x^3 = 0.$$ 

The two limits agree, consequently:

$$h'(0) = \lim_{x \to 0} \frac{h(x) - h(0)}{x - 0} = 0.$$ 

21. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable at every point and satisfy $f'(x) > 0$ for all values of $x$. Prove that the equation $f(x) = 0$ can have at most one solution. If $f''(x) > 0$ for all values of $x$, show $f(x) = 0$ can have at most 2 solutions. State and prove the generalization involving $n$-th derivatives. [Hint: Use Rolle’s theorem.]

If the equation $f(x) = 0$ has two distinct solutions, say $a$ and $b$, then: $f(a) = f(b) = 0$. By Rolle’s Theorem we can find a $\xi$ between them with $f'(\xi) = 0$. But this contradicts $f'(x) > 0$ for all $x$.

If the equation $f(x) = 0$ has three distinct solutions, say $a$, $b$ and $c$, then $f(a) = f(b) = f(c) = 0$. We can assume that the points are ordered as $a < b < c$. Otherwise
we relabel them. Then we apply Rolle’s theorem to the intervals $[a, b]$ and $[b, c]$. We get two numbers $\xi_1 \in (a, b)$ and $\xi_2 \in (b, c)$ such that

$$f'(\xi_1) = 0 = f'(\xi_2).$$

The two points $\xi_1$ and $\xi_2$ are distinct, as $b$ lies between them. We apply Rolle’s theorem to the function $f'(x)$ and the interval $[\xi_1, \xi_2]$. We deduce the existence of a point $\lambda \in (\xi_1, \xi_2)$ with $f''(\lambda) = 0$. But this contradicts $f''(x) > 0$.

If the function has $n$ derivatives on $\mathbb{R}$ and $f^{(n)}(x) > 0$, then the equation $f(x) = 0$ has at most $n$ solutions. We prove the result as follows: If $f(x) = 0$ has $n+1$ distinct solutions, we order them and label the points as $x_1 < x_2 < \cdots < x_{n+1}$. Application of Rolle’s theorem for the function $f$ on the successive intervals $[x_1, x_2], [x_2, x_3], \ldots [x_n, x_{n+1}]$ produces $n$ distinct points $y_1 < y_2 < \cdots < y_n$ where the derivative $f'(x) = 0$. Because $f'(y_i) = 0$ for $i = 1, 2, \ldots n$, we can apply Rolle’s Theorem to $f'$ on the intervals $[y_1, y_2], [y_2, y_3], \ldots [y_{n-1}, y_n]$ to deduce the existence of $n-1$ points, where $f''(x) = 0$. We repeat the process. At every stage we produce $n-k+1$ points where $f^{(k)}(x) = 0$. Here $k = 1, 2, \ldots n$. When $k = n$, we end up with one point $\xi$ with $f^{(n)}(\xi) = 0$. But this contradicts $f^{(n)}(x) > 0$.

A second proof can be done by induction.

22. If $f : [0, 1] \to [0, 1]$ is continuous on $[0, 1]$, and differentiable at least on the open interval $(0, 1)$ with $f'(x) \neq 1$, then there exists a unique $\xi \in [a, b]$ with $f(\xi) = \xi$.

The existence of (at least one) such $\xi$ follows from the Intermediate Value Theorem as in 9.16 of the book. Suppose there were two distinct such numbers, which we label $\xi_1$ and $\xi_2$. We can also order them so that $\xi_1 < \xi_2$. Then we apply Rolle’s Theorem on the interval $[\xi_1, \xi_2]$ for the function

$$g(x) = f(x) - x.$$

We have

$$g(\xi_1) = f(\xi_1) - \xi_1 = 0, \quad g(\xi_2) = f(\xi_2) - \xi_2 = 0.$$

This implies that there exists a number $\xi$ in the interval $(\xi_1, \xi_2)$ with $g'(\xi) = 0$. But $g'(x) = f'(x) - 1$, so $f'(\xi) - 1 = 0 \implies f'(\xi) = 1$. This contradicts $f'(x) \neq 1$.

23. Let $f : \mathbb{R} \to \mathbb{R}$ with

$$|f(x) - f(y)| \leq |x - y|^n, \quad x, y \in \mathbb{R}, \quad n > 1.$$ 

Show that $f$ is a constant function.

We need to show that it is differentiable and $f'(x) = 0$ for all $x \in \mathbb{R}$. We have

$$\left| \frac{f(y) - f(x)}{y - x} \right| \leq \frac{|y - x|^n}{|y - x|} = |y - x|^{n-1} \to 0, \quad y \to x,$$

as $n > 1 \implies n - 1 > 0$. We now use the sandwich theorem to deduce that

$$\lim_{y \to x} \frac{f(y) - f(x)}{y - x} = 0, \quad i.e. \quad f'(x) = 0.$$
24. If
\[ \frac{a_0}{1} + \frac{a_1}{1} + \frac{a_2}{3} + \cdots + \frac{a_n}{n+1} = 0 \]
then there exists a \( x \in [0, 1] \) with \( a_0 + a_1x + a_2x^2 + \cdots a_n x^n = 0 \).
Consider the function
\[ f(x) = a_0x + a_1\frac{x^2}{2} + \cdots + a_n\frac{x^{n+1}}{n+1}. \]
This is a polynomial and consequently it is continuous and differentiable everywhere on \( \mathbb{R} \). We have
\[ f(0) = 0, \quad f(1) = \frac{a_0}{1} + \frac{a_1}{1} + \frac{a_2}{3} + \cdots + \frac{a_n}{n+1} = 0. \]
Rolle’s theorem applied to \( f(x) \) on the interval \( [0, 1] \) implies that we can find an \( x \in (0, 1) \) with \( f'(x) = 0 \). Since
\[ f'(x) = a_0 + a_1x + \cdots a_n x^n \]
we get the result.

25. If \( f \) is differentiable on \( (0, \infty) \) and \( f'(x) = \frac{1}{x} \) for \( x > 0 \) and \( f(1) = 0 \), show that
\[ f(xy) = f(x) + f(y), \quad x, y > 0. \]

Fix \( y > 0 \). We consider the function
\[ g(x) = f(xy). \]
This is also differentiable and
\[ g'(x) = f'(xy)(xy)' = f'(xy)y = \frac{1}{xy} \cdot y = \frac{1}{x} = f'(x) \]
using the chain rule. Moreover, \( g(1) = f(y) \). Since \( g'(x) = f'(x) \) the functions differ by a constant, i.e. we can find \( c \) with
\[ g(x) = f(x) + c. \]
We compute \( c \) by plugging \( x = 1 \), which gives \( f(y) = g(1) = f(1) + c = 0 + c = c \).
This implies \( g(x) = f(x) + f(y) \), as required.
Quiz 1

True or False? Write T for True and F for False next to the statement.

1. $\frac{-1}{0} = -\infty$. FALSE: you cannot divide by 0.

2. If $x < y$, then $x^2 < y^2$. FALSE, this is true only for $0 < x < y$ (Example 1.6). Counterexample: $-5<3$ but $(-5)^2 = 25 > 3^2 = 9$.

3. If $x < y$ and $y < z$, then $x < z$. TRUE, this is Rule I on page 4.

4. For any real numbers $a$ and $b$ we have $|-ab| = |a| \cdot |b|$. TRUE:

   $$|-ab| = |ab| = |a| \cdot |b|,$$

   since a number and its opposite have the same absolute value and we used Th. 1.16 from the book.

5. For any real numbers $a$ and $b$ we have $|a + b| < |a| + |b|$. FALSE. Be careful the triangle inequality says $|a + b| \leq |a| + |b|$. Equality can happen, for instance when $a$ and $b$ are positive numbers: $|3 + 5| = 8 = |3| + |5|$.

6. $\sqrt{3}$ is irrational. TRUE, this is exercise 1.20 (6) in the book.

7. $\sqrt{9}$ is irrational. FALSE, $\sqrt{9} = 3$, which is a rational number.

8. We always have $|r|^2 = r^2$. TRUE, this is on page 10, just before 1.15.

Quiz 1: Solutions

True or False? Write T for True and F for False next to the statement.

1. $\frac{-1}{0} = -\infty$. FALSE: you cannot divide by 0.

2. If $x < y$, then $x^2 < y^2$. FALSE, this is true only for $0 < x < y$ (Example 1.6). Counterexample: $-5<3$ but $(-5)^2 = 25 > 3^2 = 9$.

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4. For any real numbers $a$ and $b$ we have $|-ab| = |a| \cdot |b|$. TRUE:

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   since a number and its opposite have the same absolute value and we used Th. 1.16 from the book.

5. For any real numbers $a$ and $b$ we have $|a + b| < |a| + |b|$. FALSE. Be careful the triangle inequality says $|a + b| \leq |a| + |b|$. Equality can happen, for instance when $a$ and $b$ are positive numbers: $|3 + 5| = 8 = |3| + |5|$.

6. $\sqrt{3}$ is irrational. TRUE, this is exercise 1.20 (6) in the book.

7. $\sqrt{9}$ is irrational. FALSE, $\sqrt{9} = 3$, which is a rational number.

8. We always have $|r|^2 = r^2$. TRUE, this is on page 10, just before 1.15.
Quiz 2

True or False? Write T for True and F for False next to the statement.

1. Every set of natural numbers which is non empty has a maximum. **FALSE:** Theorem 3.5 says that it has a MINIMUM. Counterexample: The set $\mathbb{N}$ itself is non empty but is unbounded above by the archimedean property 3.3.

2. The set of natural numbers is unbounded above. **TRUE:** This is the archimedean property 3.3.

3. $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$. **TRUE:** this is example 3.9.

4. Every set of real numbers which is bounded above has a smallest upper bound. **FALSE.** Every nonempty set of real numbers which is bounded above has a smallest upper bound. This is the continuum property. The empty set $\emptyset$ has no smallest upper bound, as all numbers are upper bounds for it. See also Homework 1 Solutions for Problem set 1: 4(vii).

5. For the set $S = [2, 3)$ we have: max $S = 3$ and sup $S = 3$. **FALSE:** This set has no maximum exactly as in 2.8(ii).

Quiz 2 Solutions

True or False? Write T for True and F for False next to the statement.

1. Every set of natural numbers which is non empty has a maximum. **FALSE:** Theorem 3.5 says that it has a MINIMUM. Counterexample: The set $\mathbb{N}$ itself is non empty but is unbounded above by the archimedean property 3.3.

2. The set of natural numbers is unbounded above. **TRUE:** This is the archimedean property 3.3.

3. $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$. **TRUE:** this is example 3.9.

4. Every set of real numbers which is bounded above has a smallest upper bound. **FALSE.** Every nonempty set of real numbers which is bounded above has a smallest upper bound. This is the continuum property. The empty set $\emptyset$ has no smallest upper bound, as all numbers are upper bounds for it. See also Homework 1 Solutions for Problem set 1: 4(vii).

5. For the set $S = [2, 3)$ we have: max $S = 3$ and sup $S = 3$. **FALSE:** This set has no maximum exactly as in 2.8(ii).
6. Suppose $S$ is a nonempty set of real numbers which is bounded above and $\xi \in \mathbb{R}$. Then

$$\sup_{x \in S} \xi x = \xi \sup_{x \in S} x.$$ 

FALSE: Theorem 2.12 imposes the condition $\xi > 0$. Without it the result is false: $S = (-\infty, 0)$ is bounded above by 0 and $\sup S = 0$. However, $(-1)S = (0, \infty)$, which is unbounded above by example 2.8 (iii).

7. $(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$. TRUE: this is the binomial theorem for $n = 4$. You can compute the binomial coefficients to get:

$$\binom{4}{0} = \frac{4!}{0!4!} = 1, \quad \binom{4}{1} = \frac{4!}{1!3!} = 4, \quad \binom{4}{2} = \frac{4!}{2!2!} = 6,$$

$$\binom{4}{3} = \frac{4!}{3!1!} = 4, \quad \binom{4}{4} = \frac{4!}{4!0!} = 1,$$

since $4! = 24$, $3! = 6$, $2! = 2$, $1! = 1$ and $0! = 1$.

8. Suppose that for each $n \in \mathbb{N}$, $P(n)$ is a statement about the natural number $n$. The principle of induction says that: Suppose that if $P(n)$ is true, then $P(n + 1)$ is true. Then $P(n)$ is true for every $n \in \mathbb{N}$.

FALSE: We are missing the first condition: $P(1)$ is true. If you want a counterexample, try $P(n) : n = n + 1$. Assuming $P(n)$ we get that $P(n + 1)$ is true simply by adding 1 to both sides. However, this statement is patently false for all numbers and, in fact, $P(1)$ fails: $1 \neq 2$. 

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Quiz 3

Multiple choice questions. Choose the write answer.

1. A sequence is convergent if and only if
   A. it is a Cauchy sequence.
   B. it is monotone.
   C. Given $\epsilon$, we can find $n$, such that $n > N$ implies $|x_n - l| < \epsilon$.
   D. Given $\epsilon$, we can find $n$, such that $n > N$ implies $|x_N - l| < \epsilon$.

2. Every sequence has
   A. an increasing subsequence.
   B. a decreasing subsequence.
   C. a monotone subsequence.
   D. a monotone subsequence, only if it is bounded.

3. The Bolzano Weierstrass theorem says:
   A. Every bounded sequence has a monotone subsequence.
   B. Every bounded sequence has a convergent subsequence.
   C. Every bounded sequence is a Cauchy sequence.
   D. Every bounded sequence has a bounded subsequence.

4. A sequence $\langle x_n \rangle$ is a Cauchy sequence, if
   A. Given $\epsilon > 0$, we can find $m$, such that for any $n > m$, $m > N$, $|x_m - x_n| < \epsilon$.
   B. Given $\epsilon$ we can find $N$, such that for any $n > N$ and $m > N$, $|x_n - x_m| < \epsilon$.
   C. Given $\epsilon > 0$, we can find $N$, such that for any $n > N$, $m > N$, $|x_m - x_n| < \epsilon$.
   D. Given $\epsilon > 0$, we can find $n$, such that for any $n > N$, $m > N$, $|x_m - x_n| < \epsilon$.

5. A Cauchy sequence
   A. may be unbounded.
   B. converges to its supremum.
   C. converges to its infimum.
   D. converges to the limit of any subsequence.

6. An increasing sequence
   A. may be unbounded.
   B. converges to its supremum.
   C. converges to its infimum.
   D. always converges.
Quiz 3: Solutions

Multiple choice questions. Choose the write answer.

1. A sequence is convergent if and only if
   **A. it is a Cauchy sequence.** Follows from Prop. 5.17, 5.19.
   B. it is monotone. Counterexample: \( x_n = (-1)^n/n \) is not monotone but has limit 0.
   C. Given \( \epsilon \), we can find \( n \), such that \( n > N \) implies \( |x_n - l| < \epsilon \). Notice that the correct one is: we can find \( N \), such that...
   D. Given \( \epsilon \), we can find \( n \), such that \( n > N \) implies \( |x_N - l| < \epsilon \). Here to make it correct you have to replace \( n > N \) by \( N > n \).

2. Every sequence has
   A. an increasing subsequence.
   B. a decreasing subsequence.
   **C. a monotone subsequence.** Theorem 5.9 says it has a monotone subsequence. We do not know in general whether it has to be increasing or decreasing.
   D. a monotone subsequence, only if it is bounded. False, since even if it is unbounded, we can apply Theorem 5.9.

3. The Bolzano Weierstrass theorem says:
   A. Every bounded sequence has a monotone subsequence. True, but this is not Bolzano-Weierstrass.
   **B. Every bounded sequence has a convergent subsequence.**
   C. Every bounded sequence is a Cauchy sequence. False, e.g. \( x_n = (-1)^n \).
   D. Every bounded sequence has a bounded subsequence. True: take the subsequence to be the whole sequence. But this is not what Bolzano Weierstrass says.

4. A sequence \( \langle x_n \rangle \) is a Cauchy sequence, if
   A. Given \( \epsilon > 0 \), we can find \( m \), such that for any \( n > m, m > N, |x_m - x_n| < \epsilon \).
   B. Given \( \epsilon \) we can find \( N \), such that for any \( n > N \) and \( m > N, |x_n - x_m| < \epsilon \). Missing \( \epsilon > 0 \).
   **C. Given \( \epsilon > 0 \), we can find \( N \), such that for any \( n > N, m > N, |x_m - x_n| < \epsilon \).**
   D. Given \( \epsilon > 0 \), we can find \( n \), such that for any \( n > N, m > N, |x_m - x_n| < \epsilon \).

5. A Cauchy sequence
   A. may be unbounded. False, by Prop. 5.18.
   B. converges to its supremum. It does not have to be increasing. Counterexample: \( x_n = (-1)^n/n \). we have sup \( x_n = 1/2 \) and inf \( x_n = -1 \), while \( \lim x_n = 0 \).
   C. converges to its infimum. It does not have to be decreasing.
   **D. converges to the limit of any subsequence.** This is part of the proof of Theorem 5.19.
6. An increasing sequence

A. **may be unbounded.** e.g. \( x_n = n \)

B. converges to its supremum. Does not have to be bounded above.

C. converges to its infimum. False.

D. always converges. \( x_n = n \) does not converge and is increasing.
Quiz 4

Multiple choice questions. Choose the write answer.

1. If $f$ and $g$ are continuous on $\mathbb{R}$. Which of the following function could be discontinuous on $\mathbb{R}$?
   A. $f + g$
   B. $f \cdot g$
   C. $f/g$
   D. $f \circ g$

2. The equation $x^5 + x + 1 = 0$ has a solution in the interval
   A. $[0, 1]$
   B. $[-1, 0]$
   C. $[-2, -1]$
   D. $[1, 2]$

3. The continuity property theorem for a functions $f$ continuous on a compact interval $[a, b]$ says:
   A. $f([a, b]) = [f(a), f(b)]$
   B. $f([a, b]) = [f(b), f(a)]$
   C. The image of $[a, b]$ under $f$ is also an interval.
   D. $f([a, b]) = [\inf_{x \in [a, b]} f(x), \sup_{x \in [a, b]} f(x)]$

4. Which one is true
   A. Let $f : [a, b] \to [a, b]$ be continuous on $(a, b)$. Then, for some $\xi \in [a, b]$, $f(\xi) = \xi$.
   B. Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$. Then, for some $\xi \in [a, b]$, $f(\xi) = \xi$.
   C. Let $f : [a, b] \to [a, b]$ be continuous on $[a, b]$. Then, for some $\xi \in [a, b]$, $f(\xi) = \xi$.
   D. Let $f : [a, b] \to [a, b]$ be continuous on $[a, b]$. Then, for some $\xi \in (a, b)$, $f(\xi) = \xi$.

5. The intermediate value theorem says: Let $f$ be continuous on an interval $I$ containing $a$ and $b$.
   A. If $\lambda$ lies between $f(a)$ and $f(b)$, then we can find a $\xi$ between $a$ and $b$ such that $\lambda = f(\xi)$.
   B. If $\lambda$ lies between $a$ and $b$, then we can find a $\xi$ between $a$ and $b$ such that $\lambda = f(\xi)$.
   C. If $f(a) < \lambda < f(b)$, then we can find a $\xi$ between $a$ and $b$ such that $\lambda = f(\xi)$.
   D. If $\lambda$ lies between $f(a)$ and $f(b)$, then we can find a $\xi$ between $a$ and $b$ such that $f(\lambda) = \xi$. 
Quiz 4 Solutions

Multiple choice questions. Choose the write answer.

1. If \( f \) and \( g \) are continuous on \( \mathbb{R} \). Which of the following function could be discontinuous on \( \mathbb{R} \)?
   A. \( f + g \)
   B. \( f \cdot g \)
   C. \( f/g \) If \( g(x) = 0 \), then the quotient may be discontinuous.
   D. \( f \circ g \)

2. The equation \( x^5 + x + 1 = 0 \) has a solution in the interval
We check the endpoints of all the intervals:

\[
\begin{align*}
  f(0) &= 1, & f(1) &= 3, & f(-1) &= -1, & f(-2) &= -33, & f(2) &= 35 \\
\end{align*}
\]

The only interval where the function changes sign is \([ -1, 0 ]\) and the intermediate value theorem guarantees the solution to exist there.

A. \([0, 1]\):
B. \([-1, 0]\)
C. \([-2, -1]\)
D. \([1, 2]\)

![Figure 9: The graph of \( x^5 + x + 1 \)](image)

3. The continuity property theorem for a functions \( f \) continuous on a compact interval \([a, b]\) says:
   A. \( f([a, b]) = [f(a), f(b)] \). True only if \( f(a) \) is the minimum and \( f(b) \) is the maximum of \( f \).
B. $f([a, b]) = [f(b), f(a)]$. True only if $f(b)$ is the minimum and $f(a)$ is the maximum of $f$.

C. The image of $[a, b]$ under $f$ is also an interval. Correct would have been **compact** interval.

D. $f([a, b]) = [\inf_{x \in [a, b]} f(x), \sup_{x \in [a, b]} f(x)]$. The image is a compact interval from the minimum to the maximum of the values of $f$.

4. Which one is true

A. Let $f : [a, b] \to [a, b]$ be continuous on $(a, b)$. Then, for some $\xi \in [a, b]$, $f(\xi) = \xi$.

Counterexample:

$$f(x) = \begin{cases} 
  x^2, & 0 < x < 1 \\
  1/2, & x = 0, 1
\end{cases}$$

This function is continuous on $(0, 1)$ but not on the compact interval $[0, 1]$. However, there is no $\xi$ with $\xi = f(\xi)$, since the equation

$$f(x) = x \iff x^2 = x \iff x^2 - x = 0 \iff x(x - 1) = 0 \iff x = 0, 1.$$ 

At $x = 0, 1$ the function is $1/2$, so these points are not fixed points.

![Graph](image1.png)

Figure 10: The graph for 4A.

B. Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$. Then, for some $\xi \in [a, b]$, $f(\xi) = \xi$.

Counterexample $f(x) = x + 1$. The equation $x + 1 = x$ has no solution. The range of the function here is too large.

C. **Let $f : [a, b] \to [a, b]$ be continuous on $[a, b]$. Then, for some $\xi \in [a, b]$, $f(\xi) = \xi$.** This is Example 9.16.

D. Let $f : [a, b] \to [a, b]$ be continuous on $[a, b]$. Then, for some $\xi \in (a, b)$, $f(\xi) = \xi$.

Counterexample: $f : [0, 1] \to [0, 1]$ be given by $f(x) = x^2$. The fixed points are at $x = 0$ and $x = 1$ only. So $\xi$ could be at the endpoints of the interval.
5. The intermediate value theorem says: Let $f$ be continuous on an interval $I$ containing $a$ and $b$.

A. If $\lambda$ lies between $f(a)$ and $f(b)$, then we can find a $\xi$ between $a$ and $b$ such that $\lambda = f(\xi)$.

B. If $\lambda$ lies between $a$ and $b$, then we can find a $\xi$ between $a$ and $b$ such that $\lambda = f(\xi)$. $\xi$ should be in the codomain of $f$, if $f(\xi) = \lambda$. So it is false.

C. If $f(a) < \lambda < f(b)$, then we can find a $\xi$ between $a$ and $b$ such that $\lambda = f(\xi)$. This would have been true if $f(a) < f(b)$. The intermediate value theorem applies even if this is not true.

D. If $\lambda$ lies between $f(a)$ and $f(b)$, then we can find a $\xi$ between $a$ and $b$ such that $f(\lambda) = \xi$. $\lambda$ should be in the domain to apply $f$ to it.
Midterm 1

1. State:
   (i) The continuum property.
   (ii) The archimedean property. (No proof is required).
   (iii) The definition of convergence of a sequence \( \langle x_n \rangle \) to a number \( l \).

2. State and prove the principle of induction.

3. (a) Give the definition of an increasing sequence.
   (b) Prove that an increasing sequence, which is bounded above, converges to its smallest upper bound.

4. Prove either one of the following:
   (i) \( \sqrt{2} \) is irrational.
   (ii) If \( x \neq 1 \), then \( \sum_{k=0}^{n} x^k = \frac{1 - x^{n+1}}{1 - x} \).

5. Do either of the following:
   (i) Define \( \binom{n}{r} \) for \( r = 0, 1, \ldots, n \). Prove that
       \[ \binom{n}{r} + \binom{n}{r-1} = \binom{n+1}{r} \).
   (ii) Suppose that \( |x| < 1 \). Prove that \( x^n \to 0 \), as \( n \to \infty \).

6. Do either of the following:
   (i) Find \( \lim_{n \to \infty} \left( \frac{1 + 2 + \cdots + n}{n^2} + \sqrt{\pi} \right) \)
   (ii) Find the sup, inf, max, min, if they exist for the set
        \( \{(x-3)^2 + 2; x \in \mathbb{R}\} \).

Midterm 1 Solutions

1. State:
   (i) The continuum property. Ans: The box at the bottom of page 14. Every non empty set of real numbers which is bounded above has a smallest upper bound. Every non empty set of real numbers which is bounded below has a largest lower bound.
   (ii) The archimedean property. (No proof is required). Ans: The statement of Theorem 3.3: The set of natural numbers is unbounded above.
   (iii) The definition of convergence of a sequence \( \langle x_n \rangle \) to a number \( l \). Ans: The box on top of page 28: Given \( \epsilon \) we can find an \( N \) such that, for any \( n > N \), \( |x_n - l| < \epsilon \).

3. (a) Give the definition of an increasing sequence. Ans: A sequence \( \langle x_n \rangle \) is increasing if
\[
x_n \leq x_{n+1}, \quad n = 1, 2, \ldots.
\]
(b) Prove that an increasing sequence, which is bounded above, converges to its smallest upper bound.

4. Prove either one of the following:
(i) \( \sqrt{2} \) is irrational.
Ans: From page 9: Let \( x = \sqrt{2} \). If \( x = m/n \), where \( m \) and \( n \) are natural numbers, with no common divisor, other than 1. Then:
\[
m^2 = 2n^2.
\]
Since the square of an even number is even and the square of an odd number is odd, we get:
\[
2n^2 \text{ even } \implies m^2 \text{ even } \implies m \text{ even }.
\]
We may write \( m = 2k \). This gives
\[
(2k)^2 = 2n^2 \implies 4k^2 = 2n^2 \implies 2k^2 = n^2.
\]
\[
2k^2 \text{ even } \implies n^2 \text{ even } \implies n \text{ even }.
\]
But then both \( m \) and \( n \) are even and they have a common factor 2.

(ii) If \( x \neq 1 \), then \( \sum_{k=0}^{n} x^k = \frac{1 - x^{n+1}}{1 - x} \).
This is exercise 3.11 (2). Use induction. \( P(1) \): \( x^0 + x^1 = \frac{1 - x^2}{1 - x} \), which is true as the left-hand side is \( 1 + x \), while the right-hand side is
\[
\frac{1 - x^2}{1 - x} = \frac{(1-x)(1+x)}{1-x} = 1 + x.
\]
Assume \( P(n) \) is true, i.e., \( \sum_{k=0}^{n} x^k = \frac{1 - x^{n+1}}{1 - x} \). We need to show \( P(n+1) \), i.e.,
\[
\sum_{k=0}^{n+1} x^k = \frac{1 - x^{n+2}}{1 - x}.
\]
We have
\[
\sum_{k=0}^{n+1} x^k = \sum_{k=0}^{n} x^k + x^{n+1} = \frac{1 - x^{n+1}}{1 - x} + x^{n+1} \quad \text{by} \ P(n)
\]
\[
= \frac{1 - x^{n+1}}{1 - x} + \frac{(1-x)x^{n+1}}{1 - x} = \frac{1 - x^{n+1} + x^{n+1} - x \cdot x^{n+1}}{1 - x} = \frac{1 - x^{n+2}}{1 - x}.
\]
5. Do either of the following:

(i) Define \( \binom{n}{r} \) for \( r = 0, 1, \ldots, n \). Prove that

\[
\binom{n}{r} + \binom{n}{r-1} = \binom{n+1}{r}.
\]

This is exercise 3.11 (4). We define \( n! = 1 \cdot 2 \cdot \cdots \cdot n \), for \( n > 1 \), and \( 0! = 1 \). Then

\[
\binom{n}{r} = \frac{n!}{r!(n-r)!}.
\]

We have:

\[
\binom{n}{r} + \binom{n}{r-1} = \frac{n!}{(r-1)!(n-r)!} \left( \frac{1}{r} + \frac{1}{n-r+1} \right) = \frac{n!}{(r-1)!(n-r)!} \left( \frac{n-r+1}{r(n-r+1)} \right) + \frac{n!}{r!(n-r+1)!} = \frac{n+1}{r!(n+1-r)!} = \binom{n+1}{r}.
\]

(ii) Suppose that \( |x| < 1 \). Prove that \( x^n \to 0 \), as \( n \to \infty \).

This is example 4.12. If \( x = 0 \), the result is obvious. So assume that \( x \neq 0 \). Let \( h = \frac{1}{|x|} - 1 > 0 \), by the condition \( |x| < 1 \). This gives \( 1 + h = \frac{1}{|x|} \). Then \( (1 + h)^n \geq 1 + nh \) by Bernoulli’s inequality. This gives

\[
|x^n| = |x|^n = \frac{1}{(1+h)^n} \leq \frac{1}{1+nh} < \frac{1}{nh}.
\]

Since the last sequence has limit 0, we get by the sandwich theorem that \( x^n \to 0 \).

6. Do either of the following:

(i) Find \( \lim_{n \to \infty} \left( \frac{1 + 2 + \cdots + n}{n^2} + \sqrt{\pi} \right) \)

By example 3.9, we have

\[
\lim_{n \to \infty} \left( \frac{1 + 2 + \cdots + n}{n^2} + \sqrt{\pi} \right) = \lim \frac{n(n+1)/2}{n^2} + \sqrt{\pi} = \lim \frac{n^2 + n}{2n^2} + \lim \sqrt{\pi} = \lim n^2(1 + 1/n) + 1 = \lim \left( \frac{1}{2} + \frac{1}{2n} \right) + 1 = 1/2 + 0 + 1 = 3/2,
\]

since by example 4.14 for any \( x > 0 \), \( \lim x^{1/n} = \lim \sqrt[n]{x} = 1 \).

(ii) Find the sup, inf, max, min, if they exist for the set

\( \{(x-3)^2 + 2; x \in \mathbb{R}\} \).

Every number \( y \geq 2 \) can be written as \( y = (x-3)^2 + 2 \), for some \( x \). We just have to solve

\[
y - 2 = (x-3)^2 \iff \sqrt{y-2} = \pm(x-3) \iff x = 3 \pm \sqrt{y-2}.
\]
This is possible for any \( y \geq 2 \), since \( y - 2 \geq 0 \) means we can find its square root.

The set is bounded below by 2: For every number \( y^2 \geq 0 \), i.e., for all \( x \) we have \((x - 3)^2 \geq 0\). This implies that \((x - 3)^2 + 2 \geq 2\). So the set in question is \([2, \infty)\). The set is unbounded above, consequently it has no sup or max. The inf is 2 and is also the minimum, as we get for \( x = 3 \): \((x - 3)^2 + 2 = (3 - 3)^2 + 2 = 0 + 2 = 2\).
1. State:
   (i) The definition of a Cauchy sequence.
   (ii) The Bolzano-Weierstrass Theorem. *(No proof is required)*.
   (iii) The definition of \( \lim_{x \to a^+} f(x) = l \).

2. Let \( f : \mathbb{R} \to \mathbb{R} \) be defined by
   \[
   f(x) = \begin{cases} 
   3 - x & (x > 1) \\
   1 & (x = 1) \\
   2x & (x < 1)
   \end{cases}
   \]
   (i) Draw a graph of \( f(x) \).
   (ii) Using only the definitions, i.e., using \( \epsilon \) and \( \delta \), show that
   \[
   \lim_{x \to 1^-} f(x) = 2, \quad \lim_{x \to 1^+} f(x) = 2.
   \]
   (iii) Is the function continuous at \( x = 1 \)?

3. Prove either one of the following:
   (i) Every sequence has a monotone subsequence.
   (ii) Suppose that \( f(y) \to l \) as \( y \to \eta \) and that \( g(x) \to \eta \), as \( x \to \xi \). Also assume that \( f \) is continuous at \( \eta \), i.e. \( l = f(\eta) \). Show that
   \[
   f(g(x)) \to l \quad \text{as} \quad x \to \xi.
   \]

4. Let \( f : (-1, \infty) \to (-\infty, 1) \) be defined by
   \[
   f(x) = \frac{x - 1}{x + 1}.
   \]
   (i) Show that \( f \) is invertible.
   (ii) Find a formula for \( f^{-1} \).
   (iii) Graph \( f \) and \( f^{-1} \).
   (iv) Find a formula for \( f \circ f \), and simplify it.

5. Consider the sequence \( x_n = \sin(n\pi/2), \) \( n = 1, 2, \ldots \). Compute \( x_1, x_2, x_3, \) and \( x_4 \). Explain why the sequence has no limit. *Hint:* If a sequence converges to \( l \), what do we know about its subsequences?
Midterm 2 Solutions

1. State:
   (i) The definition of a Cauchy sequence.
   Given $\epsilon > 0$, there exists $N$, such that
   \[ n > N \text{ and } m > N \implies |x_n - x_m| < \epsilon. \]
   (ii) The Bolzano-Weierstrass Theorem. (No proof is required).
   Every bounded sequence has a convergent subsequence.
   (iii) The definition of $\lim_{x \to a^+} f(x) = l$.
   Given $\epsilon > 0$, there exists $\delta > 0$, such that
   \[ a < x < a + \delta \implies |f(x) - l| < \epsilon. \]

2. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by
   \[ f(x) = \begin{cases} 
   3 - x & (x > 1) \\
   1 & (x = 1) \\
   2x & (x < 1) 
   \end{cases} \]
   (i) Draw a graph of $f(x)$.
   See Fig. 12

Figure 12: The square in the middle represents the point $f(1) = 1$ on the graph

(ii) Using only the definitions, i.e., using $\epsilon$ and $\delta$, show that
   \[ \lim_{x \to 1^-} f(x) = 2, \quad \lim_{x \to 1^+} f(x) = 2. \]
(a)  \( \lim_{x \to 1^-} f(x) = 2 \)

Given \( \epsilon > 0 \), one should find \( \delta > 0 \) such that

\[
1 - \delta < x < 1 \implies |f(x) - 2| < \epsilon
\]

We have for \( x < 1 \)

\[
|f(x) - 2| < \epsilon \iff 2x - 2 < \epsilon \iff 2|x - 1| < \epsilon \iff |x - 1| < \frac{\epsilon}{2}
\]

\[
\iff -\frac{\epsilon}{2} < x - 1 < \frac{\epsilon}{2} \iff 1 - \frac{\epsilon}{2} < x < 1 + \frac{\epsilon}{2}
\]

To match with \( 1 - \delta < x < 1 \) we choose \( \delta = \epsilon/2. \) Then \( 1 - \delta < x < 1 \) implies \( 1 - \epsilon/2 < x < 1 + \epsilon/2, \) which by the above inequalities implies \( |f(x) - 2| < \epsilon. \)

(b)  \( \lim_{x \to 1^+} f(x) = 2 \)

Given \( \epsilon > 0 \), one should find \( \delta > 0 \) such that

\[
1 < x < 1 + \delta \implies |f(x) - 2| < \epsilon
\]

We have for \( x > 1 \)

\[
|f(x) - 2| < \epsilon \iff 3 - x - 2 < \epsilon \iff |1 - x| < \epsilon \iff |x - 1| < \epsilon \iff -\epsilon < x - 1 < \epsilon \iff 1 - \epsilon < x < 1 + \epsilon
\]

To match with \( 1 < x < 1 + \delta \) we choose \( \delta = \epsilon. \) Then \( 1 < x < 1 + \delta \) implies \( 1 - \epsilon < x < 1 + \epsilon, \) which by the above inequalities implies \( |f(x) - 2| < \epsilon. \)

(iii) Is the function continuous at \( x = 1? \)

The function is not continuous at \( 1, \) since \( f(1) = 1, \) while

\[
\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = \lim_{x \to 1^-} f(x) = 2.
\]

3. Prove either one of the following:

(i) Every sequence has a monotone subsequence.

A point \( x_N \) is called a peak point for the sequence, if

\[
n > N \implies x_n \leq x_N.
\]

There are two possibilities. Either the sequence has infinitely many peak points, or finitely many peak points.

Case 1: There are infinitely many peak points, i.e. we can find a sequence of integers \( n_1, n_2, \ldots, \) which is increasing \( (n_1 < n_2 < \ldots) \) such that the points \( x_{n_1}, x_{n_2}, \ldots \) are all peak points.

Then \( x_{n_1} \geq x_{n_2}, \) as \( x_{n_1} \) is a peak point. Also, \( x_{n_2} \geq x_{n_3}, \) as \( x_{n_2} \) is a peak point. We continue this way to get

\[
x_{n_1} \geq x_{n_2} \geq x_{n_3} \geq \cdots
\]

i.e. we produced a subsequence which is decreasing.
Case 2: There are only finitely many peak points. Then we can find a largest subscript \( N \), such that \( x_N \) is the last peak point. Set \( n_1 = N + 1 \). Since \( x_{N+1} \) is not a peak point, we can find a subscript \( n_2 > n_1 \) with \( x_{n_2} > x_{n_1} \). Since \( x_{n_2} \) is not a peak point we can find a subscript \( n_3 > n_2 \) with \( x_{n_3} > x_{n_2} \). We continue this way to produce an increasing subsequence 
\[ x_{n_1} < x_{n_2} < x_{n_3} < \cdots. \]

(ii) Suppose that \( f(y) \to l \) as \( y \to \eta \) and that \( g(x) \to \eta \), as \( x \to \xi \). Also assume that \( f \) is continuous at \( \eta \), i.e. \( l = f(\eta) \). Show that 
\[ f(g(x)) \to l \quad \text{as} \quad x \to \xi. \]

Given \( \epsilon > 0 \) we must find \( \delta > 0 \) such that 
\[ 0 < |x - \xi| < \delta \implies |f(g(x)) - l| < \epsilon. \]

We know that \( f(y) \to f(\eta) \), as \( y \to \eta \). This means that for the given \( \epsilon > 0 \), we can find a \( \Delta > 0 \) such that 
\[ |y - \eta| < \Delta \implies |f(y) - f(\eta)| < \epsilon. \]  \hspace{1cm} (4)

Notice that the continuity of \( f \) at \( \eta \) allows to write \( |y - \eta| < \Delta \), rather than \( 0 < |y - \eta| < \Delta \).

We know that \( g(x) \to \eta \), as \( x \to \xi \). For the \( \Delta > 0 \) found above we can find a \( \delta > 0 \) such that 
\[ 0 < |x - \xi| < \delta \implies |g(x) - \eta| < \Delta. \]

We see that for \( 0 < |x - \xi| < \delta \) we can set \( y = g(x) \) in Eq. (4) to conclude: 
\[ |f(g(x)) - f(\eta)| < \epsilon. \]
Since \( l = f(\eta) \), this completes the proof.

4. Let \( f : (-1, \infty) \to (-\infty, 1) \) be defined by 
\[ f(x) = \frac{x - 1}{x + 1}. \]

(i) Show that \( f \) is invertible.

We prove that it is one-to-one.

\[ f(a) = f(b) \iff \frac{a - 1}{a + 1} = \frac{b - 1}{b + 1} \iff (a-1)(b+1) = (b-1)(a+1) \iff ab-b+a-1 = ab-a+b-1 \]
\[ \iff a - b = -a + b \iff 2a = 2b \iff a = b. \]

We prove it is onto: Given \( y \in (-\infty, 1) \) we have to find \( x \in (-1, \infty) \) with \( f(x) = y \).

We solve this equation:
\[ f(x) = y \iff \frac{x - 1}{x + 1} = y \iff x - 1 = xy + y \iff x - xy = y + 1 \iff x(1-y) = y + 1 \iff x = \frac{y + 1}{1 - y}. \]

If \( y < 1 \), then the denominator is positive. We need to show that this \( x \) found above is \( > -1 \). This amounts to 
\[ x > -1 \iff \frac{y + 1}{1 - y} > -1 \iff y + 1 > y - 1 \iff 1 > -1, \]
which is true.

(ii) Find a formula for \( f^{-1} \).
The above calculation gives

\[
f^{-1}(y) = \frac{y + 1}{1 - y} \text{ or } f^{-1}(x) = \frac{x + 1}{1 - x}.
\]

(iii) Graph \( f \) and \( f^{-1} \).
See Fig. 13

(iv) Find a formula for \( f \circ f \), and simplify it.

\[
(f \circ f)(x) = f(f(x)) = f((x-1)/(x+1)) = \frac{(x-1)/(x+1) - 1}{(x-1)/(x+1) + 1} = \frac{(x-1) - (x+1)}{(x-1) + (x+1)} = \frac{-2}{2x} = -\frac{1}{x}.
\]

5. Consider the sequence \( x_n = \sin(n\pi/2), \ n = 1, 2, \ldots \). Compute \( x_1, x_2, x_3, \) and \( x_4 \).
Explain why the sequence has no limit. \textit{Hint:} If a sequence converges to \( l \), what do we know about its subsequences?

\[
x_1 = \sin(\pi/2) = 1, \ x_2 = \sin(\pi) = 0, \ x_3 = \sin(3\pi/2) = -1, \ x_4 = \sin(2\pi) = 0
\]

and the same pattern repeats itself:

\[
x_{4n+1} = 1, \ x_{4n+2} = 0, \ x_{4n+3} = -1, \ x_{4n} = 0.
\]

In particular, the subsequences \( x_{4n+1} \) and \( x_{4n+3} \) are constant (resp. equal to 1 and \(-1\)) and have different limits (resp. equal to 1 and \(-1\)). But, if a sequence converges, say to \( l \), then all its subsequences converge to the same limit \( l \). So the sequence \( x_n \) does not converge.
Final Exam

1. State:
   (i) The intermediate value theorem.
   (ii) The sandwich (squeeze) theorem for sequences.
   (iii) The definition of $f'(\xi)$
   (iv) The definition of $x_n \to l$ as $n \to \infty$

2. Do one of the following:
   (a) Use the definition of the limit to show that $\lim_{n \to \infty} \left(1 + \frac{1}{n}\right) = 1$.
   (b) Use the definition of the limit to show that $\lim_{x \to 0^+} x^{1/2} = 0$.

3. Do one of the following:
   (a) For $x > 0$ compute the derivative $D\sqrt{x + \sqrt{x + \sqrt{x}}}$.
   (b) Suppose that $x > -1$ and $x \neq 0$. Prove by induction that, for any natural number $n \geq 2$, $(1 + x)^n > 1 + nx$.

4. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by
   \[ f(x) = \begin{cases} 
   x^{-1} & (x \geq 1) \\
   2 - x & (x < 1) 
   \end{cases} \]
   (i) Draw a graph of $f(x)$.
   (ii) Show that the function $f$ is continuous at $\xi = 1$.
   (iii) Show that the function is differentiable at $\xi = 1$ and compute $f'(1)$. Use the definition of the derivative.

5. State and prove the Mean Value Theorem.

6. Prove either of the following two theorems:
   (a) Let $f$ be continuous on the compact interval $[a, b]$. Then $f$ is bounded on $[a, b]$.
   (b) Let $f$ be continuous on the compact interval $[a, b]$. Then $f$ achieves a maximum value $d$ on $[a, b]$.

7. Let $f : [0, 1] \to [0, 1]$ be continuous on $[0, 1]$.
   (a) Prove that for some $\xi \in [0, 1]$ we have $f(\xi) = 1 - \xi$.
   (b) Assume, moreover, that $f$ is differentiable on $[0, 1]$ and that $f'(x) \neq -1$ for all $x$ in $[0, 1]$. Show that the $\xi$ you found in (a) is unique, i.e., there is exactly one $\xi \in [0, 1]$ with $f(\xi) = 1 - \xi$. 

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Final Exam: Solutions

1. State:
   (i) The intermediate value theorem.
   Let \( f \) be continuous on an interval containing \( a \) and \( b \). If \( \lambda \) lies between \( f(a) \) and \( f(b) \) we can find a \( \xi \) between \( a \) and \( b \) with \( f(\xi) = \lambda \). (9.10)
   (ii) The sandwich (squeeze) theorem for sequences.
   Suppose that \( y_n \to l \) as \( n \to \infty \) and \( z_n \to l \) as \( n \to \infty \). If \( y_n \leq x_n \leq z_n \), \( n = 1, 2, \ldots \) then \( x_n \to l \) as \( n \to \infty \). (4.10)
   (iii) The definition of \( f'(\xi) \)
   \[
   f'(\xi) = \lim_{x \to \xi} \frac{f(x) - f(\xi)}{x - \xi} \tag{10.1}
   
   (iv) The definition of \( x_n \to l \) as \( n \to \infty \)
   Given \( \epsilon > 0 \) we can find \( N \) such that
   \[
   n > N \implies |x_n - l| < \epsilon \quad (4.4)
   
2. Do one of the following:
   (a) Use the definition of the limit to show that \( \lim \left( 1 + \frac{1}{n} \right) = 1 \).
   Given \( \epsilon > 0 \) we must find \( N \) such that
   \[
   n > N \implies \left| 1 + \frac{1}{n} - 1 \right| < \epsilon
   
   \] i.e. \( 1/n < \epsilon \). This is equivalent to \( n > 1/\epsilon \). So it suffices to take \( N = 1/\epsilon \).
   (b) Use the definition of the limit to show that \( \lim_{x \to 0^+} x^{1/2} = 0 \).
   Given \( \epsilon > 0 \) we must find \( \delta > 0 \) such that
   \[
   0 < x < \delta \implies |x^{1/2} - 0| < \epsilon.
   
   But for positive \( x \)
   \[
   |x^{1/2} - 0| < \epsilon \iff x^{1/2} < \epsilon \iff x < \epsilon^2
   
   So it suffices to take \( \delta = \epsilon^2 \).
3. Do one of the following:
   (a) For \( x > 0 \) compute the derivative \( D\sqrt{x + \sqrt{x + \sqrt{x}}} \). Repeated use of the chain rule gives
   \[
   \left( \sqrt{x + \sqrt{x + \sqrt{x}}} \right)' = \frac{d}{dx} \left( x + (x + x^{1/2})^{1/2} \right)^{1/2} = \frac{1}{2} \left( x + (x + x^{1/2})^{1/2} \right)^{-1/2} \frac{d}{dx} \left( x + (x + x^{1/2})^{1/2} \right)
   
   \]
\[
\frac{1}{2} (x + (x + x^{1/2})^{1/2})^{-1/2} \left( 1 + \frac{1}{2} (x + x^{1/2})^{-1/2} \frac{d}{dx} (x + x^{1/2}) \right)
\]

\[
= \frac{1}{2} (x + (x + x^{1/2})^{1/2})^{-1/2} \left( 1 + \frac{1}{2} (x + x^{1/2})^{-1/2} \left( 1 + \frac{1}{2} x^{-1/2} \right) \right)
\]

(b) Suppose that \( x > -1 \) and \( x \neq 0 \). Prove by induction that, for any natural number \( n \geq 2 \), \((1+x)^n > 1+nx\).

\( P(2) \): \((1+x)^2 > 1+2x \), since \((1+x)^2 = 1+2x+x^2 > 1+2x \), as \( x^2 > 0 \) for \( x \neq 0 \).

Assume \( P(n) \): \((1+x)^n > 1+nx \). We try to prove \( P(n+1) \): \((1+x)^{n+1} > 1+(n+1)x \).

We have

\[(1+x)^{n+1} = (1+x)^n(1+x) > (1+nx)(1+x)\]

using the inductive hypothesis and the fact that \( 1+x > 0 \). Then

\[(1+x)^{n+1} > 1+nx+x+nx^2 > 1+nx+x = 1+(n+1)x,\]

as \( nx^2 > 0 \).

4. Let \( f : \mathbb{R} \to \mathbb{R} \) be defined by

\[ f(x) = \begin{cases} 
  x^{-1} & (x \geq 1) \\
  2 - x & (x < 1)
\end{cases} \]

(i) Draw a graph of \( f(x) \). (ii) Show that the function \( f \) is continuous at \( \xi = 1 \).

![Figure 14: The graph of \( f(x) \)](image)

\[ \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} x^{-1} = 1^{-1} = 1. \]

\[ \lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} 2 - x = 2 - 1 = 1. \]

Since

\[ \lim_{x \to 1^+} f(x) = \lim_{x \to 1^-} f(x) = f(1) = 1 \]
(iii) Show that the function is differentiable at $\xi = 1$ and compute $f'(1)$. Use the definition of the derivative.

\[
\lim_{x \to 1^+} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1^+} \frac{x^{-1} - 1}{x - 1} = \lim_{x \to 1^+} \frac{1}{x - 1} = -1.
\]

\[
\lim_{x \to 1^-} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1^-} \frac{2 - x - 1}{x - 1} = \lim_{x \to 1^-} \frac{1}{x - 1} = \lim_{x \to 1^-} -1 = -1.
\]

Since both limits agree the function is differentiable at 1 and $f'(1) = -1$.

5. State and prove the Mean Value Theorem.

(11.6) Let $f(x)$ be continuous on the closed interval $[a, b]$ and differentiable on the open interval $(a, b)$ (at least). Then there exist a $\xi \in (a, b)$ with

\[
f'(\xi) = \frac{f(b) - f(a)}{b - a}.
\]

The proof that follows is somehow similar to the one in Binmore. We consider the line that passes through $(a, f(a))$ and $(b, f(b))$. We find its equation using the point-slope formula

\[
y - f(a) = \frac{f(b) - f(a)}{b - a} (x - a) \Leftrightarrow y = f(a) + \frac{f(b) - f(a)}{b - a} (x - a).
\]

We now consider the vertical distance from the line to $f(x)$. This is given by the function

\[
g(x) = f(x) - \left( f(a) + \frac{f(b) - f(a)}{b - a} (x - a) \right).
\]
We apply Rolle’s theorem to this function. It is continuous on \([a, b]\), as \(f(x)\) is and the linear function is continuous. Also it is differentiable on \((a, b)\), as \(f(x)\) is and the same applies to the linear function. Moreover, by the choice of \(g(x)\) to measure the vertical distance from \(f(x)\) to the line passing through \((a, f(a))\) and \((b, f(b))\), we have
\[
g(a) = g(b) = 0.
\]
Moreover,
\[
g'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}.
\]
Rolle’s theorem produces a \(\xi \in (a, b)\) with
\[
g'(\xi) = 0 \iff f'(\xi) - \frac{f(b) - f(a)}{b - a} = 0 \iff f'(\xi) = \frac{f(b) - f(a)}{b - a}.
\]

6. Prove either of the following two theorems:

The proofs below are variations of (9.11) and (9.12) in Binmore.

(a) Let \(f\) be continuous on the compact interval \([a, b]\). Then \(f\) is bounded on \([a, b]\).

We will prove that \(f\) is bounded above. If it is not, given \(n\), we can find a number \(x_n \in [a, b]\) with \(f(x_n) > n\). These numbers form a sequence \(\langle x_n \rangle, \ n = 1, 2, \ldots\). This sequence is bounded, as all the terms are in \([a, b]\). By the Bolzano-Weierstrass theorem, it has a convergent subsequence, call it \(x_{n_r}, \ r = 1, 2, \ldots\). Call its limit \(\xi\). As \(a \leq x_{n_r} \leq b\), we also have \(a \leq \xi = \lim x_{n_r} \leq b\). By the continuity of \(f(x)\) we have
\[
f(\xi) = \lim f(x_{n_r}).
\]
On the other hand, \(f(x_{n_r}) > n_r \to \infty\). But this means that the sequence \(f(x_{n_r})\) is unbounded, while it converges, which means it is bounded. This is a contradiction. So \(f(x)\) is bounded above. For the bound below, we can use \(-f(x)\).

(b) Let \(f\) be continuous on the compact interval \([a, b]\). Then \(f\) achieves a maximum value \(d\) on \([a, b]\).

By (a) we know that \(f(x)\) is bounded above. By the continuum property of the real numbers,
\[
S = \{f(x); x \in [a, b]\}
\]
has a supremum, call it \(d\). We must find a \(\xi \in [a, b]\) with \(f(\xi) = d\). Consider \(d - 1/n < d\). Since \(d\) is the smallest upper bound of \(S\), \(d - 1/n\) is not an upper bound. So we can find \(x_n \in [a, b]\) with \(f(x_n) > d - 1/n\). This produces a sequence \(\langle x_n \rangle, n = 1, 2, \ldots\), with \(a \leq x_n \leq b\). This sequence is bounded, so it has a convergent subsequence \(x_{n_r}, \ r = 1, 2, \ldots\) by the Bolzano-Weierstrass theorem. Call its limit \(\xi\). As \(a \leq x_{n_r} \leq b\), we also have \(a \leq \xi = \lim x_{n_r} \leq b\). By the continuity of \(f(x)\)
\[
f(\xi) = \lim f(x_{n_r}).
\]
Now we have
\[
d \geq f(x_{n_r}) > d - \frac{1}{n_r},
\]
where \(1/n_r \to 0\). By the sandwich theorem \(d \geq f(\xi) \geq d\). So \(f(\xi) = d\) as required.
7. Let \( f : [0, 1] \to [0, 1] \) be continuous on \([0, 1]\).

(a) Prove that for some \( \xi \in [0, 1] \) we have \( f(\xi) = 1 - \xi \).

We use the intermediate value theorem for the function

\[ g(x) = f(x) - (1 - x) \]

on the interval \([0, 1]\). We compute

\[ g(0) = f(0) - (1 - 0) = f(0) - 1 \leq 0, \quad \text{as } f(x) \leq 1, \]
\[ g(1) = f(1) - (1 - 1) = f(0) \geq 0, \quad \text{as } f(x) \geq 0. \]

The function \( g(x) \) changes sign at the endpoints, so by the intermediate value theorem we can find a \( \xi \in [0, 1] \) with

\[ g(\xi) = 0 \iff f(\xi) - (1 - \xi) = 0 \iff f(\xi) = 1 - \xi. \]

(b) Assume, moreover, that \( f \) is differentiable on \([0, 1]\) and that \( f'(x) \neq -1 \) for all \( x \) in \([0, 1]\). Show that the \( \xi \) you found in (a) is unique, i.e., there is exactly one \( \xi \in [0, 1] \) with \( f(\xi) = 1 - \xi \).

Proof by contradiction: Assume there exist two distinct \( \xi_1 \) and \( \xi_2 \) with \( f(\xi_1) = 1 - \xi_1 \) and \( f(\xi_2) = 1 - \xi_2 \). We can label them in such a way that \( \xi_1 < \xi_2 \). Then \( g(\xi_1) = 0 = g(\xi_2) \). We notice that, since \( f \) is continuous and differentiable, the function \( g \) is continuous on the closed interval \([\xi_1, \xi_2]\) (in fact on the whole interval \([0, 1]\)) and differentiable on it as well with

\[ g'(x) = f'(x) - (-1) = f'(x) + 1. \]

We apply Rolle’s theorem on the interval \([\xi_1, \xi_2]\) for the function \( g \). We get the existence of a point \( \xi \) with \( g'(\xi) = 0 \). But

\[ g'(\xi) = 0 \iff f'(\xi) + 1 = 0 \iff f'(\xi) = -1. \]

This contradicts the assumption \( f'(x) \neq -1 \).