Unsolvability of the Weighted Region Shortest Path Problem☆

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Abstract

Let $\mathcal{S}$ be a subdivision of the plane into polygonal regions, where each region has an associated positive weight. The *weighted region shortest path problem* is to determine a shortest path in $\mathcal{S}$ between two points $s, t \in \mathcal{S}$, where the distances are measured according to the weighted Euclidean metric - the length of a path is defined to be the weighted sum of (Euclidean) lengths of the sub-paths within each region. We show that this problem cannot be solved in the *Algebraic Computation Model over the Rational Numbers* ($\mathsf{ACM}\mathbb{Q}$). The $\mathsf{ACM}\mathbb{Q}$ can compute exactly any number that can be obtained from the rationals $\mathbb{Q}$ by applying a finite number of operations from $+, -, \times, \div, \sqrt[k]{}$, for any integer $k \geq 2$. Our proof uses Gröbner bases and Galois theory, and is based on Bajaj's technique.

Keywords: computational geometry, weighted region shortest paths, Galois theory, unsolvability

1. Introduction

The weighted region shortest path problem is one of the classical path problems in Computational Geometry and has been studied over the last two decades. It was originally introduced in [2] as a generalization of the two-dimensional shortest path problem with obstacles. There are several well known approximation algorithms for this problem; see, e.g., [2–4]. In this paper, we show that determining the exact shortest path distance in this setting is an unsolvable problem in an algebraic model of computation, confirming the suspicion expressed in [2, Section 4]. Thus, we provide further justification for the search for approximate solutions as opposed to exact ones.

The algebraic complexity of geometric optimization problems was first studied by Bajaj, who showed that Euclidean shortest paths among polyhedral obstacles in three dimensions [5] and solutions to the Weber problem and its variations [6] cannot be expressed as finite algebraic expressions. More recently, the algebraic complexity of semi-definite programming [7] and shortest paths through certain cube complexes [8] were investigated. We employ Bajaj's technique [6] to show that the weighted region shortest path problem is unsolvable. In a nutshell, the technique is as follows.

As a consequence of the fundamental theorem of Galois [9], we know that there is no general formula to solve a polynomial equation of degree $d \geq 5$ *using radicals*. However, there are some polynomial equations of degree $d \geq 5$ that can be solved by radicals. The *Galois group* $\mathrm{Gal}(p)$ of an irreducible polynomial p over $\mathbb{Q}$ determines the solvability of p by radicals: A polynomial equation $p(x) = 0$ is solvable by radicals if and only if $\mathrm{Gal}(p)$ is *solvable* (refer to [9]).

We will present an instance of the weighted region shortest path problem such that solving this instance exactly within $\mathsf{ACM}\mathbb{Q}$ is equivalent to the statement that the polynomial equation $p_{12}(x) = 0$ in Equation (7) is solvable. However, we will show that the Galois group of p_{12} is S_{12} (i.e., the symmetric group over 12 elements) up to isomorphism. This is proved using the following theorem.[1]

Theorem 1 (Bajaj [6]). *Let p be a polynomial of even degree $d \geq 6$. Suppose that there are three prime numbers q_1, q_2 and q_3 that do not divide the discriminant $\Delta(p)$ of p, such that*

$$p(x) \equiv p_d(x) \pmod{q_1} , \tag{1}$$

$$p(x) \equiv p_1(x)p_{d-1}(x) \pmod{q_2} , \tag{2}$$

$$p(x) \equiv p_1'(x)p_2(x)p_{d-3}(x) \pmod{q_3} , \tag{3}$$

where $p_d(x)$ is an irreducible polynomial of degree d modulo q_1; $p_{d-1}(x)$ (respectively $p_1(x)$) is an irreducible polynomial of degree $d-1$ (respectively of degree 1) modulo q_2; $p_{d-3}(x)$ (respectively $p_1'(x)$ and $p_2(x)$) is an irreducible polynomial of degree $d-3$ (respectively of degree 1 and of degree 2) modulo q_3. Then $\mathrm{Gal}(p) \cong S_d$.

If $d \geq 5$ is odd, the same result holds if we replace (3) by

$$p(x) \equiv p_2(x)p_{d-2}(x) \pmod{q_4} ,$$

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[1] Alternatively, it can be verified using symbolic computation software. For example, GAP uses the algorithm from [10] to test the solvability of polynomials up to degree 15 via the command `isSolvable`, and MAGMA implements an extension of the algorithm in [11], that works for polynomials of arbitrary degree, limited only by time and space constraints.

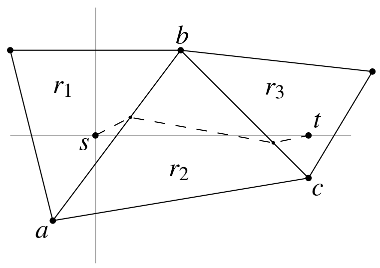

Figure 1: The weighted region shortest path problem with two bends.

where q_4 is a prime number such that $q_4 \nmid \Delta(p)$ and $p_{d-2}(x)$ (respectively $p_2(x)$) is an irreducible polynomial of degree $d-2$ (respectively of degree 2) modulo q_4.

Observe that (1) implies that $p(x)$ is irreducible over $\mathbb{Q}$, which implies that $\mathrm{Gal}(p)$ is a *transitive* group. (2) and (3) guarantee the existence of a $(d-1)$-cycle and an element with cycle decomposition $(2, d-3)$ in $\mathrm{Gal}(p)$. These two elements, together with the transitivity of $\mathrm{Gal}(p)$, imply that $\mathrm{Gal}(p) \cong S_d$.

Lemma 2 ([9, Chapter 4]). *A symmetric group S_n over n elements is solvable if and only if $n \leq 4$.*

If a problem is solvable with in ACM$\mathbb{Q}$, then we can express its solution as a finite sequence of the allowed operations on the rational input data. For practical applications however, we may need to rely on approximations of such an explicit representation, due to the occurrence of roots. The latter can hardly be avoided for the weighted region shortest path problem, as the length of a path is the weighted sum of Euclidean distances. A problem may be unsolvable in ACM$\mathbb{Q}$ even though its solution can be approximated with sufficient precision in practice. Nonetheless, we use the ACM$\mathbb{Q}$ as a viewpoint to gain insights about algebraic complexity and the applicability of symbolic computation. One advantage of the latter is the reusability of a result without cascaded approximation error. Unsolvability on the other hand concludes any search for a closed formula for solutions and provides further justification for the employment of approximation approaches.

2. Unsolvability

Consider the situation depicted in Fig. 1, where $s = (0, 0)$ is the source and $t = (5, 0)$ is the target. The three regions r_1, r_2 and r_3 have weights $w_1 = 1$, $w_2 = 2$ and $w_3 = 3$, respectively. Region r_2 is the triangle $\triangle abc$ where $a = (-1, -2)$, $b = (2, 2)$ and $c = (5, -1)$. The shortest path (identified by a dashed line) goes through the two edges ba and bc. The equations of the line segments ba and bc are

$$ba : (x, y) = (2, 2) + \lambda_1(-3, -4),$$
$$bc : (x, y) = (2, 2) + \lambda_2(3, -3),$$

where $\lambda_1, \lambda_2 \in [0, 1]$. The function to optimize is therefore

$$\phi(\lambda_1, \lambda_2) = \sqrt{25\lambda_1^2 - 28\lambda_1 + 8} + 2\sqrt{18\lambda_2^2 - 6\lambda_2\lambda_1 + 25\lambda_1^2} + 3\sqrt{18\lambda_2^2 - 30\lambda_2 + 13}.$$

The candidates (λ_1, λ_2) that minimize ϕ are $(0, 0)$, $(1, 1)$ and the solutions of the following system of equations

$$\frac{\partial}{\partial \lambda_1}\phi(\lambda_1, \lambda_2) = 0, \qquad \frac{\partial}{\partial \lambda_2}\phi(\lambda_1, \lambda_2) = 0, \tag{4}$$

where $\lambda_1, \lambda_2 \in [0, 1]$. Computing (4), we get

$$\frac{25\lambda_1 - 14}{\sqrt{25\lambda_1^2 - 28\lambda_1 + 8}} + \frac{50\lambda_1 - 6\lambda_2}{\sqrt{18\lambda_2^2 - 6\lambda_2\lambda_1 + 25\lambda_1^2}} = 0,$$
$$\frac{36\lambda_2 - 6\lambda_1}{\sqrt{18\lambda_2^2 - 6\lambda_2\lambda_1 + 25\lambda_1^2}} + \frac{54\lambda_2 - 45}{\sqrt{18\lambda_2^2 - 30\lambda_2 + 13}} = 0,$$

and this system of equations leads to the following one:

$$15100\lambda_1^2 - 52500\lambda_1^3 + 46875\lambda_1^4 - 3624\lambda_1\lambda_2 + 12600\lambda_1^2\lambda_2 -11250\lambda_1^3\lambda_2 - 3240\lambda_2^2 + 11592\lambda_1\lambda_2^2 - 10350\lambda_1^2\lambda_2^2 = 0, \tag{5}$$
$$5573\lambda_1^2 - 726\lambda_1\lambda_2 - 13380\lambda_1^2\lambda_2 + 2178\lambda_2^2 + 1800\lambda_1\lambda_2^2 +8028\lambda_1^2\lambda_2^2 - 5400\lambda_2^3 - 1080\lambda_1\lambda_2^3 + 3240\lambda_2^4 = 0. \tag{6}$$

The standard technique to isolate λ_1 and λ_2 in this system is to use Buchberger's algorithm [12] from Gröbner bases theory. Gröbner bases theory is a generalization of the ideas from linear algebra for solving a system of linear equations, namely bases and Gaussian elimination, to systems of multivariate polynomials. Just as the solution set to a system of linear equations forms a vector space, the solution set to a system of polynomials forms a *variety*. The goal is to find a set of polynomials whose solution set is the variety of the original polynomials, but which have good computational properties. This is analogous to finding a basis for the nullspace of a system of linear equations. A *Gröbner basis* is such a set, and can be found by running Buchberger's algorithm, which is a generalization of Gaussian elimination. The algorithm has different outcomes depending on the choice of monomial ordering used, i.e., ordering first by total degree of monomial versus degree of first variable. If a lexicographic ordering is used, then the Gröbner basis has the property that it contains a polynomial in only the last variable, a polynomial in only the last two variables, etc., and thus can be used to solve the original system of polynomial equations. This algorithm is implemented in many symbolic computation software such as Mathematica and Singular. Using the command

```
eqn1 = 15100*lambda1^2-52500*lambda1^3
+46875*lambda1^4-3624*lambda1*lambda2
+12600*lambda1^2*lambda2-11250*lambda1^3*lambda2
-3240*lambda2^2+11592*lambda1*lambda2^2
-10350*lambda1^2*lambda2^2;

eqn2 = 5573*lambda1^2-726*lambda1*lambda2
-13380*lambda1^2*lambda2+2178*lambda2^2
+1800*lambda1*lambda2^2+8028*lambda1^2*lambda2^2
-5400*lambda2^3-1080*lambda1*lambda2^3
+3240*lambda2^4;
```

```
GB = GroebnerBasis[{eqn1,eqn2},{lambda1,lambda2}];

p15 = GB[[1]]
p14 = GB[[2]]
pp14 = GB[[3]]
```

in Mathematica, we obtain the following system of three equations[2], which is equivalent to (5) and (6).

$$p_{15}(\lambda_2) = 0, \quad p_{14}(\lambda_1, \lambda_2) = 0, \quad p'_{14}(\lambda_1, \lambda_2) = 0,$$

where p_{15} is a polynomial of degree 15 in λ_2, and p_{14} and p'_{14} are polynomials of total degree 14 in λ_1 and λ_2; refer to the appendix for the coefficients of these polynomials. The polynomial p_{15} factors into $\lambda_2^3 \cdot p_{12}(\lambda_2)$, where p_{12} is the degree-12 polynomial

$$\begin{aligned} p_{12}(x) = &-745794461011616 \\ &+ 7579514247189376x \\ &- 35835761266110832x^2 \\ &+ 106594990715823828x^3 \\ &- 227453553923353605x^4 \\ &+ 372947142859928208x^5 \\ &- 484843926044252448x^6 \\ &+ 504065307805559136x^7 \\ &- 416814671916200304x^8 \\ &+ 267526211688938880x^9 \\ &- 125338196832117120x^{10} \\ &+ 37669520655360000x^{11} \\ &- 5350784184000000x^{12}. \end{aligned} \tag{7}$$

Theorem 3. *The weighted region shortest path problem cannot be solved exactly within* ACM$\mathbb{Q}$.

Proof. Following the notation of Theorem 1, and the above example, we have $p(x) = p_{12}(x)$, $d = 12$ and $\Delta(p) = 2^{117} \cdot 3^{144} \cdot 5^{49} \cdot 7^{16} \cdot 47 \cdot 53^2 \cdot 241 \cdot 5573^2 \cdot 9067 \cdot 13417^2 \cdot 35993^2 \cdot 5242801 \cdot q_{96}$, where q_{96} is a prime number with 96 digits. With $q_1 = 61$, $q_2 = 89$ and $q_3 = 139$, one finds

$$\begin{aligned} p(x) &\equiv 33 + 19x + 40x^2 + 15x^3 + 23x^4 + 33x^5 + 56x^6 \\ &\quad +39x^7 + 57x^8 + 26x^9 + 49x^{10} + 56x^{11} \\ &\quad +17x^{12} \pmod{61}, \\ p(x) &\equiv 38(x+32)(1 + 84x + 57x^2 + 16x^3 + 71x^4 \\ &\quad +86x^5 + 12x^6 + 54x^7 + 12x^8 + 22x^9 \\ &\quad +82x^{10} + x^{11}) \pmod{89}, \\ p(x) &\equiv 95(x+35)(58 + 41x + x^2)(61 + 62x + 42x^2 \\ &\quad +3x^3 + 106x^4 + 57x^5 + 28x^6 + 29x^7 \\ &\quad +106x^8 + x^9) \pmod{139}, \end{aligned}$$

so that $\mathrm{Gal}(p) \cong S_{12}$ by Theorem 1. Lemma 2 tells us that S_{12} is non-solvable. With numerical methods, one easily finds that the minimum of the function $\phi(\lambda_1, \lambda_2)$ for $\lambda_1, \lambda_2 \in [0,1]$ is at $(\lambda_1, \lambda_2) \approx (0.39398, 0.72450)$, which is a solution of the system of equations (5) and (6). Hence, if the minimum of $\phi(\lambda_1, \lambda_2)$ can be computed in ACM$\mathbb{Q}$, then we can solve the equation $p_{12}(\lambda_2) = 0$ in this model as well. However, we showed above that this equation cannot be solved within ACM$\mathbb{Q}$. Therefore, in general, the weighted region shortest path problem cannot be solved exactly within ACM$\mathbb{Q}$. □

Remark 1. Let $\mathcal{P}$ be a problem that can be translated into a (system of) polynomial equation(s), and assume that we want to use Theorem 1 to prove that $\mathcal{P}$ cannot be solved exactly within ACM$\mathbb{Q}$. In general, $\mathcal{P}$ admits infinitely many different instances leading to infinitely many different polynomial equations. Our experience shows that most of the time, Theorem 1 applies on the first instance of $\mathcal{P}$ we can think of. Otherwise, one can use a symbolic computation software as a black box and compute Gal(p). To use Theorem 1, we need to find three prime numbers that satisfy the constraining properties. Bajaj [6] explains why trying $d + 1$ prime numbers that do not divide $\Delta(p)$ will most likely be sufficient. As for the factorization of a polynomial modulo a prime number, refer to [13] for standard algorithms that perform this task.

3. Generalization of the Counterexample

We have shown that one instance of the weighted region shortest path problem is unsolvable, which shows this problem is unsolvable in general. However, it would be useful to know how often an instance of the weighted region shortest path problem is unsolvable. If we know the sequence of regions that the shortest path goes through, then we know that the path itself is made up of a sequence of line segments passing through the interiors of the prescribed regions and bending only on the boundaries of the regions. Furthermore, the shortest path is locally optimal between any two bendpoints. That is, if we treat the bendpoints u_i and u_{i+3} as fixed, then the intermediate bendpoints u_{i+1} and u_{i+2} must be optimal with respect to u_i and u_{i+3}. This implies that any instance of the weighted region shortest path problem in which the shortest path goes through at least three regions, will contain a generalization of the given counter-example. In particular, the equations to optimize will have the same form, but different coefficients. A polynomial of degree 5 or higher is unsolvable if its coefficients are algebraically independent, i. e., not related by an algebraic expression. This will be true in the general instance of the problem, except in very specific cases, and thus a generic instance of the weighted region shortest path problem in which the path passes through at least three regions is usually unsolvable.

4. Conclusions and Future Work

We demonstrated that even a toy example of the weighted region shortest path problem cannot be solved in ACM$\mathbb{Q}$. Thus, we cannot expect this to be the case for inputs of realistic size

[2] The coefficients in this system of equations involve large integers or fractions thereof. Among the many examples that we studied there was none involving only fractions of small integers.

or for instances of more general versions, such as the flow path problem [14] or the anisotropic shortest path problem [15]. The method we employed, Bajaj's technique, will be a useful tool-kit to prove similar unsolvability results and guide more realistic analysis of problems in computational geometry with algebraic components.

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Appendix

Here we present the missing polynomials from Section 2. In Section 2 we work with the following system of equations, refer to (5) and (6):

$$\begin{aligned} 15100\lambda_1^2 - 52500\lambda_1^3 + 46875\lambda_1^4 - 3624\lambda_1\lambda_2 + 12600\lambda_1^2\lambda_2 & \\ -11250\lambda_1^3\lambda_2 - 3240\lambda_2^2 + 11592\lambda_1\lambda_2^2 - 10350\lambda_1^2\lambda_2^2 &= 0, \\ 5573\lambda_1^2 - 726\lambda_1\lambda_2 - 13380\lambda_1^2\lambda_2 + 2178\lambda_2^2 + 1800\lambda_1\lambda_2^2 & \\ +8028\lambda_1^2\lambda_2^2 - 5400\lambda_2^3 - 1080\lambda_1\lambda_2^3 + 3240\lambda_2^4 &= 0. \end{aligned}$$

One can verify that
$c_{1,1}(\lambda_1,\lambda_2)p_{15}(\lambda_2) + c_{1,2}(\lambda_1,\lambda_2)p_{14}(\lambda_1,\lambda_2) + c_{1,3}(\lambda_1)p'_{14}(\lambda_1,\lambda_2) = 15100\lambda_1^2 - 52500\lambda_1^3 + 46875\lambda_1^4 - 3624\lambda_1\lambda_2 + 12600\lambda_1^2\lambda_2 - 11250\lambda_1^3\lambda_2 - 3240\lambda_2^2 + 11592\lambda_1\lambda_2^2 - 10350\lambda_1^2\lambda_2^2$

and

$c_{2,1}(\lambda_1,\lambda_2)p_{15}(\lambda_2) + c_{2,2}(\lambda_1,\lambda_2)p_{14}(\lambda_1,\lambda_2) + c_{2,3}p'_{14}(\lambda_1,\lambda_2) = 5573\lambda_1^2 - 726\lambda_1\lambda_2 - 13380\lambda_1^2\lambda_2 + 2178\lambda_2^2 + 1800\lambda_1\lambda_2^2 + 8028\lambda_1^2\lambda_2^2 - 5400\lambda_2^3 - 1080\lambda_1\lambda_2^3 + 3240\lambda_2^4$,

where $c_{1,1}$, $c_{1,2}$, $c_{1,3}$, $c_{2,1}$, $c_{2,2}$ and $c_{2,3}$ are the polynomials defined below. These polynomials act as coefficients in the Gröbner basis decomposition of (7) and (8).[3] For completeness we present the expressions for p_{15}, p_{14}, p'_{14}, $c_{1,1}$, $c_{1,2}$, $c_{1,3}$, $c_{2,1}$, $c_{2,2}$ and $c_{2,3}$ in the following. They were obtained using the command

```
coeff1 = PolynomialReduce[eqn1,GB,{lambda1,lambda2}];
coeff2 = PolynomialReduce[eqn2,GB,{lambda1,lambda2}];

c11 = coeff1[[1,1]]
c12 = coeff1[[1,2]]
c13 = coeff1[[1,3]]
c21 = coeff2[[1,1]]
c22 = coeff2[[1,2]]
c23 = coeff2[[1,3]]
```

in Mathematica.

$$\begin{aligned} p_{15}(\lambda_2) = -745794461011616\lambda_2^3 + 7579514247189376\lambda_2^4 \\ -35835761266110832\lambda_2^5 + 106594990715823828\lambda_2^6 \\ -227453553923353605\lambda_2^7 + 372947142859928208\lambda_2^8 \\ -484843926044252448\lambda_2^9 + 504065307805559136\lambda_2^{10} \\ -416814671916200304\lambda_2^{11} + 267526211688938880\lambda_2^{12} \\ -125338196832117120\lambda_2^{13} + 37669520655360000\lambda_2^{14} \\ -5350784184000000\lambda_2^{15} \ , \end{aligned}$$

$$\begin{aligned} p_{14}(\lambda_1,\lambda_2) = 105729774924783745419466675940843936580037645149375000\lambda_1\lambda_2 \\ +583318116370559329033532399612501188569438459993750000\lambda_2^2 \\ +3191528352522480397679021215391861367791043461691924375 5216\lambda_2^3 \\ -25201142969097288290723992266864491165166230146853609453 9878\lambda_2^4 \\ +92860220006629316065843813996778721286898973816482702422 9933\lambda_2^5 \\ -21966161872469769599337035409153905591581218286983455423 91864\lambda_2^6 \end{aligned}$$

[3]This gives an idea of what happens in the GroebnerBasis command of Mathematica we use as a black box. The motivated reader can then verify that it is indeed a Gröbner basis.

$$
\begin{aligned}
&+382029359593801708444983508921913043022697203649740149276364 8\lambda_2^7\\
&-514724990155175908386893607752215793927741938831874963701958 4\lambda_2^8\\
&+543051732912372604845087191965523509563481393240440208153713 6\lambda_2^9\\
&-447649163415269818665147736417615680161093876973555276039628 8\lambda_2^{10}\\
&+280809614209824171580423614629181268338257052648834875971008 0\lambda_2^{11}\\
&-122587068151462564191217187295276547260491026332963838239744 0\lambda_2^{12}\\
&+306749854236551468421323717467174706882079158592688624320000\lambda_2^{13}\\
&-279336516856898429681635850743256332958899563839326080000 00\lambda_2^{14}
\end{aligned}
$$

$$
\begin{aligned}
p'_{14}(\lambda_1,\lambda_2) = {}& 70146670911407120621748545835514673638160690049698437500\lambda_1^2\\
&+77829571698952984571659874085456423425325513463190625000\lambda_2^2\\
&+1967442819599289640882020241213004852423810166341983589713028 8\lambda_2^3\\
&-1734024226358219382382822455988804014002018443081734693417175 70\lambda_2^4\\
&+7108087449183899732796406430377948523682480935963849431783356 95\lambda_2^5\\
&-1848495940038094028760273377816610630800391315995664891469379 328\lambda_2^6\\
&+3490323643365282372310104591221221710426293218755520480244326 176\lambda_2^7\\
&-5092688129421818549868308151967715449042539879654775671186912 576\lambda_2^8\\
&+5857423633966761925194768993035993547574537163254572999092886 864\lambda_2^9\\
&-5313768168139984371216684362279682435709813721681196965720980 608\lambda_2^{10}\\
&+3744215513660392636633822124715545501508588463991283980043861 120\lambda_2^{11}\\
&-1941045179621232293942897106578581468007054103277286731011210 240\lambda_2^{12}\\
&+6495574627235958416332398784848976705676243399376911053507200 00\lambda_2^{13}\\
&-1026037024653801795735589836944400642632603599472790055680000 00\lambda_2^{14}
\end{aligned}
$$

$$
\begin{aligned}
c_{1,1}(\lambda_1,\lambda_2) \quad = \quad & -\frac{348084626316961436129137910942153157467444434097733879244}{206605129308830190284414955379989351464020644304423323771 09375}\\
&+\frac{593714604220850343167278987089337500210001375886247705 86\lambda_1}{826420517235320761137659821519957405856082577217693295084375}\\
&+\frac{3265341211484356879650125088466391587960613348585940119860 9\lambda_2}{5904668644298003335666895109398362336969164362607954786545156 25}\\
&+\frac{590535103948393492084700746749478260178616695751542861411864\lambda_1\lambda_2}{153521384751748086727339272844357420761198273427806824450174062 5}\\
&+\frac{399781462396058763828633526960882718604242460958040052646038782702261321845751574396308446931330907369187 6\lambda_2^2}{998387347890416338007736738048544907350686995120783880891102789230696705057664894091651424364879638671875}\\
&+\frac{4203283634696226352163700844869381701523649447368774251865417021792828158352045190290028221176709936237 31\lambda_1\lambda_2^2}{26623662610411102346872979681294530862684986536554236823762741046151912134871063842444037983063457031250}\\
&-\frac{1674001062610987555831491233169312133339816726438996613318848192432606283190917411423617990514384172818269 3\lambda_2^3}{798709878312333070406189390438835925880549596096627104712882231384557364046131915273321139491903710937500}\\
&-\frac{288601384259637521264639961031220706398358671729634599946131946645077432509082470954523809512840863950601\lambda_1\lambda_2^3}{2662366261041110234687297968129453086268498653655423682376274104615191213487106384244403798306345703125}\\
&+\frac{526908716520993534616434192550884300175691110437058607860235102881312828163976845488463094765317173222440 98\lambda_2^4}{998387347890416338007736738048544907350686995120783880891102789230696705057664894091651424364879638671875}\\
&+\frac{461460785499147999022922533136202044876597186063131553974254295653128950455084907102566401502705577334904 4\lambda_1\lambda_2^4}{133118313052055511734364898406472654313424932682771184118813705230759560674355319212220189915317285156 25}\\
&-\frac{204445849220909586318663349274528332088975763635054739458027159488708931650370669370462092510945236557235 6\lambda_2^5}{221863855086759186223941497344121090522374887804618640198022842051265934457258865353700316525528808593 75}\\
&-\frac{642678436965082938453129857831883317196333619304503989932564055222982494466466288947855682885806552725104\lambda_1\lambda_2^5}{88745542034703674489576598937648436208949955121847456079209136820506373782903546141480126610211523437 5}\\
&+\frac{135049930972641405134451332571442519263511278573001922099932685601243404307848109452176944799199225549456 72\lambda_2^6}{110931927543379593111970748672060545261187443902309320099011421025632967228629432676850158262764404296875}\\
&+\frac{796104779949943243732998610777099924997865691512355409024251784868764340517929500263004712211185998974 96\lambda_1\lambda_2^6}{704329698688124400710925388394035208007539326363868699041342355718304553832567826519683544525488281 25}\\
&-\frac{880350149468811060134398498015496747305796478892793960926002106382986709885250467313197663226050218990916\lambda_2^7}{739546183622530620746471657813736968407916292682062133993409473504219781524196217845667721751762695312 5}\\
&-\frac{133894928486758497497027609236593217541892441545464624430374117862518232853394541512658824606776262546448\lambda_1\lambda_2^7}{986061578163374160995295543751649291210555056909416178657879298005626375365594957127556962335683593 75}\\
&+\frac{107201679142294886193287061919996740878358906772694586795284036136426308952589618913148748169680104982777 6\lambda_2^8}{123257697270421770124411942968956161401319382113677022332234912250703296920699369640944620291960449218 75}\\
&+\frac{206301342303169290012829562820334333878285470206371670043111037403102139570619389751957954234718968561728\lambda_1\lambda_2^8}{164343596360562360165882590625274881868425842818236029776313216334271062560932492854592827055947265625}\\
&-\frac{733318939714866727463321034994481031696660122715812549896904926199605111402319898901945657909820505052 8\lambda_2^9}{164343596360562360165882590625274881868425842818236029776313216334271062560932492854592827055947265625}
\end{aligned}
$$

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\begin{aligned}
& - \frac{5842252857136128637028464493144343976481081355463060545575964252413707382692256446752412590564549293952\lambda_1\lambda_2^9}{657374385442249440663530362501099527473703371272944119105252865337084250243729971418371308223789 0625} \\
& + \frac{1190539981458930454133484186483856316642707878214973052110299774570670159561822511875053101622834611456\lambda_2^{10}}{1173882831146874001184875647323392013345898877273114498402237259530507589720946377532805907542480 46875} \\
& + \frac{1474176285370751436351327905452873031335930673953752372925667773848225207537086374209387989125865588812 8\lambda_1\lambda_2^{10}}{3286871927211247203317651812505497637368516856364720595526264326685421251218649857091856541118945 3125} \\
& + \frac{1025821990111637770939124504934451361725439790421205131784186782889410137175358739774701250366854899 2\lambda_2^{11}}{657374385442249440663530362501099527473703371272944119105252865337084250243729971418371308223789 0625} \\
& - \frac{4896159132430043434253506750381881094145217744008575292187465230695488465835247802040597068779076096\lambda_1\lambda_2^{11}}{3756425059669996803791602071434854442706876407273966394887159230497624287107028408104978904135937 5} \\
& - \frac{1096566642606616450621723878078722318839514722840732559074647410116484671004291354222239705886545 92\lambda_2^{12}}{2629497541768997762654121450004398109894813485091776476421011461348337000974919885673485232895156 25} \\
& + \frac{1362561779075167939907254875746005426466681908878950541702384762191111978586673246670243058766262272\lambda_1\lambda_2^{12}}{1051799016707599105061648580001759243957925394036710590568404584539334800389967954269394093158062 5}
\end{aligned}
$$

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\begin{aligned}
c_{1,2}(\lambda_1, \lambda_2) \quad = \quad & - \frac{4}{29174882705514278537380429343499982500010387734375} \\
& + \frac{4\lambda_1}{839125197815744011265608539213047115714584485312 5} \\
& - \frac{\lambda_1^2}{23495505538840832315437039097965319240008365588 75} \\
& + \frac{56\lambda_2}{45252823167920480607538174793275040883179377600781 25} \\
& - \frac{2\lambda_1\lambda_2}{181011292671681922430152699173100163532717510403125} \\
& - \frac{128941145237124769199859635305749680305333783282444647652 4\lambda_2^2}{13780553877768034589956200058641870934735979725334840796851616850897827507041710639093036817138671875} \\
& - \frac{881208008001617589176619470404635095586789855794426615536\lambda_1\lambda_2^2}{23886293388131259955924080101645909620209031523913724047876135874889567678872298441094597149707031 25} \\
& + \frac{100104910199735108377228017796118728885162299181846238444362784\lambda_2^3}{142626763984345191143962391149792129621526713588683411555871827032956672151094984870235917766411376953125} \\
& + \frac{4485216437101899963363298564354284573434952446828769011624398 9\lambda_1\lambda_2^3}{13311831305205551173436489840647265431342493268277118411881370523075956067435531921222018991531728515625} \\
& - \frac{4876581191898522706758849950335321699414112438875117370593688011\lambda_2^4}{1996774695780832676015473476097089814701373990241567761782205578461393410115329788183302848729759277343750} \\
& - \frac{3789078079071961015468835913900450326071116050977094616386217 83\lambda_1\lambda_2^4}{266236626104111023468729796812945308626849865365542368237627410461519121348710638424440379830634570312 50} \\
& + \frac{7252727452524780707699583365792102684721522980755676523452031787\lambda_2^5}{13311831305205551173436489840647265431342493268277118411881370523075956067435531921222018991531728515625 00} \\
& + \frac{3812403038680002355434839117830971313145058078123175147804 99\lambda_1\lambda_2^5}{100618528384017771530132198342005029715362760909124099863048907959757793404652546645669077789355468 75} \\
& - \frac{9916289656602528391916096342461566847434233572187324856027722 32\lambda_2^6}{1109319275433795931119707486720605452611874439023093200990114210256329672286294326768501582627644042968 75} \\
& - \frac{3592266072208771859658644221528651922267284455022905921986453 2\lambda_1\lambda_2^6}{493030789081687080497647771875824645605277528454708089328939649002813187682797478563778481167841796875} \\
& + \frac{787531140259867964361125726853524749447462881179154717682552 4\lambda_2^7}{69768507888917983089289779039031789472444933271892654150321648443794319011716624325062992618090820312 5} \\
& + \frac{3349783655602494576837723408384827145283373594241447543506 24\lambda_1\lambda_2^7}{310082257284079924841287906840141286544199703430634018445873993083530306718740552555835522747070312 5} \\
& - \frac{13527704894733370973118137392866388833491108764549639954543489 6\lambda_2^8}{12325769727042177012441194296895616140131938211367702233223491225070329692069936964094462029196044921875} \\
& - \frac{2074377423001307479983383256742891132207217268146015220416008 8\lambda_1\lambda_2^8}{164343596360562360165882590625274881868425842818236029776313216334271062560932492854592827055947265625} \\
& + \frac{333498872154389482086012223323073766121178147419113905210449 16\lambda_2^9}{410858990901405900414706476563187204671064607045590074440783040835677656402331232136482067639868164062 5} \\
& + \frac{191161441129827238839476815443061684941198621195519537353449 44\lambda_1\lambda_2^9}{164343596360562360165882590625274881868425842818236029776313216334271062560932492854592827055947265625} \\
& - \frac{3539243555147295348605333510971745669224656078127544019050592\lambda_2^{10}}{821717981802811800829412953126374409342129214091180148881566081671355312804662464272964135279736328125} \\
& - \frac{2739744140033244093067587044676159080444195885460004075585728\lambda_1\lambda_2^{10}}{328687192721124720331765181250549763736851685636472059552626432668542125121864985709185654111894531 25} \\
& + \frac{1066477863794275970535511864661975185928974135578308898621216\lambda_2^{11}}{821717981802811800829412953126374409342129214091180148881566081671355312804662464272964135279736328125} \\
& + \frac{1450731338524741728053771720851960811999518725478845225078144\lambda_1\lambda_2^{11}}{328687192721124720331765181250549763736851685636472059552626432668542125121864985709185654111894531 25} \\
& - \frac{1047272019835227068841754063531047164396511254536016355 84\lambda_2^{12}}{65737438544224944066353036250109952747370337127294411910525286533708425024372997141837130822378906 25} \\
& - \frac{3993351236023846747132261855560655997799244966188439693056\lambda_1\lambda_2^{12}}{262949754176899776265412145000439810989481348509177647642101146134833700097491988567348523289515625} \\
& - \frac{210050999202776149064205930110277659269034604537048764 16\lambda_2^{13}}{262949754176899776265412145000439810989481348509177647642101146134833700097491988567348523289515625} \\
& + \frac{2610032551144506583257731259020843680802368753069028505 6\lambda_1\lambda_2^{13}}{105179901670759910506164858000175924395792539403671059056840458453933480038996795426939409315806 25}
\end{aligned}
$$

$$
\begin{aligned}
c_{1,3}(\lambda_1) \quad = \quad & \frac{4}{46454748947951735511091752208950114992159397383906 25} \\
& - \frac{4\lambda_1}{133612706497918324993806753972408902167925123904187 5}
\end{aligned}
$$

$$+\frac{\lambda_1^2}{3741155781941713099826589111227449260701903469317\,25}$$

$$
\begin{aligned}
c_{2,1}(\lambda_1,\lambda_2) \quad = \quad & -\frac{173749420341951244147801077050932338202138500426745437 9}{267745945831568123462068546757168647350546122770251730468 75} \\
& +\frac{6984999706624872891977374662406622929851427856\lambda_1}{4894896987258506732382716478742774841668409497656 25} \\
& -\frac{32479916934040638099306914256055206306083126001054707739983 7\lambda_2}{149214815611932915205410801107770087168459354219861289390234375} \\
& -\frac{1085744746914220343781840463774779936244754729459714233820573699969814837087116039196103065284322039300 59\lambda_2^2}{1990524449010938329660340008470492468350752626992810337323011322907463973239358203424549762475585937500} \\
& +\frac{17212451196303986486160686873979101114233474369906862771258241272285950747385324425092745610681420306289 89\lambda_2^3}{5175363567428439657116884022023280417711956830181306877039829439559406330422331328903829382436523437500} \\
& -\frac{6292194617618569677544559110190839285533798574031772383201479160285125289909172330229004225762763163162054\lambda_2^4}{646920445928554957139610502752910052213994603772663359629978679944925791302791416112978672804565 4296875} \\
& +\frac{2119350754377374074362989770091706918141045103334553071210674422944095600091748456476223412508707584217 6\lambda_2^5}{1105846916117187960922411115816940260194862570551561298512784068281924429577421224124749868041992187 5} \\
& -\frac{2048900347584421139704921500496758707041163976669162703902520784546477432981126524452485626956217742528956\lambda_2^6}{7188004954761721745995672252810111691266606708585148440333096443832508792253237956810874142272949218 75} \\
& +\frac{1711984786055084382417285239675644718815228240136329493242676318487494192479483853051609602541319436348\lambda_2^7}{5265937695796133147249576741985429810451726526436006183394209848961544902749624876784523181152343 75} \\
& -\frac{6849220810027268844100708795649711303762432455070894576459404212702788111492975381122265008193370720517 44\lambda_2^8}{2396001651587240581998557417603370563755535569528382813444365481277502930751079318936958047424316406 25} \\
& +\frac{1836204521035133524204821089977640251344407813968849068151979062233858274305022278035726301905157609129 6\lambda_2^9}{958400660634896232799422967041348225502214227811353125377746192511001172300431727574783218969726562 5} \\
& -\frac{4256228437617481829169448876769837346351468723601290866909455852925066473911298699787807743715911760294 4\lambda_2^{10}}{4792003303174481163997114835206741127511071139056765626888730962555005861502158637873916094848632812 5} \\
& +\frac{8797080365616773073144986317508749035741783058187089102310083430374922185607804052228124617258787545 6\lambda_2^{11}}{3833602642539584931197691868165392902008856911245412501510984770044004689201726910299132875878906 25} \\
& -\frac{3242137440411836018743043509085400945798625751860539772325972847856430808353524492090257188346562 56\lambda_2^{12}}{1533441057015833972479076747266157160803542764498165000604393908017601875680690764119653150351562 5}
\end{aligned}
$$

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\begin{aligned}
c_{2,2}(\lambda_1,\lambda_2) \quad = \quad & -\frac{121}{4894896987258506732382716478742774841668409497656 25} \\
& -\frac{446\lambda_1}{978979397451701346476543295748554968333681899531 25} \\
& +\frac{989592\lambda_2}{384347311439537948626690897910882680567803513755968 75} \\
& +\frac{1338\lambda_1\lambda_2}{4894896987258506732382716478742774841668409497656 25} \\
& +\frac{14794748936587908136221025221789956964014910528529350270 76\lambda_2^2}{1160811853451561021244590889562013371997119332088037609240945056423696018845848584448194281005859375} \\
& -\frac{1061048524566622078712196031209350648031187304116279289489902 1\lambda_2^3}{9952622245054691648301700042352462341753763134964051686615056614537319866196791017122748812377929687 50} \\
& +\frac{10788307085719461602681116078355038278838750955715330196894855 81\lambda_2^4}{2587681783714219828558442011011640208855978415090653438519914719779703165211165664451914691218261718 7500} \\
& -\frac{8993929770619580635610760536006763450609866292778305198009590 51\lambda_2^5}{8625605945714066095194806703372134029519928050302178128399715732599010550703885548173048970727539062 500} \\
& +\frac{105444061645863500354965488853128280325479961133154287087053 72\lambda_2^6}{5529234580585939804612055579084701300974312852757806492563920341409622147887106120623749340209960937 5} \\
& -\frac{12216006663489086761221975470838541655035504100166928086495 52\lambda_2^7}{4520757833183472796223693240761076535387802961374307195178048077882081001417130790447090655517578125} \\
& +\frac{55592732660413845526237051904924613699727089374517959548563 48\lambda_2^8}{1843078193528646601537351859694900433658104284252602164187973447136540715962368706874583113403320312 5} \\
& -\frac{90791749361021531695109271636113777876894207694414523920203 16\lambda_2^9}{3422859502267486545712224882290529376793622242183404019206236401825004186787256169909940067749023437 5} \\
& +\frac{123546955586293484529724021197786476890907990308902950728339 2\lambda_2^{10}}{6845719004534973091424449764581058753587244484366808038412472803650008373574512339819880135498046875} \\
& -\frac{42798645812438526660929846193711212178584824616081567997453 12\lambda_2^{11}}{4792003303174481163997114835206741127511071139056765626888730962555005861502158637873916094848632812 5} \\
& +\frac{31287763280700793162093870893470695453402928391915167616\lambda_2^{12}}{1117668408903669076150930573809152449565264405610907434842852702636736060991757116705286552734375} \\
& -\frac{62104224444801488868521385424448631322890416289068508288\lambda_2^{13}}{1533441057015833972479076747266157160803542764498165000604393908017601875680690764119653150351562 5}
\end{aligned}
$$

$$c_{2,3} \quad = \quad \frac{1}{349635499089893338027336891338769631547743535546875}.$$