Approximating the Maximum Rectilinear Crossing Number

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1. INTRODUCTION

The problem of drawing a graph in the plane with a minimum number of edge crossings—called the *crossing number* of a graph—is a well-studied problem which dates back to the first half of the twentieth century, as mentioned in [11], and was formulated in full generality in [3]. It was shown that this problem is NP-Complete [4], and that it remains so even when restricted to cubic graphs [5]. Many variants have been studied, differing in both the methods of counting edge crossings and the types of drawings admitted in the problem domain [9, 8, 10].

Recently there has been interest in characterizing both the minimum and *maximum* number of edge crossings possible in particular graph classes for various edge crossing variants (e.g., [7, 12, 6])—in particular, the *maximum rectilinear crossing number* (MRCN) of a graph G, denoted by $\overline{\mathrm{CR}}(G)$ (e.g., [2]). There exist multiple terminologies and notations for this problem (as in, e.g., [12, 6]) but we adopt the conventions of [2].

In this paper we investigate the corresponding algorithmic problem, where, given a graph G we seek a straight-line drawing maximizing the number of edge crossings. We prove that the problem is NP-hard. Then we present an efficient derandomization of a known randomized 1/3-approximation algorithm [12], extended to a more general edge-weighted setting.

We note the distinction between $\overline{\mathrm{CR}}(G)$ and $\overline{\mathrm{CR}}^\circ(G)$, where the latter denotes the maximum crossing number taken only over *convex* straight-line drawings. But since all convex drawings which preserve vertex ordering also preserve edge crossings, we will only examine the simple convex drawing where the vertices of G are placed along the circumference of a circle of arbitrary size. In fact the statement that $\overline{\mathrm{CR}}(G) = \overline{\mathrm{CR}}^\circ(G)$ is an open conjecture, first posed in [2].

A number of approximation results are known for MRCN in the special cases where G is k-colorable, for $k \in \{2, 3, 4\}$, or where G is a triangulation [6]. In [12] a related optimization problem is considered, motivated by the "Planarity Game", in which, given a drawing of a planar graph with many edge crossings, the player tries to rearrange the vertices in order to eliminate all edge crossings. It is shown in that paper that the problem of finding a drawing that is maximally difficult for the game player—in the sense of requiring the largest number of single-vertex moves to remove all edge crossings—is NP-Hard.

2. PROBLEM AND HARDNESS

We now formally define the problem and state our results. Let $G = (V, E)$ be an unweighted simple graph with $|V| = n$. Let $D(G)$ denote the set of all possible straight-line drawings of G. Then $D(V)$ and $D(E)$ denote the sets of vertices and edges of G respectively, taken in their relative positions in D. For $D \in D(G)$, let $\mathrm{cross}(D)$ denote the number of edge-crossings in D. The goal in the MRCN problem is to find $\overline{\mathrm{CR}}(G) := \max_{D\in D(G)} \mathrm{cross}(D)$ as well as the corresponding D which maximizes this quantity.

We also define the weighted version of MRCN. Let $w : E \to \mathbb{R}$ be an assignment of weights to each edge in G. For a drawing $D \in D(G)$, denote the crossing of $e, f \in D(E)$ by $e \oslash f$. We define the *crossing-cost* function $c : D(E) \times D(E) \to \mathbb{R}$ as $c(e, f) := w(e)w(f)$ if $e \oslash f$ and otherwise $c(e, f) := 0$. For $D \in D(G)$, define its *cost* to be $\mathrm{cost}(D) := \sum_{e,f\in D(E)} c(e, f)$. Then we want to find $\overline{\mathrm{CR}}_w(G) := \max_{D\in D(G)} \mathrm{cost}(D)$ as well as the maximizing D. Note that this cost definition is consistent with the graph G being complete, where "missing" edges have weight 0.

Addtionally, we will examine the relationship of $\overline{\mathrm{CR}}(G)$ to $\overline{\mathrm{CR}}^\circ(G) := \max_{D\in D^\circ(G)} \mathrm{cross}(D)$ where $D^\circ(G)$ denotes the set of all possible *convex* straight-line drawings of G.

We now prove the hardness of this problem in the unweighed setting.

In the decision version of the problem, we ask if there exists a straight-line drawing $D \in D(G)$ with $\mathrm{cross}(D) \geq k$ for some nonnegative integer k.

It is clear that in the setting restricted to convex drawings, i.e., for $\overline{\mathrm{CR}}^\circ(G)$, the problem lies in NP. This is because in this case we can actually limit our attention to *circle drawings*, which are drawings of G in which all vertices lie on the circumference of a unit circle. This restriction is without loss of generality since clearly any convex drawings which preserve relative vertex ordering also preserve edge crossings.

In the more general setting where any drawing is permitted, however, it is not clear whether the problem is even in NP, since conceivably the coordinates of a "bad" drawing could be arbitrarily large. This issue would be dissolved if the conjecture $\overline{\mathrm{CR}}(G) = \overline{\mathrm{CR}}^\circ(G)$ were proven. The crossing number minimization problem *is* known to be in NP [4], but unfortunately the techniques used to prove this do not seem to work for the maximization problem. (Briefly, they guess the k crossings, insert new vertices at the cross points, and test this new graph G' for planarity. Unfortunately, examples can be constructed in which G' is planar and yet $\mathrm{Cr}(G) < k$, where $\mathrm{Cr}(G)$ denotes the crossing number of G.) Therefore we prove only that the problem is NP-hard.

THEOREM 1. *MRCN is NP-hard, both under the requirement that drawings be convex and in the general case.*

PROOF. (Sketch) We reduce from Max Cut.

Given the Max Cut instance G, add to it (say) n^3 new disjoint edges M on $2n^3$ new vertices to obtain MRCN instance G'. Now consider (near-)optimal solutions to this instance. The $2n^3$ new vertices should be drawn in such a way that all $\binom{n^3}{2}$ pairs cross, say, near-vertically.

Now consider the placement of the n original vertices among the edges M. First we argue that without loss of generality we may assume these vertices are placed above all intersections of M and below the upper endpoints of the edges M. Therefore the n vertices of V can be pictured as placed within n^3+1 columns, with the edges M as column separators. We then argue that any solution in which some vertices lie within "interior" columns (number 2 through n^3-1) can be improved by moving each of these vertices to one of the two "exterior" columns (number 1 or n^3+1). The locations of the vertices of G in the first or last columns then corresponding to a cut of G.

Finally, note that the resulting drawing can be made convex by stretching the edges as needed. $\square$

3. ALGORITHMS

In the special case of convex drawings (or under the assumption that the conjecture $\overline{\mathrm{CR}}(G) = \overline{\mathrm{CR}}^{\circ}(G)$ is true), an optimal drawing can be found by brute-force enumeration, though absent this case it is not even clear how to solve the problem optimally by brute force; either way, the problem is NP-hard. Therefore we turn to approximation algorithms.

DEFINITION 1. *Let $[n] = \{1, 2, ..., n\}$. Let $S_n = \{\sigma : [n] \to [n]\}$ be the set of all permutations of $[n]$.*

The following natural algorithm appeared in [12], where it was shown that the algorithm provides a 1/3-approximation guarantee:

Algorithm 1 Randomized

1: Choose a uniform-random permutation $\sigma \in S_n$
2: In order of $\sigma(\cdot)$, place the vertices clockwise on a circle centered at the origin, i.e., $v_{\sigma(1)}, v_{\sigma(2)}, ..., v_{\sigma(n)}$

We remark that the analysis is tight, as there exists an infinite family of instances on which the approximation factor 1/3 is achieved. Consider, for example, graphs consisting of n vertices and only two disjoint edges, on vertices, say, (a, b) and (b, c). Then it can easily be verified that exactly one third of all possible permutations achieve the one possible edge crossing. We note that the guarantee also holds in the weighted setting for MRCN. We also note that it holds in both convex and non-convex settings since the solution constructed is convex, and yet the comparison guarantee is with respect to any possible drawing.

In [12] it was stated without details that the above algorithm can be derandomized via the method of conditional expectations (see, e.g., [1], Chapter 15). However, it turns out that proving this claim is nontrivial, and we now show how to accomplish the derandomization.

THEOREM 2. *There exists a deterministic 1/3-approximation for (weighted) MRCN.*

PROOF. (Sketch) We derandomize the above algorithm by the method of conditional expectations. We place each vertex on the unit circle sequentially, in each round choosing a location maximizing the conditional expectation of the solution value. Placement of a vertex v in round i of the algorithm is represented by an angle θ_i. Thus each placement at stage $i > 2$ will satisfy $\theta_i \in (\theta_h, \theta_{h+1})$ where θ_h, θ_{h+1} represent the locations of two previously placed vertices which are adjacent on the circle.

The main ingredient of the proof is to partition the set of edges in G into various groups at each stage i, depending on the vertex v_i being placed currently. We then calculate the number of crossings between each group in polynomial time for each possible placement region (θ_h, θ_{h+1}). Thus we can determine the placement maximizing the conditional expectation and maintain the required expected value. $\square$

Finally, we note two other natural algorithms we have not yet analyzed:

- Greedy: Construct a drawing by placing vertices iteratively, each time placing the new vertex in a location maximizing the number of new edge crossings.
- k-Local search: Given an arbitrary drawing on the unit circle, repeatedly make moves repositioning k of the vertices in order to increase the number of edge crossings.

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