Square Line-of-Sight Blocking*

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1. INTRODUCTION

We consider the following coverage problem: given a collection of n axis-parallel rectangles lying within a unit square, choose a minimum-cardinality subset of rectangles whose *projections* in the x and y dimensions cover the corresponding sides of the square. That is, projecting a rectangle (downward) along the y dimension produces an interval on the x axis; in a feasible solution the union of these intervals must equal $[0, 1]$. Similarly, projecting the rectangles (leftward) along the x dimension yields intervals on the y axis, which must jointly cover $[0, 1]$. Intuitively, the chosen set of rectangles block all axis parallel lines of sight through the unit square. We call this problem SQUARE-LOS-COVER. This problem is known to admit a simple 2-approximation which solves two corresponding 1-dimensional subproblems and takes the union.[1] It was previously not known, however, whether the problem was hard. In this abstract we show that the problem is indeed NP-hard.

We also consider a natural 3D generalization of the problem where we are given a collection of cuboids lying within the unit cube, and we seek a minimum-cardinality subset of them whose projections in the x, y, and z dimensions fully cover the corresponding (2D) faces of the cube (CUBE-LOS-COVER). The 2D algorithm above suggests a natural strategy for trying to obtain a 3-approximation for this variation: optimally solve the three 2D subproblems and take their union. Unfortunately, however, we show that the 2D subproblem (find a minimum collection of rectangles lying within a unit square covering the square's *area*) is also hard.

2. HARDNESS OF SQUARE-LOS-COVER

We show the problem is NP-hard by reduction from 3-Dimensional Matching (3DM), first performing the reduction of [2] from 3DM to the *Hypergraph Assignment* problem (HA), and then transforming the resulting HA instance into a SQ.-LOS-COVER instance. A 3DM instance is specified by disjoint sets X, Y, and Z, with $|X| = |Y| = |Z| = n$, and a subset of legal triples $T \subseteq X \times Y \times Z$, with $|T| = m$. An HA problem instance is specified by a *bipartite graph* (see Fig. 2(a)) with two disjoint sets of vertices V_1, V_2 with $|V_1| = |V_2|$ and a set of hyperedges $E \subseteq \{P(V_1) \setminus \varnothing\} \times \{P(V_2) \setminus \varnothing\}$. The decision problem asks whether there exists a *perfect matching*, i.e., a partition of $V_1 \cup V_2$ into a disjoint set $E' \subseteq E$.

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[1]The unit-weight 1D subproblem is solvable by a simple greedy algorithm; for weighted, it is an instance of shortest path on a weighted interval graph [1].

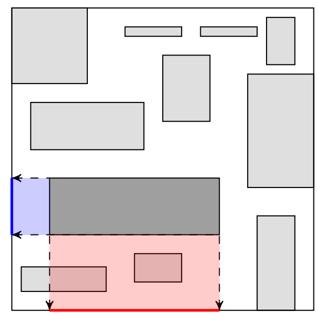


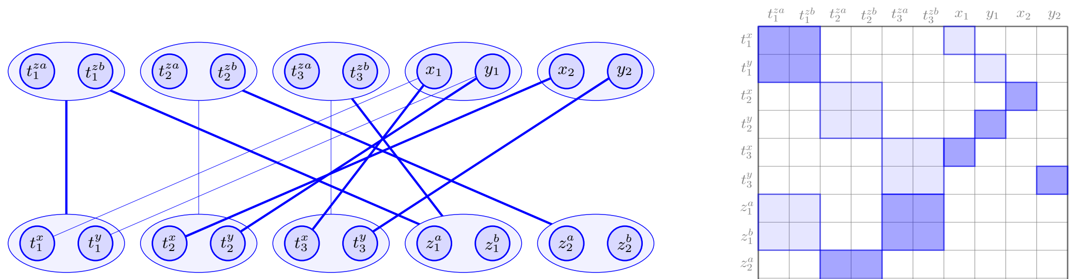
Figure 1: An example problem instance, indicating the x and y projections of the shaded rectangle.

The transformation of [2] works as follows. For each triple $t_i \in T$, we introduce nodes $t_i^{za}, t_i^{zb} \in V_1$ and $t_i^x, t_i^y \in V_2$, and hyperedge $(\{t_i^{za}, t_i^{zb}\}, \{t_i^x, t_i^y\})$. For each $x_j \in X$ and $y_j \in Y$ we introduce corresponding nodes in V_1; for each node $z_j \in Z$ we introduce nodes z_j^a, z_j^b in V_2. Then $|V_1| = |V_2| = 2(m+n)$. For each appearance of an x_j (respectively, y_j) in some triple t_i, we add an edge (x_j, t_i^x) (respectively, (y_j, t_i^y)); and for each appearance of a z_j in some triple t_i, we add a hyperedge $(\{t_i^{za}, t_i^{zb}\}, \{z_j^a, z_j^b\})$. Note that all hyperedges produced are of size either 1×1 or 2×2. Note also that the 2×2 edges partition each node set into disjoint pairs. A subset of triples T' then form a 3DM iff doing the following produces a perfect matching: for each $t = (x, y, z) \in T'$, select hyperedges (x, t^x), (y, t^y), and $(\{t^{za}, t^{zb}\})$; and for each $t \notin T'$, select $(\{t^{za}, t^{zb}\}, \{t^x, t^y\})$.

Our argument uses an optimization version of the hypergraph problem where we wish to cover all nodes with hyperedges, *without* requiring them to be disjoint, i.e., the SET COVER problem instance corresponding to the hypergraph.

LEMMA 2.1. *Suppose an HA instance I' produced by the [2] reduction admits a perfect matching. Then an optimal set cover solution to I' will necessarily be a perfect matching.*

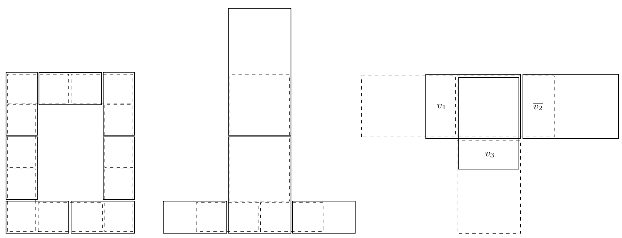
Let I be an instance of 3DM, and let I' be the resulting hypergraph instance constructed by the reduction of [2]. Since the hyperedges of I' partition its nodes into disjoint pairs, we may assume that each such pair's nodes are drawn adjacent to one another, as in Fig. 2(a). From I', we then construct a SQUARE-LOS-COVER problem instance I'' as follows. We superimpose an $n \times n$ mesh on the unit square,



(a) Resulting hypergraph instance, with "top" nodes V_1 and "bottom" nodes V_2, and with hyperedges corresponding to T' shown bold.

(b) Resulting problem instance, with boxes corresponding to the bold hyperedges shown shaded.

Figure 2: Reducing from 3DM instance $T = \{(x_1, y_1, z_1), (x_2, y_1, z_2), (x_1, y_2, z_1)\}$ (with optimal solution $T' = \{(x_2, y_1, z_2), (x_1, y_2, z_1)\}$) into a hypergraph (via the construction of [2]), and then to SQUARE-LOS-COVER.



(a) Variable. (b) + connector. (c) Clause.

Figure 3: Gadgets used in subproblem reduction.

where the columns correspond to the nodes V_1 and the rows correspond to V_2 (see Fig. 2 for an example). For each 1×1 edge between some $c \in V_1$ and $r \in V_2$, we add a rectangle in cell (r, c) of the square; similarly, for hyperedges between some $\{c_1, c_2\} \subset V_1$ and $\{r_1, r_2\} \subset V_2$, we add a rectangle spanning the four cells $\{r_1, r_2\} \times \{c_1, c_2\}$.

By construction we have a one-to-one correspondence between feasible solutions to I' and feasible solutions to I''. Thus if we can compute an optimal box cover solution to I'', then we can transform that solution to a set cover of the same size to I', and by Lemma 2.1 that set cover will be a perfect matching iff I' admits a perfect matching.

THEOREM 2.1. *SQUARE-LOS-COVER is NP-hard, even when restricted to nonoverlapping, grid-aligned rectangles of sizes 1×1 and 2×2.*

3. SUBPROBLEM OF CUBE-LOS-COVER

We show that the 2D subproblem of CUBE-LOS-COVER is hard by reduction from rectlinear PLANAR 3-SAT. Given an instance of PLANAR 3-SAT with n variables and m clauses we define three types of gadgets to represent variables, clauses, and connections between them. The basic idea is to have two kinds of rectangles (solid and dashed) whose selection corresponds to assignments of *true* or *false* values to variables. Variable gadgets (see Fig. 3(a)) are simply two series of small rectangles covering the border of the same larger rectangle such that only two optimal ways to cover that area exist, selecting all rectangles of one kind and none of the other. This selection of one kind of rectangle is propagated by a connector gadget, one of which is inserted into the variable gadget for each occurrence of the variable in a clause. (Details omitted). Chains of either kind of rectangles lead from this larger rectangle to the clause gadget, with the chain of the same kind as the protruding part of the connector gadget leading slightly further. The clause gadget (see Fig. 3(c)) simply consists of a central square area where these slightly longer chains end and overlap.

The hardness result is obtained by considering the number of rectangles needed to cover the area of all gadgets and paths between them. Let k_i be the number of solid rectangles in the variable gadget for variable v_i (there are also k_i dashed rectangles). Let $k_{i,j}$ be the number of solid rectangles (and also the number of dashed ones) in the path from the gadget for variable v_i to the gadget for clause C_j (this is 0 if v_i is not a part of C_j). Then we can show the following:

LEMMA 3.1. *A satisfying assignment for the given instance of PLANAR 3-SAT exists iff the area of all gadgets and paths can be covered using* $\sum_{i=1}^{n}\left(k_i + \sum_{j=1}^{m} k_{i,j}\right)$ *rectangles.*

By tiling all empty space not covered by our gadgets with dummy forced-choice rectangles, we obtain:

THEOREM 3.2. *The 2D subproblem of CUBE-LOS-COVER is NP-hard.*

4. DISCUSSION

Very little is known in terms of positive results for these problems, aside from the simple 2-approximation for SQUARE-LOS-COVER. Since the problem subsumes edge cover on bipartite graphs (when edges are all 1×1), greedy algorithms are unlikely to provide a better approximation. We conjecture that a k-local search algorithm will provide a $1+1/k$-approximation. As for the CUBE-LOS-COVER subproblem, the PTAS of [3] can be applied in the special case where all rectangles are squares of fixed size. For the general problem, however, no approximability is known aside from a $O(\log n)$-approximation by reduction to set cover.

5. REFERENCES

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