(Appendix A)

Begriffsschrift in Modern Notation:

(1) TO (51)

Richard Mendelsohn

Lehman College and the Graduate Center, CUNY

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Frege signed the preface to *Begriffsschrift* in December of 1878, and it was published the following year by George Olms. Frege did little to connect up his own work with his contemporaries, either with the logical achievements of Boole, or the mathematical investigations of Dedekind. The only explicit references are to philosophers—Aristotle, Leibniz and Kant. His previous work, which consisted mainly of reviews, gave no indication of the direction and creativity of his thinking. Like Athena, emerging full-grown from Zeus's brow, Frege's remarkable work bore no evidence of the genesis and growth of the ideas presented therein. There is little surprise to the reception his contemporaries gave *Begriffsschrift:* they did not know what to make of it.

Here are some of the achievements of *Begriffsschrift:*

First, Frege synthesized the two otherwise opposed traditions—the Stoic logic of the propositional connectives and the Aristotelian treatment of the quantifiers—into one system, and extended the Aristotelian treatment to include relations as well as properties. His function/argument analysis of propositions supplanted the subject/predicate distinction of traditional analysis, creating one of the first extensions of mathematical forms of analysis to domains other than arithmetic and geometry.

Second, propositions (1), (2), (8), (28), (31) and (41), together with the Rule of Inference, *Modus Ponens*—and a suitable substitution rule that is employed but never precisely stated—constitutes a complete and consistent axiomatization of truth-functional logic.[1]

Third, Frege provides two rules for the universal quantifier: Universal Instantiation is given in proposition (58), and Universal Generalization is given in the informal explanation of his notation for quantification. When appended to the axioms of truth-functional logic, these constitute a complete and consistent axiomatization of first-order logic.

Fourth, propositions (52) and (54), when appended to the other rules, constitute a complete, consistent axiomatization of first-order logic *with* identity.[2]

Fifth, higher-order quantification is introduced and used—to define the ancestral relation, a key in the definition of mathematical induction—though it is not clearly

[1] In the preface, Frege notes that he could reduce his axioms further by combining (31) and (41) as a biconditional. But this sort of reduction carries no intellectual interest, for it is the number of axioms he is after, not the quality, viz. the independence. Actually, it is well known that propositions (1), (2) and (28) are sufficient.

[2] They are not clearly quantificational as opposed to truth-functional because Frege used the identity sign both for identity and the biconditional.

delimited from first-order quantification since proposition (58) is apparently intended to serve for any order quantifier, and not just the first-order quantifier in whose terms it is explicitly stated.

It is no small measure of the greatness of the work that his contemporaries just did not understand what he had done.[3] Part of the reason for the lack of understanding was his incredibly cumbersome notation. As Schröder said, it appeared to copy the Japanese form of writing vertically. Frege wanted to capture the way in which a proof is written, with each step in the proof on a different line. We, today, take each step in the proof on a single line; Frege, however, broke up each line so that each of its logically significant components appeared on a different line. This has proved practically unreadable. Hence this appendix which is designed to enable the reader to understand the logical apparatus of *Begriffsschrift* by rewriting the proofs in modern notation.

We use $\neg$ for the negation sign and $\supset$ for the material conditional, with appropriate parentheses, braces or brackets for ease of reading. We use lower case a, b, c, ... for propositional variables. Our substitution notation will be $X^{a,b,c,\ldots}_{x,y,z\ldots}$ which designates the result of replacing a for x, b for y, c for z, ... in X. Each line in the proof is numbered and the proposition from which it results by substitution is written on the right. There are only two lines in each proof, because each inference is obtained by *Modus Ponens*.

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[3]This is evident from the reviews of the work, some of which have been included in Terry Bynum's edition of *Begriffsschrift*.

Proposition 1

$$a \supset (b \supset a)$$

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Proposition 2

$$(c \supset (b \supset a)) \supset ((c \supset b) \supset (c \supset a))$$

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Proposition 3

$$(b \supset a) \supset [(c \supset (b \supset a)) \supset ((c \supset b) \supset (c \supset a))]$$

Proof: By Modus Ponens from

1. $(c \supset (b \supset a)) \supset ((c \supset b) \supset (c \supset a))$ (2)
2. $[((c \supset (b \supset a)) \supset ((c \supset b) \supset (c \supset a))] \supset$
   $[(b \supset a) \supset ((c \supset (b \supset a)) \supset ((c \supset b) \supset (c \supset a))]$ $(1)_{a,b}^{(c\supset(b\supset a))\supset((c\supset b)\supset(c\supset a)),b\supset a}$

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Proposition 4

$$[(b \supset a) \supset (c \supset (b \supset a))] \supset [(b \supset a) \supset ((c \supset b) \supset (c \supset a))]$$

Proof: By Modus Ponens from

1. $(b \supset a) \supset [(c \supset (b \supset a)) \supset ((c \supset b) \supset (c \supset a))]$ (3)
2. $[(b \supset a) \supset ((c \supset (b \supset a)) \supset ((c \supset b) \supset (c \supset a))] \supset$
   $[((b \supset a) \supset (c \supset (b \supset a))) \supset ((b \supset a) \supset ((c \supset b) \supset (c \supset a)))]$ $(2)_{a,b,c}^{(c\supset b)\supset(c\supset a),c\supset(b\supset a),b\supset a}$

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Proposition 5

$$(b \supset a) \supset ((c \supset b) \supset (c \supset a))$$

Proof: By Modus Ponens from

1. $(b \supset a) \supset (c \supset (b \supset a))$ $\quad (1)^{a \supset b, c}_{a,b}$
2. $[(b \supset a) \supset (c \supset (b \supset a))] \supset [(b \supset a) \supset ((c \supset b) \supset (c \supset a))]$ $\quad (4)$

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Proposition 6

$$(c \supset (b \supset a)) \supset (c \supset ((d \supset b) \supset (d \supset a)))$$

Proof: By Modus Ponens from

1. $(b \supset a) \supset ((d \supset b) \supset (d \supset a))$ $\quad (5)^{d}_{c}$
2. $((b \supset a) \supset ((d \supset b) \supset (d \supset a))) \supset$
   $((c \supset (b \supset a)) \supset (c \supset ((d \supset b) \supset (d \supset a))))$ $\quad (5)^{(d \supset b) \supset (d \supset a), b \supset a}_{a,b}$

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Proposition 7

$$(b \supset a) \supset ((d \supset (c \supset b)) \supset (d \supset (c \supset a)))$$

Proof: By Modus Ponens from

1. $(b \supset a) \supset ((c \supset b) \supset (c \supset a))$ (5)
2. $((b \supset a) \supset ((c \supset b) \supset (c \supset a))) \supset$
   $((b \supset a) \supset ((d \supset (c \supset b)) \supset (d \supset (c \supset a))))$ $(6)_{a,b,c}^{c \supset a, c \supset b, b \supset a}$

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PROPOSITION 8

$$(d \supset (b \supset a)) \supset (b \supset (d \supset a))$$

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PROPOSITION 9

$$(c \supset b) \supset ((b \supset a) \supset (c \supset a))$$

Proof: By Modus Ponens from

1. $(b \supset a) \supset ((c \supset b) \supset (c \supset a))$ (5)
2. $((b \supset a) \supset ((c \supset b) \supset (c \supset a))) \supset$
   $((c \supset b) \supset ((b \supset a) \supset (c \supset a)))$ $(8)_{a,b,d}^{c \supset a, c \supset b, b \supset a}$

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PROPOSITION 10

$$((e \supset (d \supset b)) \supset a) \supset ((d \supset (e \supset b)) \supset a)$$

Proof: By Modus Ponens from

1. $(d \supset (e \supset b)) \supset (e \supset (d \supset b))$ $(8)^{b,e}_{a,b}$

2. $((d \supset (e \supset b)) \supset (e \supset (d \supset b))) \supset$
   $(((e \supset (d \supset b)) \supset a) \supset ((d \supset (e \supset b)) \supset a))$ $(9)^{e\supset(d\supset b),d\supset(e\supset b)}_{b,c}$

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PROPOSITION 11

$$((c \supset b) \supset a)) \supset (b \supset a)$$

Proof: By Modus Ponens from

1. $b \supset (c \supset b)$ $(1)^{b,c}_{a,b}$

2. $(b \supset (c \supset b)) \supset (((c \supset b) \supset a) \supset (b \supset a))$ $(9)^{c\supset b,b}_{b,c}$

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PROPOSITION 12

$$(d \supset (c \supset (b \supset a))) \supset (d \supset (b \supset (c \supset a)))$$

Proof: By Modus Ponens from

1. $(c \supset (b \supset a) \supset (b \supset (c \supset a))$ $(8)^{c}_{d}$

2. $((c \supset (b \supset a)) \supset (b \supset (c \supset a))) \supset$
   $((d \supset (c \supset (b \supset a))) \supset (d \supset (b \supset (c \supset a))))$ $(5)^{b\supset(c\supset a),c\supset(b\supset a),d}_{a,b,c}$

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Proposition 13

$$(d \supset (c \supset (b \supset a))) \supset (b \supset (d \supset (c \supset a)))$$

Proof: By Modus Ponens from

1. $(d \supset (c \supset (b \supset a))) \supset (d \supset (b \supset (c \supset a)))$ (12)
2. $((d \supset (c \supset (b \supset a))) \supset (d \supset (b \supset (a \supset c)))) \supset$
   $((d \supset (c \supset (b \supset a))) \supset (b \supset (d \supset (c \supset a))))$ $(12)^{c\supset a, d, d\supset(c\supset(b\supset a))}_{a,c,d}$

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Proposition 14

$$(e \supset (d \supset (c \supset (b \supset a)))) \supset (e \supset (b \supset (d \supset (c \supset a))))$$

Proof: By Modus Ponens from

1. $(d \supset (c \supset (b \supset a))) \supset (b \supset (d \supset (c \supset a)))$ (13)
2. $((e \supset (c \supset (b \supset a))) \supset (b \supset (d \supset (c \supset a)))) \supset$
   $((e \supset (d \supset (c \supset (b \supset a)))) \supset (e \supset (b \supset (d \supset (c \supset a)))$ $(5)^{b\supset(d\supset(c\supset a)), d\supset(c\supset(b\supset a)), e}_{a,b,c}$

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Proposition 15

$$(e \supset (d \supset (c \supset (b \supset a)))) \supset (b \supset (e \supset (d \supset (c \supset a))))$$

Proof: By Modus Ponens from

1. $(e \supset (d \supset (c \supset (b \supset a)))) \supset (e \supset (b \supset (d \supset (c \supset a))))$ (14)

2. $((e \supset (d \supset (c \supset (b \supset a)))) \supset (e \supset (b \supset (d \supset (c \supset a))))) \supset$
   $((e \supset (d \supset (c \supset (b \supset a)))) \supset (b \supset (e \supset (d \supset (c \supset a)))))$ $(12)_{a,c,d}^{d \supset (c \supset a), e, (e \supset (d \supset (c \supset (b \supset a)))}$

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PROPOSITION 16

$$(e \supset (d \supset (c \supset (b \supset a)))) \supset (e \supset (d \supset (b \supset (c \supset a))))$$

Proof: By Modus Ponens from

1. $(e \supset (d \supset (c \supset (b \supset a)))) \supset (e \supset (b \supset (d \supset (c \supset a))))$ (12)

2. $((d \supset (c \supset (b \supset a))) \supset (d \supset (b \supset (c \supset a)))) \supset$
   $((e \supset (d \supset (c \supset (b \supset a)))) \supset (e \supset (d \supset (b \supset (c \supset a)))))$ $(5)_{a,b,c}^{d \supset (b \supset (c \supset a)), d \supset (c \supset (b \supset a)), e}$

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PROPOSITION 17

$$(d \supset (c \supset (b \supset a))) \supset (c \supset (b \supset (d \supset a)))$$

Proof: By Modus Ponens from

1. $(d \supset (c \supset (b \supset a))) \supset (c \supset (d \supset (b \supset a)))$ $(8)_{a,b}^{b \supset a, c}$

2. $((d \supset (c \supset (b \supset a))) \supset (c \supset (d \supset (b \supset a)))) \supset$
   $((d \supset (c \supset (b \supset a))) \supset (c \supset (b \supset (d \supset a))))$ $(16)_{c,d,e}^{d,c,d \supset (c \supset (b \supset a))}$

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Proposition 18

$$(c \supset (b \supset a)) \supset ((d \supset c \supset (b \supset (d \supset a)))$$

Proof: By Modus Ponens from

1. $(c \supset (b \supset a)) \supset ((c \supset d) \supset (d \supset (b \supset a)))$ $\qquad (5)^{b\supset a,c,d}_{a,b,c}$
2. $((c \supset (b \supset a)) \supset ((c \supset d) \supset (d \supset (b \supset a))) \supset$
   $((c \supset (b \supset a)) \supset ((d \supset c \supset (b \supset (d \supset a))))$ $\qquad (16)^{d,d\supset c,c\supset(b\supset a)}_{c,d,e}$

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Proposition 19

$$(d \supset (c \supset b)) \supset ((b \supset a) \supset (d \supset (c \supset a)))$$

Proof: By Modus Ponens from

1. $(c \supset b) \supset ((b \supset a) \supset (c \supset a))$ $\qquad (9)$
2. $((c \supset b) \supset ((b \supset a) \supset (c \supset a))) \supset$
   $((d \supset (c \supset b)) \supset ((b \supset a) \supset (d \supset (c \supset a))))$ $\qquad (18)^{c\supset a,b\supset a,c\supset b}_{a,b,c}$

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Proposition 20

$$(e \supset (d \supset (c \supset b))) \supset ((b \supset a) \supset (e \supset (d \supset (c \supset a))))$$

Proof: By Modus Ponens from

1. $(d \supset (c \supset b)) \supset ((b \supset a) \supset (d \supset (c \supset a)))$ (19)

2. $((d \supset (c \supset b)) \supset ((b \supset a) \supset (d \supset (c \supset a)))) \supset$

$((e \supset (d \supset (c \supset b))) \supset ((b \supset a) \supset (e \supset (d \supset (c \supset a)))))$ $(18)^{d\supset(c\supset a), b\supset a, d\supset(c\supset b), e}_{a,b,c,d}$

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Proposition 21

$$((d \supset b) \supset a)) \supset ((d \supset c) \supset ((c \supset b) \supset a)))$$

Proof: By Modus Ponens from

1. $(d \supset c) \supset ((c \supset b) \supset (d \supset b))$ $(9)^{b,c,d}_{a,b,c}$

2. $((d \supset c) \supset ((c \supset b) \supset (d \supset b))) \supset$

$(((d \supset b) \supset a)) \supset ((d \supset c) \supset ((c \supset b) \supset a))))$ $(19)^{d\supset b, c\supset b, d\supset c}_{b,c,d}$

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Proposition 22

$$(f \supset (e \supset (d \supset (c \supset (b \supset a))))) \supset (f \supset (e \supset (d \supset (b \supset (c \supset a)))))$$

Proof: By Modus Ponens from

1. $(e \supset (d \supset (c \supset (b \supset a)))) \supset (e \supset (d \supset (b \supset (c \supset a))))$ (16)

2. $((e \supset (d \supset (c \supset b \supset a)))) \supset (e \supset (d \supset (b \supset (c \supset a))))) \supset$

$[(f \supset (e \supset (d \supset (c \supset (b \supset a))))) \supset$

$(f \supset (e \supset (d \supset (b \supset (c \supset a)))))]$ $(5)^{e\supset(d\supset(b\supset(c\supset a))), e\supset(d\supset(c\supset(b\supset a))), f}_{a,b,c}$

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Proposition 23

$$(d \supset (c \supset (b \supset a))) \supset ((e \supset d) \supset (c \supset (b \supset (e \supset a)))$$

Proof: By Modus Ponens from

1. $(d \supset (c \supset (b \supset a))) \supset ((e \supset d) \supset (c \supset (e \supset (b \supset a)))$ $(18)^{b\supset a,c,d,e}_{a,b,c,d}$
2. $((d \supset (c \supset (b \supset a))) \supset ((e \supset d) \supset (c \supset (e \supset (b \supset a)))) \supset$
   $((d \supset (c \supset (b \supset a))) \supset ((e \supset d) \supset (c \supset (b \supset (e \supset a))))$ $(22)^{e,c,e\supset d,d\supset(c\supset(b\supset a))}_{c,d,e,f}$

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Proposition 24

$$(c \supset a) \supset (c \supset (b \supset a))$$

Proof: By Modus Ponens from

1. $(c \supset a) \supset (b \supset (c \supset a)$ $(1)^{c\supset a}_{a}$
2. $((c \supset a) \supset (b \supset (c \supset a))) \supset ((c \supset a) \supset (c \supset (b \supset a)))$ $(12)^{c,b,c\supset a}_{b,c,d}$

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Proposition 25

$$(d \supset (c \supset a)) \supset (d \supset (c \supset (b \supset a)))$$

Proof: By Modus Ponens from

1. $(c \supset a) \supset (c \supset (b \supset a))$ (24)
2. $((c \supset a) \supset (c \supset (b \supset a))) \supset$
   $((d \supset (c \supset a)) \supset (d \supset (c \supset (b \supset a))))$ $(5)^{c\supset(b\supset a),c\supset a,d}_{a,b,c}$

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PROPOSITION 26

$$b \supset (a \supset a)$$

Proof: By Modus Ponens from

1. $a \supset (b \supset a)$ (1)
2. $(a \supset (b \supset a)) \supset (b \supset (a \supset a))$ $(8)_d^a$

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PROPOSITION 27

$$a \supset a$$

Proof: By Modus Ponens from

1. $a \supset (b \supset a)$ (1)
2. $(a \supset (b \supset a)) \supset (a \supset a)$ $(26)_b^{a \supset (b \supset a)}$

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PROPOSITION 28

$$(b \supset a) \supset (\neg a \supset \neg b)$$

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Proposition 29

$$(c \supset (b \supset a)) \supset (c \supset (\neg a \supset \neg b))$$

Proof: By Modus Ponens from

1. $(b \supset a) \supset (\neg a \supset \neg b)$ (28)
2. $((b \supset a) \supset (\neg a \supset \neg b)) \supset ((c \supset (b \supset a)) \supset (c \supset (\neg a \supset \neg b)))$ $(5)^{\neg a \supset \neg b, b \supset a}_{a,b}$

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Proposition 30

$$(b \supset (c \supset a)) \supset (c \supset (\neg a \supset \neg b))$$

Proof: By Modus Ponens from

1. $(c \supset (b \supset a)) \supset (c \supset (\neg a \supset \neg b))$ (29)
2. $((c \supset (b \supset a)) \supset (c \supset (\neg a \supset \neg b))) \supset$
   $((b \supset (c \supset a)) \supset (c \supset (\neg a \supset \neg b)))$ $(10)^{c \supset (\neg a \supset \neg b), a, b, c}_{a,b,d,e}$

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Proposition 31

$$\neg \neg a \supset a$$

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Proposition 32

$$((\neg b \supset a) \supset (\neg a \supset \neg \neg b)) \supset ((\neg b \supset a) \supset (\neg a \supset b))$$

Proof: By Modus Ponens from

1. $\neg\neg b \supset b$ $\quad (31)^{b}_{a}$
2. $(\neg\neg b \supset b) \supset$
   $(((\neg b \supset a) \supset (\neg a \supset \neg\neg b)) \supset ((\neg b \supset a) \supset (\neg a \supset b)))$ $\quad (7)^{b,\neg\neg b,\neg a,\neg b \supset a}_{a,b,c,d}$

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PROPOSITION 33

$$(\neg b \supset a) \supset (\neg a \supset b)$$

Proof: By Modus Ponens from

1. $((\neg b \supset a) \supset (\neg a \supset \neg\neg b)) \supset ((\neg b \supset a) \supset (\neg a \supset b))$ $\quad (32)$
2. $(\neg b \supset a) \supset (\neg a \supset \neg\neg b)$ $\quad (28)^{\neg b}_{b}$

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PROPOSITION 34

$$(c \supset (\neg b \supset a)) \supset (c \supset (\neg a \supset b))$$

Proof: By Modus Ponens from

1. $(\neg b \supset a) \supset (\neg a \supset b)$ $\quad (33)$
2. $((\neg b \supset a) \supset (\neg a \supset b)) \supset$
   $((c \supset (\neg b \supset a)) \supset (c \supset (\neg a \supset b)))$ $\quad (5)^{\neg a \supset b,\neg b \supset a}_{a,b}$

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Proposition 35

$$(c \supset (\neg b \supset a)) \supset (\neg a \supset (c \supset b))$$

Proof: By Modus Ponens from

1. $(c \supset (\neg b \supset a)) \supset (c \supset (\neg a \supset b))$ (34)
2. $((c \supset (\neg b \supset a)) \supset (c \supset (\neg a \supset b))) \supset$
   $((c \supset (\neg b \supset a)) \supset (\neg a \supset (c \supset b)))$ $(12)^{b, \neg a, c \supset (\neg b \supset a)}_{a,b,d}$

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Proposition 36

$$a \supset (\neg a \supset b)$$

Proof: By Modus Ponens from

1. $a \supset (\neg b \supset a)$ $(1)^{\neg b}_{b}$
2. $(a \supset (\neg b \supset a)) \supset (a \supset (\neg a \supset b))$ $(34)^{a}_{c}$

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Proposition 37

$$((\neg c \supset b) \supset a) \supset (c \supset a)$$

Proof: By Modus Ponens from

1. $c \supset (\neg c \supset b)$ $(36)^{c}_{a}$
2. $(c \supset (\neg c \supset b)) \supset (((\neg c \supset b) \supset a) \supset (c \supset a))$ $(9)^{\neg c \supset b}_{b}$

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Proposition 38

$$\neg a \supset (\neg a \supset b)$$

Proof: By Modus Ponens from

1. $a \supset (\neg a \supset b)$ (36)
2. $(a \supset (\neg a \supset b)) \supset (\neg a \supset (a \supset b))$ $(8)^{b,\neg a,a}_{a,b,d}$

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Proposition 39

$$(\neg a \supset a) \supset (\neg a \supset b)$$

Proof: By Modus Ponens from

1. $\neg a \supset (a \supset b)$ (38)
2. $(\neg a \supset (a \supset b)) \supset ((\neg a \supset a) \supset (\neg a \supset b))$ $(2)^{b,a,\neg a}_{a,b,c}$

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Proposition 40

$$\neg b \supset ((\neg a \supset a) \supset a)$$

Proof: By Modus Ponens from

1. $(\neg a \supset a) \supset (\neg a \supset b)$ (39)
2. $((\neg a \supset a) \supset (\neg a \supset b)) \supset (\neg b \supset ((\neg a \supset a) \supset a))$ $(35)^{b,a,\neg a \supset a}_{a,b,c}$

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PROPOSITION 41

$$a \supset \neg\neg a$$

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PROPOSITION 42

$$\neg\neg(a \supset a)$$

Proof: By Modus Ponens from

1. $a \supset a$ (27)
2. $(a \supset a) \supset \neg\neg(a \supset a)$ $(41)_{a}^{a \supset a}$

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PROPOSITION 43

$$(\neg a \supset a) \supset a$$

Proof: By Modus Ponens from

1. $\neg\neg(a \supset a)$ (42)
2. $\neg\neg(a \supset a) \supset ((\neg a \supset a) \supset a)$ $(40)_{b}^{\neg(a \supset a)}$

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Proposition 44

$$(\neg\, a \supset c) \supset ((c \supset a) \supset a))$$

Proof: By Modus Ponens from

1. $(\neg\, a \supset a) \supset a$ (43)
2. $((\neg\, a \supset a) \supset a) \supset ((\neg\, a \supset c) \supset ((c \supset a) \supset a)))$ $(21)^{a,\neg a}_{b,d}$

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Proposition 45

$$((\neg\, c \supset a) \supset (\neg\, a \supset c) \supset ((\neg\, c \supset a) \supset ((c \supset a) \supset a))$$

Proof: By Modus Ponens from

1. $(\neg\, a \supset c) \supset ((c \supset a) \supset a))$ (44)
2. $((\neg\, a \supset c) \supset ((c \supset a) \supset a))) \supset$
   $(((\neg\, c \supset a) \supset (\neg\, a \supset c) \supset ((\neg\, c \supset a) \supset ((c \supset a) \supset a)))$ $(5)^{(c\supset a)\supset a,\neg a\supset c,\neg c\supset a}_{a,b,c}$

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Proposition 46

$$(\neg\, c \supset a) \supset ((c \supset a) \supset a)$$

Proof: By Modus Ponens from

1. $((\neg\, c \supset a) \supset (\neg\, a \supset c)) \supset ((\neg\, c \supset a) \supset ((c \supset a) \supset a))$ (45)
2. $(\neg\, c \supset a) \supset (\neg\, a \supset c)$ $(33)^{c}_{b}$

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Proposition 47

$$(\neg c \supset b) \supset ((b \supset a) \supset ((c \supset a) \supset a))$$

Proof: By Modus Ponens from

1. $(\neg c \supset a) \supset ((c \supset a) \supset a)$ (46)
2. $((\neg c \supset a) \supset ((c \supset a) \supset a)) \supset$
   $((\neg c \supset a) \supset ((b \supset a) \supset ((c \supset a) \supset a)))$ $(21)^{(c\supset a)\supset a, a, \neg c, b}_{a,b,d,c}$

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Proposition 48

$$(d \supset (\neg c \supset b)) \supset ((b \supset a) \supset ((c \supset a) \supset (d \supset a)))$$

Proof: By Modus Ponens from

1. $(\neg c \supset b) \supset ((b \supset a) \supset ((c \supset a) \supset a))$ (47)
2. $((\neg c \supset b) \supset ((b \supset a) \supset ((c \supset a) \supset a))) \supset$
   $((d \supset (\neg c \supset b)) \supset ((b \supset a) \supset ((c \supset a) \supset (d \supset a))))$ $(23)^{c\supset a, b\supset a, \neg c\supset b, d}_{b,c,d,e}$

* * * * *

Proposition 49

$$(\neg c \supset b) \supset ((c \supset a) \supset ((b \supset a) \supset a))$$

Proof: By Modus Ponens from

1. $(\neg c \supset b) \supset ((b \supset a) \supset ((c \supset a) \supset a))$ (47)

$((\neg c \supset b) \supset ((b \supset a) \supset ((c \supset a) \supset a))) \supset$

$((\neg c \supset b) \supset ((c \supset a) \supset ((b \supset a) \supset a)))$ $(12)^{c\supset a, b\supset a, \neg c\supset b}_{b,c,d}$

* * * * *

PROPOSITION 50

$$(c \supset a) \supset ((b \supset a) \supset ((\neg c \supset b) \supset a))$$

Proof: By Modus Ponens from

1. $(\neg c \supset b) \supset ((c \supset a) \supset ((b \supset a) \supset a))$ (49)
2. $((\neg c \supset b) \supset ((c \supset a) \supset ((b \supset a) \supset a))) \supset$

   $((c \supset a) \supset ((b \supset a) \supset ((\neg c \supset b) \supset a)))$ $(17)^{b\supset a, c\supset a, \neg c\supset b}_{b,c,d}$

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