C1M4

Inverse Functions and Logarithms

Each summer a new group of incoming students is inducted into the U.S. Naval Academy, they become Fouth Class Midshipmen or *plebes*, and identification numbers called *alpha numbers* are assigned. Since this year is 2000, and it is hoped that these students will graduate in 2004, each number assigned begins with 04. So a typical alpha might be 047854, except that 7854 is a larger number than would be needed. Unless an error has been made, there is a $one - to - one$ relationship between a set called Plebes and a set called Alphas. If numbers are assigned to names then there is a function F so that

$$F : \textit{Plebes} \longrightarrow \textit{Alphas}$$

and, for example

$$F(\text{John Doe}) = 041721$$

And, unless two plebes are mistakenly assigned the same number, there is a unique association that allows one to identify the plebe if you have their alpha number. This means that there is an *inverse function*

$$F^{-1} : \textit{Alphas} \longrightarrow \textit{Plebes}$$

and

$$F^{-1}(041721) = \text{John Doe}$$

The discussion above is a very simplistic example of how functions and inverse functions relate. More precisely, with $\iff$ meaning "if and only if",

$$f^{-1}(x) = y \quad \iff \quad f(y) = x$$

You have studied exponential functions recently and probably noted that when $a > 1$, then a^x is an increasing function. We sometimes use $\equiv$ to denote a definition or an equivalent statement.

$$f \text{ is increasing} \quad \equiv \quad u < v \Rightarrow f(u) < f(v)$$

It is easy to see that when a function is increasing then it has an inverse. And, inverses of exponential functions are called *logarithms*. This leads to the very important relationship

$$\log_a(x) = y \quad \iff \quad a^y = x$$

In calculus, the most important base is e and we call that logarithm the *natural logarithm* and identify it by ln.

$$\log_e(x) \equiv \ln(x)$$

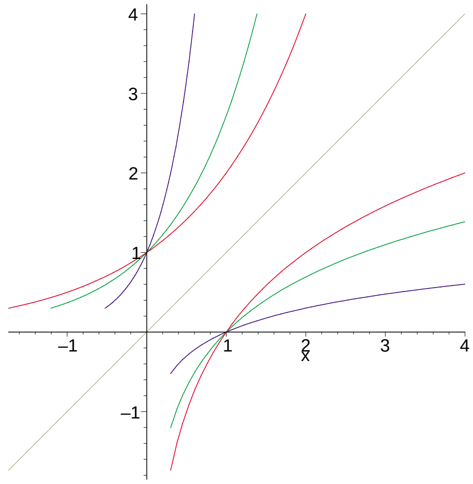
From which it follows that

♠ $e^{\ln(x)} = x \qquad e^{-\ln(x)} = \dfrac{1}{x} \qquad a^b = e^{b\ln(a)} \qquad \ln(e^x) = x \qquad \ln(e) = 1 \qquad \ln(e^2) = 2 \qquad \ln(e^3) = 3$

The graph of an inverse function is related to the graph of the function by reflection about the line $y = x$. In this next example we will plot several logarithmic functions and the related exponentials.

Maple Example Plot $\log_2(x)$, $\ln(x)$, $\log_{10}(x)$, their inverses 2^x, e^x, 10^x, and the line $y = x$.

```
> with(plots):
> A:=plot([log[2](x),ln(x),log[10](x)],x=(.3)..4,color=[red,green,blue]):
> B1:=plot(2^x,x=log[2](.3)..log[2](4),color=red):
> B2:=plot(exp(x),x=ln(.3)..ln(4),color=green):
> B3:=plot(10^x,x=log[10](.3)..log[10](4),color=blue):
> B4:=plot(x,x=log[2](.3)..4,color=khaki,scaling=constrained):
> display(A,B1,B2,B3,B4);
```



Maple Example: Plot $y = 3^x$ and its tangent line at $x = 0$ on the same coordinate axes.

Do you remember how we approximated the slope of a^x at $x = 0$ in **C1M3**? We defined a function

$$F := x \longrightarrow \frac{1}{2}\frac{a^x - a^{-x}}{x}$$

that calculated the slope of the line segment that joins the two points $P(-x, a^{-x})$ and $Q(x, a^x)$ spaced equally from $x = 0$. Then we selected different values of a and looked at values of this slope function as x got closer to 0. We are going to repeat this process and then compare our approximations with certain values.

```
> F:=x->(a^x-a^(-x))/(2*x);
```

$$F := x \longrightarrow \frac{1}{2}\frac{a^x - a^{-x}}{x}$$

```
> a:=2; seq(evalf(F(1/2^i),14),i=9..15);
```

$$a := 2$$

.69314739230, .69314723351, .6931471938, .6931471839, .6931471815, .6931471806, .693147181

```
> evalf(ln(2),14);
```

.69314718055995

```
> a:=3; seq(evalf(F(1/2^i),14),i=9..15);
```

$$a := 3$$

1.09861313170, 1.09861249940, 1.0986123413, 1.0986123019, 1.0986122923, 1.0986122897, 1.098612289

```
> evalf(ln(3),14);
```

1.0986122886681

```
> a:=10; seq(evalf(F(1/2^i),14),i=9..15);
```

$$a := 10$$

2.30259285469, 2.30258703341, 2.3025855781, 2.3025852143, 2.3025851234, 2.3025851002, 2.302585095

```
> evalf(ln(10),14);
```

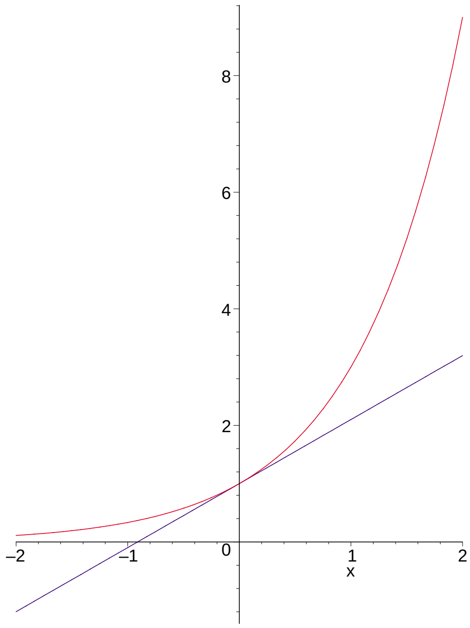
2.3025850929940

Although we have not proved it (yet), we are *strongly* suspicious that at $x = 0$ the slope of the line tangent to $y = a^x$ is $\ln(a)$. We will use this value for m in the equation for the line, $y - y_0 = m(x - x_0)$.

```
> eq1:=y-1=ln(3)*(x-0);
```

$$eq1 := y - 1 = \ln(3)\, x$$

```
> plot([3^x,ln(3)*x+1],x=-2..2,color=[red,blue]);
```



C1M4 Problems: Use Maple to display the following graphs:

1. $y = 5^x$ and $y = \log_5(x)$
2. $y = \log_{10}(x) + .01$ and $y = \dfrac{\ln(x)}{\ln(10)}$, $.2 \le x \le 5$ $\qquad \log_a(x) = \frac{\ln(x)}{\ln(a)}$
3. $y = \ln(2x) + .02$ and $y = \ln(x) + \ln(2)$, $.2 \le x \le 5$ $\qquad \ln(ab) = \ln(a) + \ln(b)$
4. $y = \ln(x^2) + .03$ and $y = 2\ln(x)$, $.2 \le x \le 5\pi$ $\qquad \ln(x^r) = r\,\ln(x)$
5. $y = 5^x$ and the line tangent at $x = 0$.