Exponential Functions

Every thirty days a biology student counted the number of bugs that lived in a colony and recorded the numbers, which we list for you.

$$124,\ 248,\ 496,\ 992,\ 1984$$

The student observed that the population was actually doubling each time, so the same data could be recorded as:

$$124,\ 124 \cdot 2,\ 124 \cdot 2^2,\ 124 \cdot 2^3,\ 124 \cdot 2^4$$

And since the data was collected every 30 days, the following formula was proposed as a model that represented the population each day, where t is the number of days.

$$P = 124 \cdot 2^{t/30}$$

Using this, the number of bugs after 47 days was estimated to be $124 \cdot 2^{47/30} \approx 367.3$. And, after 12 thirty day periods there would be $124 \cdot 2^{360/30} = 507{,}904$ bugs.

The idea of doubling something and recording the result is not a new one. Imagine putting a penny on the corner square of a chessboard and having someone put double the previous amount on a square for the next 63 days. How much would be accumulated? The amount placed each day, where you put the penny down on the 't^{th}' day would be

$$Q(t) = (.01)2^t$$

This means that $\$1{,}342{,}177.28$ — over a million dollars, would be placed on the 27th day and the total after all 64 squares are accounted for would be approximately $1.844674407 \times 10^{17}$ dollars. This is roughly equivalent to 184,467 trillion dollars.

Suppose that we consider a function $f(x) = a^x$ for some $a > 0$. It is interesting to note that for any two values of x that are a fixed distance apart, like x and $x + h$, you get the same ratio

$$\frac{f(x+h)}{f(x)} = \frac{a^{x+h}}{a^x} = a^h = \text{constant}$$

for their functional values. When the first example involving the bugs was constructed it was decided that it was easiest to double the number of bugs every 30 days. But, suppose that the bugs increased by a factor of 1.375 every 30 days. Then we would have gotten $P = 124 \cdot (1.375)^{t/30}$ as our population function.

Maple Example: Suppose that a disease hits a flock of critters. Every week a count is taken to see how many remain. It is suspected that the relationship is exponential. If it is, find an expression which provides the number present at any time t in weeks. The table shows the data.

Week	Critters
0	1165
1	728
2	455
3	284
4	177
5	111
6	69

We begin by establishing a list of our data. Then, we check out the ratios of successive terms by using the `seq` command, which yields a list for each integer that is specified. The `[i]` refers to the ith member of the list *dataset*. For example, `dataset[3]=455`.

```
> dataset:=[1165,728,455,284,177,111,69];
```
$$dataset := [1165, 728, 455, 284, 177, 111, 69]$$
```
> seq(dataset[i]/dataset[i-1],i=2..7);
```
$$\frac{728}{1165},\ \frac{5}{8},\ \frac{284}{455},\ \frac{177}{284},\ \frac{37}{59},\ \frac{23}{37}$$

In Maple, you may use `%` to refer to the result(s) of the last command executed. It is extremely useful, but **BEWARE!** if you skip around in a worksheet because the order of execution of commands can produce

some erroneous results. That is why it is important to 'execute the worksheet' before you hand it in. Then, it is important to look back over it to see if errors have occurred. After examining the results, we select $a = .625$ as a potential base.

```
>  evalf(%);
```
$$.6248927039, .6250000000, .6241758242, .6232394366, .6271186441, .6216216216$$

```
>  A0:=1165;  a:=.625;
```
$$A0 := 1165$$
$$a := .625$$

```
>  Q:=t->A0*(a)^t;
```
$$Q := \to A0\, a^t$$

```
>  seq(Q(i),i=0..6);
```
$$1165., 728.125, 455.078125, 284.4238281, 177.7648925, 111.1030579, 69.43941117$$

The function Q that is defined seems to reflect the number of critters very well.

It might be interesting to look at functions of the form $y = a^x$ for different positive values of a.

```
>  plot([sqrt(3)^x,2^x,sqrt(5)^x,3^x],x=-1..2,color=[navy,red,green,blue]);
```
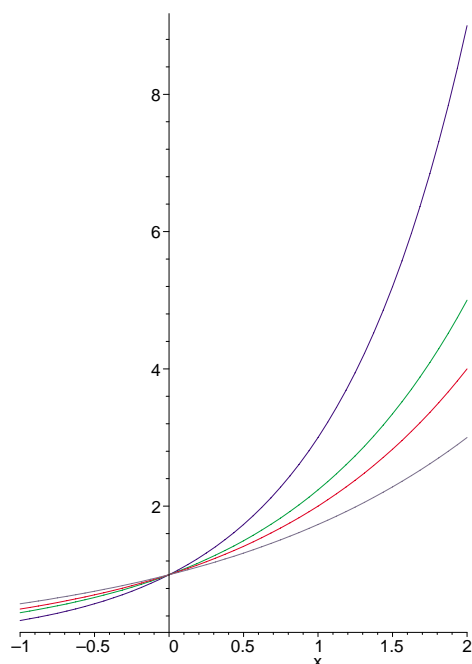

We recall that $\sqrt{3} < 2 < \sqrt{5} < 3$, so that when $0 < x$ the graphs maintain the related inequalities, $(\sqrt{3})^x < 2^x < (\sqrt{5})^x < 3^x$. But wait!, It gets better. In a worksheet, enter the following three lines and execute them:

```
>  with(plots):
>  G:=a^x;
>  animate(G,x=-1..2,a=1/2..4);
```

And what will appear? You will obtain the expression G and a plot which shows $y = \left(\dfrac{1}{2}\right)^x$. Move the cursor to the plot and 'click' on it. The *toolbar* at the top of the screen, which is below the *menu bar*, will be replaced by what looks like the buttons on a tape player. Click on the triangular 'play' button. What happens? You should see G plotted for values of the parameter a as a varies from $1/2$ to 4.

From this animation we see that there should be some value of a for which the slope of a^x at $x = 0$ should equal 1. We will use brute force and admittedly prior knowledge to try to find that special value of a to get a slope closer and closer to 1. We begin by defining a function F that produces the slope of the line that joins the two points $P(-x, a^{-x})$ and $Q(x, a^x)$ spaced equally from $x = 0$. As x gets smaller, these two points get closer together and the slope determined should be closer to that of a line tangent to the graph of $y = a^x$ for $x = 0$.

Q(x,f(x))

P(x,f(-x))

x
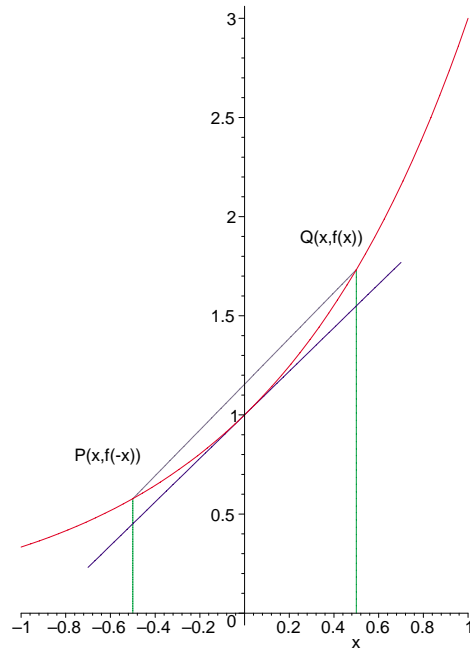
The `seq` command produces a sequence of values of F with $x = \frac{1}{2^i}$ for $i = 3$ to $i = 10$.

It is easier to obtain the lines below than it looks. Once you have entered the command line where `a:=0`, simply copy and paste to lines below, change the value of a, and hit $<$Enter$>$.

```
> F:=x->(a^x-a^(-x))/(2*x);
```

$$F := x \longrightarrow \frac{1}{2}\frac{a^x - a^{-x}}{x}$$

```
> a:=2;  seq(evalf(F(1/2^i)),i=3..10);
```

$$a := 2$$

$$.694014760, .693364012, .69320139, .69316073, .69315056, .6931480, .6931474, .6931474$$

```
> a:=3;  seq(evalf(F(1/2^i)),i=3..10);
```

$$a := 3$$

$$1.102068589, 1.099475751, 1.09882812, 1.09866622, 1.09862576, 1.0986156, 1.0986131, 1.0986123$$

```
> a:=2.7;  seq(evalf(F(1/2^i)),i=3..10);
```

$$a := 2.7$$

$$.9958055410, .9938898465, .9934112640, .9932916545, .9932617665, .9932542335, .9932524545, .9932519425$$

```
> a:=2.7183;  seq(evalf(F(1/2^i)),i=3..10);
```

$$a := 2.7183$$

$$1.002612939, 1.000657870, 1.000169456, 1.000047386, 1.000016870, 1.000009280, 1.000007398, 1.000006810$$

```
> a:=2.718281828;  seq(evalf(F(1/2^i)),i=3..10);
```

$$a := 2.718281828$$

$$1.002606202, 1.000651170, 1.000162760, 1.000040704, 1.000010157, 1.000002496, 1.000000742, .9999998975$$

```
> a:=exp(1);  seq(evalf(F(1/2^i)),i=3..10);
```

$$a := e$$

$$1.002606202, 1.000651170, 1.00016276, 1.00004071, 1.00001016, 1.0000025, 1.0000007, .9999999$$

In the **C1M0 − Introduction to Maple**, we showed e or `exp(1)` to 90 places. There are several ways to define e, but for now it suffices to regard it as that number a for which $y = a^x$ has a slope of 1 at $x = 0$. The function $y = e^x$ occurs frequently in growth and decay problems and its importance to Calculus and the Sciences cannot be overstated. Frequently students underestimate how rapidly e^x grows as x increases. For this reason and the fact that $e^3 = 20.08553692\ldots$ we provide this 'Rule of Thumb':
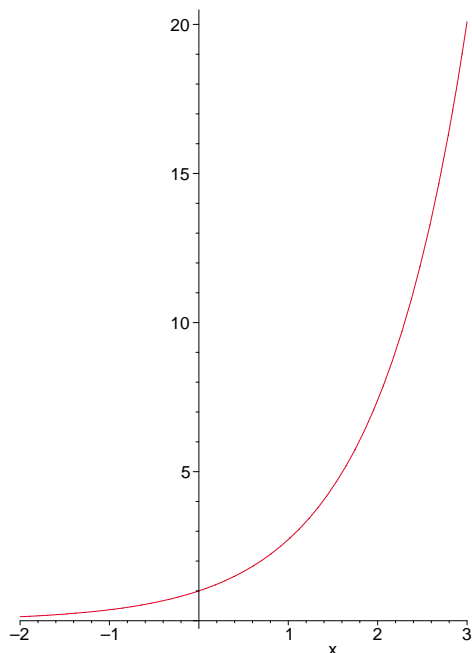
$$e^3 \approx 20$$

Think about it! This means that $e^6 = (e^3)^2$ must be over 400 and e^9 must be over 8000. After we provide the following reminder, we will plot e^x.

♠ The exponential function, e^x, is accessed by `exp(x)` in Maple, which does not recognize e as any

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particular number.

```
>  plot(exp(x),x=-2..3);
```

Maple Example: It is known that radioactive materials satisfy an equation of the form $A = A_0 e^{kt}$ where A_0 is the initial amount and k is a negative constant. Suppose we have 2.837 grams of an unstable compound whose half-life is 137 days. How much will be present after 97 days and when will we have exactly 1 gram remaining?

First, we must find the value of k, using the half-life of the compound.

```
>  A0:=2.837;
```
$$A0 := 2.837$$

```
>  A:=A0*exp(k*t);
```
$$A := 2.837 e^{(kt)}$$

When $t = 137$, the amount present is one-half the original amount. We set this up as an equation, which we can solve for k.

```
>  eq:=A0/2=subs(t=137,A);
```
$$eq := 1.418500000 = 2.837 e^{(137k)}$$

```
>  k:=solve(eq,k);
```
$$k := -.005059468471$$

```
>  A;
```
$$2.837 e^{(-.005059468471t)}$$

We see that Maple is using the value of k. Substituting $t = 97$ we find the amount at that time.

```
>  A97:=subs(t=97,A);
```
$$A97 := 2.837 e^{(-.4907684417)}$$

```
>  A97:=evalf(A97);
```
$$A97 := 1.736686025$$

That gives the amount at 97 days. Now to find when there is one gram we must set the amount equal to 1, and solve for t.

```
>  eq1:=1=A;
```
$$eq1 := 1 = 2.837 e^{(-.005059468471t)}$$

```
>  solve(eq1,t);
```
$$206.0981626$$

C1M3 Problems: Use Maple to solve the following problems.

1. Plot the graph of the expression A in the last example for $0 \leq t \leq 250$.

2. Plot e^x and $1 + x + x^2/2 + x^3/6$ on the same coordinate axes for $-3 \leq x \leq 3$.

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3. The command `evalf(solve(2+x=exp(x)));` finds all solutions of the equation $2 + x = e^x$. Use a similar command to find all solutions of $2x^2 = e^x$.

4. Plot $2x^2$ and e^x on the same coordinate axes for $-1 \leq x \leq 3$.

5. The data shown may be related by an exponential function. Determine a function that fits this data.

Time	Reading
1.2	6.292161022
1.35	6.783009413
1.5	7.312148644
1.65	7.882565768
1.8	8.497480852
1.95	9.160365159