C2M2

Rational Fractions or Partial Fraction Decompositions

When you were learning basic algebra there were probably problems assigned on adding fractions which had constants, linear expressions, and quadratic expressions in x in the numerators and denominators. Our objective here is to take the answers to those questions and find the fractions that were added together. The need here is to break down a complicated fraction into several simple ones whose anti-derivatives are (much!) easier to find. To approach this systematically we separate the problems into groups determined by the nature of the denominators of the fractions. Recall that $P(x) = x^2 - a^2$ can be factored into $P(x) = (x-a)(x+a)$, so we say that $P(x)$ is *reducible*. Likewise, $Q(x) = x^2 + a^2$ can **NOT** be factored, so $Q(x)$ is *irreducible*. There are different approaches to to these problems and we will use substitution as our method. Two principles from algebra are applicable here. The first states that if two polynomials in x agree for all values of x, then the polynomials have the same coefficients, that is, they are identical. The second states that $x-r$ is a factor of the polynomial $P(x)$ if and only if $P(r) = 0$.

In the following, $P(x)$ is a polynomial in x and the degree of P is less than the degree of the denominator. For cases where the degree of P is greater than or equal to the degree of the denominator, one must divide the polynomials and then work with the remainder. Note that we are forcing the left and right sides to agree for all x, so we may use $\equiv$ instead of $=$.

I. The denominator has distinct linear factors. The expression on the left breaks down as shown:

$$\frac{P(x)}{(x-r_1)(x-r_2)\cdots(x-r_n)} \equiv \frac{a_1}{x-r_1} + \frac{a_2}{x-r_2} + \cdots + \frac{a_n}{x-r_n}$$

and the problem is reduced to determining the coefficients $a_1, a_2, \ldots, a_n$. We are assuming that no two factors are the same.

Example: Decompose $\dfrac{7x-11}{(x-1)(x+1)(x-2)}$. Each factor occurs with a constant numerator.

$$\frac{7x-11}{(x-1)(x+1)(x-2)} = \frac{a}{x-1} + \frac{b}{x+1} + \frac{c}{x-2} \qquad \text{multiply through by the LHS denominator}$$

$$\begin{aligned}
7x-11 &= a(x+1)(x-2) + b(x-1)(x-2) + c(x-1)(x+1) && \text{substitute } x=1\\
-4 &= a(2)(-1) && \text{so } a=2 \quad \text{substitute } x=-1\\
-18 &= b(-2)(-3) && \text{so } b=-3 \quad \text{substitute } x=2\\
3 &= c(1)(3) && \text{so } c=1
\end{aligned}$$

From this we conclude:

$$\frac{7x-11}{(x-1)(x+1)(x-2)} = \frac{2}{x-1} - \frac{3}{x+1} + \frac{1}{x-2} \text{ for all values of } x \text{ except } 1,\ -1, \text{ and } 2.$$

II. The denominator has linear factors with repeat(s). In this case, each factor occurs as a denominator up to the power to which it occurs in the original denominator. It is easiest to explain with an example. Suppose the original denominator is $(x-1)^3(x+2)^2(x-4)$ then the decomposition will look like:

$$\frac{a}{x-1} + \frac{b}{(x-1)^2} + \frac{c}{(x-1)^3} + \frac{d}{x+2} + \frac{e}{(x+2)^2} + \frac{f}{x-4}$$

Example: Decompose $\dfrac{3x^2-7x+1}{(x-1)^2(x+2)}$.

$$\frac{3x^2-7x+1}{(x-1)^2(x+2)} = \frac{a}{x-1} + \frac{b}{(x-1)^2} + \frac{c}{x+2} \qquad \text{multiply by LHS denominator}$$

$$\begin{aligned}
3x^2-7x+1 &= a(x-1)(x+2) + b(x+2) + c(x-1)^2 && \text{substitute } x=1\\
-3 &= b(3) && \text{so } b=-1, \text{ substitute that value and simplify}\\
3x^2-7x+1 &= a(x-1)(x+2) - (x+2) + c(x-1)^2 && \text{move } (x+2) \text{ to the other side}\\
3x^2-6x+3 &= a(x-1)(x+2) + c(x-1)^2 && \text{both sides must be divisible by } x-1, \text{ so divide}\\
3x-3 &= a(x+2) + c(x-1) && \text{use } x=1 \text{ again}\\
0 &= 3a && \text{so } a=0 \quad \text{use } x=-2\\
-9 &= c(-3) && \text{so } c=3
\end{aligned}$$

And we conclude:

$$\frac{3x^2-7x+1}{(x-1)^2(x+2)} = \frac{-1}{(x-1)^2} - \frac{3}{x+2}$$

III. The denominator has an irreducible quadratic factor. A quadratic factor requires a linear numerator such as $ax+b$.

Example: Decompose $\dfrac{x^2+5}{(x-1)(x^2+2)}$.

$$\begin{aligned}
\frac{x^2+5}{(x-1)(x^2+2)} &= \frac{a}{x-1} + \frac{bx+c}{x^2+2} && \text{multiply by LHS denominator}\\
x^2+5 &= a(x^2+2)+(bx+c)(x-1) && \text{substitute } x=1\\
6 &= a(3) && \text{so } a=2, \text{ substitute that value and simplify}\\
x^2+5 &= 2x^2+4+(bx+c)(x-1) &&\\
-x^2+1 &= (bx+c)(x-1) && \text{both sides must be divisible by } x-1, \text{ so divide}\\
-x-1 &= bx+c && b=-1,\ c=-1
\end{aligned}$$

Our decomposition is:

$$\frac{x^2+5}{(x-1)(x^2+2)} = \frac{2}{x-1} - \frac{x+1}{x^2+2}$$

IV. The denominator has a repeated linear factor and an irreducible quadratic factor. The method illustrated here is the easiest one for this case. **This type of problem will be on the Final Exam**.

Example: Decompose $\dfrac{8x}{(x-1)^2\,(x^2+3)}$.

$$\begin{aligned}
\frac{8x}{(x-1)^2\,(x^2+3)} &= \frac{a}{x-1} + \frac{b}{(x-1)^2} + \frac{cx+d}{x^2+3} && \text{multiply by LHS denominator}\\
8x &= a(x-1)\left(x^2+3\right)+b\left(x^2+3\right)+(cx+d)(x-1)^2 && \text{substitute } x=1\\
8 &= 4b && \text{so } b=2, \text{ substitute that value}\\
8x &= a(x-1)\left(x^2+3\right)+2x^2+6+(cx+d)(x-1)^2 && \text{move } 2x^2+6 \text{ to the LHS}\\
-2x^2+8x-6 &= a(x-1)\left(x^2+3\right)+(cx+d)(x-1)^2 && x-1 \text{ is a factor of RHS and LHS, } \div\\
-2(x-3) &= a\left(x^2+3\right)+(cx+d)(x-1) && \text{substitute } x=1\\
4 &= 4a && \text{so } a=1, \text{ substitute that value}\\
-2x+6 &= x^2+3+(cx+d)(x-1) && \text{move } x^2+3 \text{ to the LHS}\\
-x^2-2x+3 &= (cx+d)(x-1) && \text{divide by } x-1\\
-x-3 &= cx+d && \text{which forces } c=-1 \text{ and } d=-3\\
\frac{8x}{(x-1)^2\,(x^2+3)} &= \frac{1}{x-1} + \frac{2}{(x-1)^2} - \frac{x+3}{x^2+3}
\end{aligned}$$

Now observe how Maple would have done the problem.

Maple Example:

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> with(student):
> convert(8*x/((x-1)^2*(x^2+3)),parfrac,x);
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$$\frac{2}{(x-1)^2} + \frac{1}{x-1} - \frac{3+x}{x^2+3}$$

C2M2 Problems

Find the partial fraction decompositions of the expressions using pencil and paper. Then, check your answers using Maple as was done in the preceding Maple Example. Do **NOT** use Maple to duplicate your pencil and paper work.

1. $\dfrac{8x+5}{(x+1)^2\left(x^2+2\right)}$

2. $\dfrac{4x^2+4x+12}{x^2\left(x^2+4\right)}$

3. $\dfrac{7x^2-17x+1}{(x-2)^2\left(x^2+1\right)}$

4. $\dfrac{x^3-2x^2+2x-1}{x^3\left(x^2+1\right)}$