C2M13

Maclaurin and Taylor Series

It is remarkable that knowing about the values of a function and its derivatives at a point provides a means of evaluating the function at points nearby. Maclaurin and Taylor series are that means. Taylor and Maclaurin series are written respectively as

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n\,!}(x-c)^n \qquad\qquad f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n\,!}x^n$$

where, by letting $c = 0$, we see that the Maclaurin series is a special case of the Taylor series. The reader is reminded that

$$\begin{aligned} 0! &= 1 \\ 1! &= 1 \cdot 0! = 1 \\ 2! &= 2 \cdot 1! = 2 \\ 3! &= 3 \cdot 2! = 6 \\ 4! &= 4 \cdot 3! = 24 \\ 5! &= 5 \cdot 4! = 120 \\ 6! &= 6 \cdot 5! = 720 \\ 7! &= 7 \cdot 6! = 5040 \\ 8! &= 8 \cdot 7! = 40320 \\ 9! &= 9 \cdot 8! = 362880 \\ 10! &= 10 \cdot 9! = 3628800 \end{aligned}$$

Just for fun, in a Maple worksheet enter `357!;`. The speed with which this computation is done is remarkable.

Frequently it is useful to write out the first few terms of a Taylor series. The result is a *Taylor Polynomial*. For example,

$$T_n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

The Maple syntax for a Taylor polynomial of degree n at $x = a$ is:

```
> taylor(f(x),x=a,n);
```

Every student should grind out a few Taylor series by hand so that they appreciate how the coefficients are determined. But, we can use Maple spreadsheets to accomplish the same thing. A worksheet entitled **Taylor Series Worksheet** follows that does exactly that. We will explain a few subtleties of the 'how's and why's' of this worksheet now.

Begin by defining the function $f(x) = \cos(2x)$ and identifying $a = \pi/6$. Recall that to open the spreadsheet you must click on **Insert** and the **Spreadsheet**. Resize it so that it shows about 10 rows and 4 columns. Enter n in cell **A1**, *function* in cell **B1** and $x = a$ in cell **C1**. Put 0 in **A2**, $f(x)$ in **B2**, and then `eval(~B2,x=a)` in **C2**. Continue by highlighting from **A2** down to **A8**, click on the button on the third row of the menus at the extreme left, insert a step size of 1, and then click **OK**. Move to cell **B3**. Enter `diff(~B2,x)` and then highlight that cell down to **B8**. Click on **Spreadsheet**, **Fill**, and **Down** and the successive derivatives should appear. Highlight from **C2** down to **C8**, and click on **Spreadsheet, Fill, Down** as before. Now the spreadsheet should be complete.

Continue by highlighting cells **C2** down to **C8**, clicking on the **Copy** button, clicking on the command line just below the spreadsheet, and then clicking on the **Paste** button. Important: put a colon at the end of MATRIX material and <Enter>. Convert the matrix to a list as shown, and then note that we have a list of a list of small matrices. So, we must look only at the first entry of our list which is `c[1]`. We want to formulate our coefficients for the Taylor series without the matrix brackets and this is done in our definition of the function b. Check out the numerator of the fraction. We have `c[n+1]` as the n^{th} matrix in c if we start with 0. That matrix has one entry, and to access that number we use `(c[n+1])[1]`.

The coefficients for the Taylor polynomial, T_6, are $b(0), b(1), b(2), b(3), b(4), b(5), b(6)$, which we enter as $b(n)$ in the summation. We used $x - a$ which is easier than $x - Pi/6$. Then, let Maple do the same thing in one line. Compare the coefficients.

Taylor Series Worksheet

```
> restart:    with(student):

> f:=x->cos(2*x);
```

$$f := x \to \cos(2x)$$

```
> a:=Pi/6:
```

	A	B	C
1	n	*function*	$x = \frac{1}{6}\pi$
2	0	$\cos(2x)$	$\frac{1}{2}$
3	1	$-2\sin(2x)$	$-\sqrt{3}$
4	2	$-4\cos(2x)$	-2
5	3	$8\sin(2x)$	$4\sqrt{3}$
6	4	$16\cos(2x)$	8
7	5	$-32\sin(2x)$	$-16\sqrt{3}$
8	6	$-64\cos(2x)$	-32

```
> MATRIX([[1/2], [-sqrt(3)], [-2], [4*sqrt(3)], [8],[-16*sqrt(3)], [-32]]):

> c:=convert(%,list);
```

$$c := \left[\left[\left[\frac{1}{2}\right], [-\sqrt{3}], [-2], [4\sqrt{3}], [8], [-16\sqrt{3}], [-32]\right]\right]$$

```
> c:=c[1];
```

$$c := \left[\left[\frac{1}{2}\right], [-\sqrt{3}], [-2], [4\sqrt{3}], [8], [-16\sqrt{3}], [-32]\right]$$

```
> b:=n->(c[n+1])[1]/n!;
```

$$b := n- > \frac{c_{n+1_1}}{n!}$$

```
> T6:=sum(b(n)*(x-a)^n,n=0..6);
```

$$T6 := \frac{1}{2} - \sqrt{3}\left(x - \frac{1}{6}\pi\right) - \left(x - \frac{1}{6}\pi\right)^2 + \frac{2}{3}\sqrt{3}\left(x - \frac{1}{6}\pi\right)^3 + \frac{1}{3}\left(x - \frac{1}{6}\pi\right)^4 - \frac{2}{15}\sqrt{3}\left(x - \frac{1}{6}\pi\right)^5 - \frac{2}{45}\left(x - \frac{1}{6}\pi\right)^6$$

```
> taylor(f(x),x=a,7);
```

$$\frac{1}{2} - \sqrt{3}\left(x - \frac{1}{6}\pi\right) - \left(x - \frac{1}{6}\pi\right)^2 + \frac{2}{3}\sqrt{3}\left(x - \frac{1}{6}\pi\right)^3 + \frac{1}{3}\left(x - \frac{1}{6}\pi\right)^4 - \frac{2}{15}\sqrt{3}\left(x - \frac{1}{6}\pi\right)^5 - \frac{2}{45}\left(x - \frac{1}{6}\pi\right)^6 + O\left(\left(x - \frac{1}{6}\pi\right)^7\right)$$

C2M13 Problem: Use Maple and the method illustrated above to find a Taylor polynomial, T_6, for $f(x) = \arctan(x)$ at $a = 1$. Your work should display the spreadsheet, T_6, and the Maple solution.