C1M14

Integrals as Area Accumulators

Most textbooks do a good job of developing the integral and this is not the place to provide that development. We will show how Maple presents Riemann Sums and the accompanying diagrams and then focus on integrals from a to x. We hear about "moving the goalpost" when standards of performance are raised, but here that is exactly what is happening because x is a variable. We will be focusing on functions that are bounded on their domain, which will be an interval $[a, b]$, and are continuous at all except possibly a finite number of points. For this reason, when we break $[a, b]$ up into n subintervals we may assume that the subintervals all have the same length, $\dfrac{b-a}{n} = \Delta x$. In more general situations, we allow the subintervals to be of random length and then force the largest interval to get small as a means of controlling the approximation in the limit which we will call a definite integral. Let's break all this down into small parts and then assemble them in a useful way.

1. Suppose that f is a bounded function on $[a, b]$, f is continuous at all but at most a finite number of points, and n is a positive integer.

2. Let $\Delta x = \dfrac{b-a}{n}$. Then there are $n+1$ points $\{x_i\}_0^n$ determined by
$x_0 = a = a + 0 \cdot \Delta x,\ \ x_1 = a + 1 \cdot \Delta x,\ \ x_2 = a + 2 \cdot \Delta x, \ldots,\ \ x_i = a + i \cdot \Delta x, \ldots,\ \ x_n = a + n \cdot \Delta x = b$

3. In each subinterval $[x_{i-1}, x_i]$, a value t_i is selected. Obviously $x_{i-1} \le t_i \le x_i$.

4. Intuitively, we think of $f(t_i) \cdot \Delta x$ as the area of a rectangle of height $f(t_i)$ and base with width Δx placed above the interval $[x_{i-1}, x_i]$.

5. The value $\displaystyle\sum_{n=1}^{\infty} f(t_i) \cdot \Delta x$ is called a *Riemann Sum* and its value provides an approximation to the signed area between $y = f(x)$ and the x-axis.

6. We write $\displaystyle\lim_{n\to\infty} \sum_{n=1}^{\infty} f(t_i) \cdot \Delta x \ = \int_a^b f(x)\, dx$

7. On subintervals where $f(x) < 0$, there will be a negative contribution, so we must be careful how we use the phrase "the integral of f is the area beneath the curve". It makes sense for *positive* f only.

Maple Examples: Maple has three plotting commands `leftbox, middlebox, rightbox` that illustrate Riemann sums. It also has three numerical commands that compute Riemann sums, `leftsum, middlesum, rightsum`. We will illustrate their use for x^2 on the interval $[1, 3]$ using 13 subintervals. These commands are all in the package `student`.

```
> restart:    with(plots):    with(student):
> leftsum(x^2,x=1..3,13);
```

$$\frac{2}{13}\left(\sum_{i=0}^{12}\left(1+\frac{2}{13}i\right)^2\right)$$

```
> value(leftsum(x^2,x=1..3,13));
```

$$\frac{1362}{169}$$

```
> evalf(leftsum(x^2,x=1..3,13));
```

$$8.059171595$$

```
> rightsum(x^2,x=1..3,13);
```

$$\frac{2}{13}\left(\sum_{i=1}^{13}\left(1+\frac{2}{13}i\right)^2\right)$$

```
> value(rightsum(x^2,x=1..3,13));
```

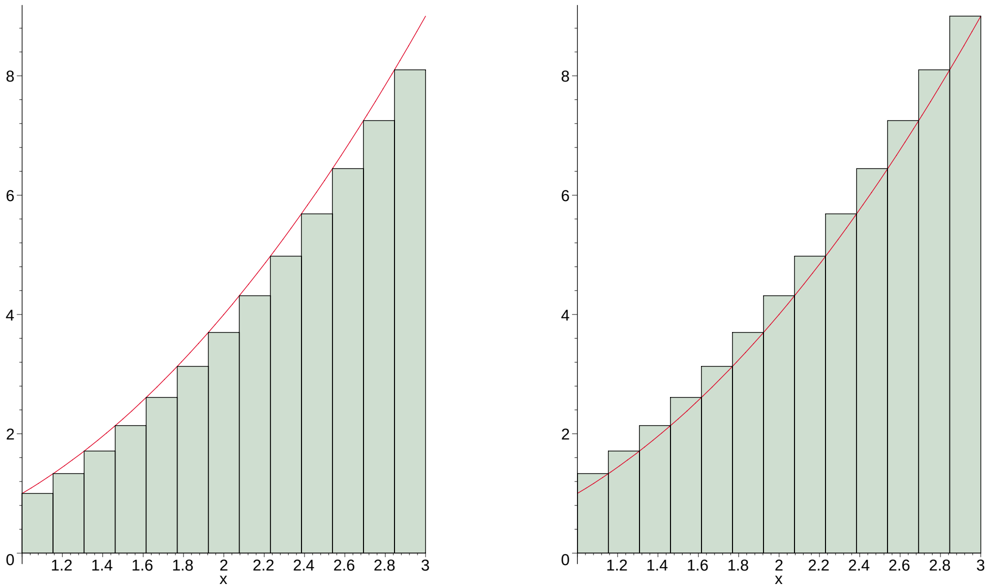
$$\frac{1570}{169}$$

```
> evalf(rightsum(x^2,x=1..3,13));
```

$$9.289940825$$

Leftbox is on the left and rightbox is on the right.

```
> leftbox(x^2,x=1..3,13);
```



Maple Animation Example: We will use the function $f(x) = x\,\sin(\pi x)$ on $[0,3]$ and set up an animation of the approximation of $\int_0^3 x\,\sin(\pi x)\,dx$ using `middlebox`. The reader is **urged** to type in the commands below or to copy and paste, and to watch the animation. The value of each Riemann sum using the midpoint of the subinterval is shown near the top of the display.

```
> restart:    with(plots):
> f:=x->x*sin(Pi*x);
```

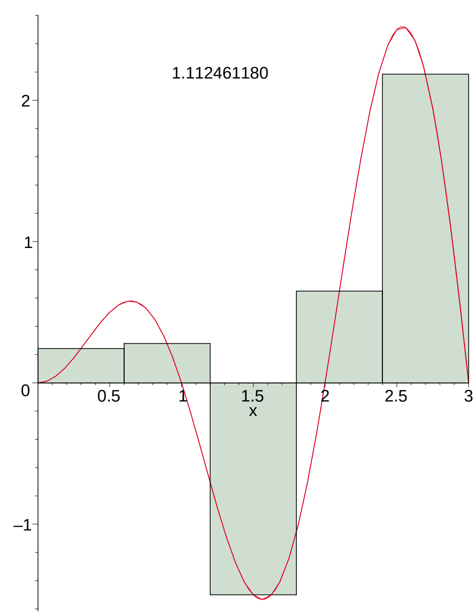
$$f := x \to x\,\sin(\pi x)$$

```
> nstart:=5;    frameno:=50;
```

$$nstart := 5$$

$$frameno := 50$$

```
> framenumbers:=[seq(nstart+i,i=0..(frameno-1))]:
> A:=display(seq(middlebox(f(x),x=0..3,i),i=framenumbers),insequence=true):
> B:=animate(f(x),x=0..3,y=-2..3,color=red, frames=frameno):
> C:=display(seq(textplot([1.3,2.2,evalf(middlesum(f(x),x=0..3,i))]),i=5..(frameno+4)),
  insequence=true):
> display(A,B,C);
```



Let's look at the last approximation and the actual answer in floating point. By using uppercase "I" in `Int` on the left we obtain an inert integral, while the lowercase "i" yields an active command "integrate it" on the right in `int`.

```
> evalf(middlesum(f(x),x=0..3,frameno+4));
```

$$.9561427706$$

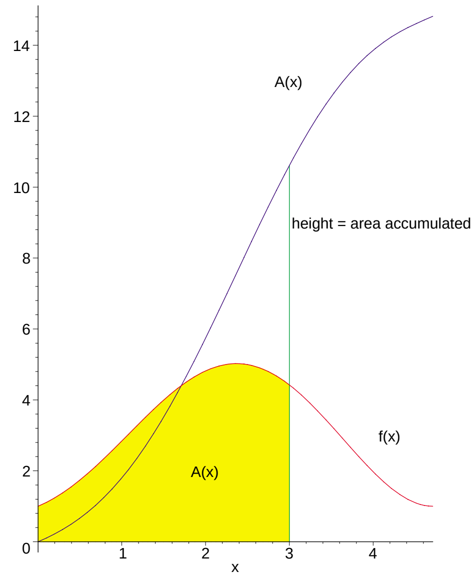
```
> Int(f(x),x=0..3)=evalf(int(f(x),x=0..3));
```

$$\int_0^3 x\,\sin(\pi x)\,dx = .9549296583$$

Play the animation! Click on the display. A box will appear around the figure and a "tape player" window will appear in the *context bar*. Click on the "Play" button and watch!

♠ It is important to note that Maple deals with definite integrals by simply including the range. For example, the value of $\int_a^b f(x)\,dx$ results from `int(f(x),x=a..b)`.

Now we will turn our attention to $f(x)$ and $A(x) = \displaystyle\int_a^x f(t)\,dt$. This integral intuitively 'accumulates area' as x moves from left to right.



We will state the first part of the **Fundamental Theorem of Calculus** and then illustrate it using a function defined in a piecewise manner.

The Fundamental Theorem of Calculus (Part One): Assume that f is continuous on $[a, b]$ and that the function A is defined by $A(x) = \displaystyle\int_a^x f(t)\,dt$ for $a \le t \le b$. Then, $A'(x) = f(x)$ for all x in (a, b). In other words, A is *an* antiderivative for f.

Sometimes we write

$$D_x\left(\int_a^x f(t)\,dt\right) = f(x)$$

Maple Example: Suppose that f is defined by:

$$f(x) = \begin{cases} 1 - x, & \text{if } x \le 1 \\ \ln(x), & \text{if } 1 < x < e \\ (x - e - 1)^2, & \text{if } e \le x \end{cases}$$

We will plot the graph of $f(x)$ and of $A(x) = \displaystyle\int_0^x f(t)\,dt$, which of course is the area accumulator function for $f(x)$. We will note that f is continuous, but is not differentiable at $x = 1$ and $x = e$. It will be important to observe that the graph of $A(x)$ is smooth at all points, indicating that A is differentiable everywhere, as it should be.

```
> restart:    with(plots):  with(student):    with(plottools):
> e:=exp(1);
```

$$e := \mathbf{e}$$

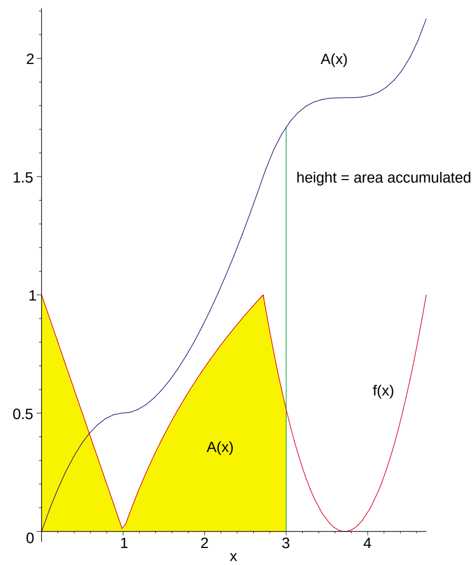
```
> f:=x->piecewise(x<=1,1-x,x>1 and x<e,ln(x),x>=e,(x-e-1)^2);
```

$$f := x \to \operatorname{piecewise}(x \le 1, 1-x, 1 < x \textbf{ and } x < e, \ln(x), e \le x, (x-e-1)^2)$$

```
> A:=x->int(f(t),t=0..x);
```

$$A := x \to \int_0^x f(t)\,dt$$

```
> A1:=plot(f(x),x=0..(e+2),color=red):
> A2:=plot(A(x),x=0..(e+2),color=blue):
> A3:=plot(f(x),x=0..3,color=yellow,filled=true):
> A4:=line([3,0],[3,A(3)],color=green):
> A5:=textplot([3.6,2,"A(x)"]):
> display(A1,A2,A3,A4,A5);
```



Without fanfare we will state the rest of the Fundamental Theorem of Integral Calculus. The same hypotheses still apply.

Fundamental Theorem of Calculus (Part Two): If F is any antiderivative of f, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Suppose that f is continuous on $[a,b]$ and that $F(x) = \displaystyle\int_a^x f(t)\,dt$. If we apply the Mean Value Theorem to F on $[a,b]$, then there is a point c in (a,b) for which

$$\frac{F(b)-F(a)}{b-a} = F'(c)$$

$$\frac{\int_a^b f(t)\,dt - \int_a^a f(t)\,dt}{b-a} = f(c)$$

$$\frac{1}{b-a}\int_a^b f(t)\,dt = f(c)$$

This last value is sometimes called the *average value of* $f(x)$ *over* $[a,b]$. We know that $F' = f$ and $\int_a^a g(x)\,dx = 0$ for any function g and these facts were used above. We summarize this as the

Mean Value Theorem for Integrals: If f is continuous on $[a,b]$, then there is a number c in $[a,b]$ such that $\displaystyle\int_a^b f(x)\,dx = f(c)(b-a)$.

Maple Example: Find a value c that satisfies the MVT for integrals for $f(x) = 3\,e^{-x} + x\,\sin(\pi x)$ on the interval $[0,3]$ and display a graph that illustrates this theorem.

```
> restart:    with(plots):    with(plottools):
> f:=x->3*exp(-x)+x*sin(Pi*x);
```

$$f := x \to 3\,\mathbf{e}^{-x} + x\,\sin(\pi x)$$

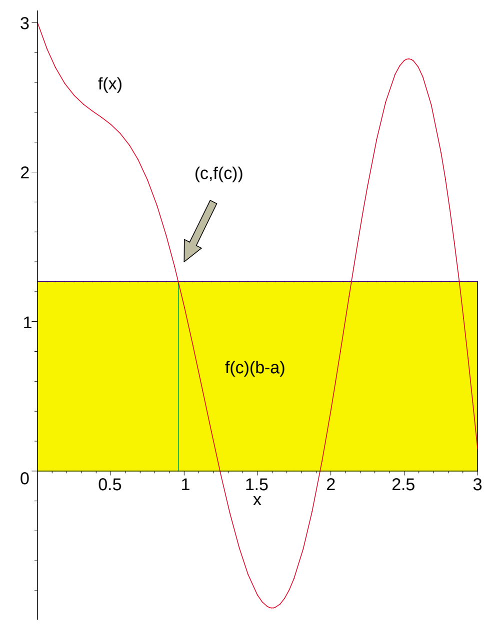
```
> A:=int(f(x),x=0..3);
```

$$A := -3\,\frac{(\mathbf{e}^{(-3)}\pi - 1 - \pi)}{\pi}$$

```
> c:=fsolve(f(x)=A/3,x,0..3);
```

$$c := .9601184662$$

```
> A1:=plot(f(x),x=0..3,color=red):
> A2:=plot(f(c),x=0..3,color=blue):
> A3:=plot(f(c),x=0..3,color=yellow,filled=true):
> A4:=line([c,0],[c,f(c)],color=green):
> A5:=textplot([.5,2.6,"f(x)"]):
> A6:=line([3,0],[3,f(c)],color=black):
> A7:=arrow([1.2,1.8],[1,1.4],.05,.13,.3,color=khaki):
> A8:=textplot([1.25,2,"(c,f(c))"]):
> A9:=textplot([1.5,.7,"f(c)(b-a)"]):
> display(A1,A2,A3,A4,A5,A6,A7,A8,A9);
```



C1M14 Problems: Use Maple to solve the problems and plot the graphs.

1. For $f(x) = 3\,e^{-x}\,sin(x)$ on $[0, \pi]$, evaluate (use `evalf`) and display graphically the left, right, and middle sums with 47 subintervals. Remember that the commands you will need are in `student`.

2. Define $g(x) = \dfrac{1}{x}$ for x in $[1/4, 5]$. Then define $G(x) = \int_1^x g(t)\,dt$. This is how $\ln(x)$ is defined in some textbooks when exponentials and logarithms are delayed until after the integral has been developed. Display g and G on the same graph and fill the graph below $g(x)$ from 1 to 3.

3. For $f(x) = x\,\sin(x^2)$ on $[0, \pi]$, use Maple to find the average value of f on this interval and display a graph that illustrates the Mean Value Theorem for Integrals.