Review of Differentiation and Anti-differentiation

As we begin Calculus II it is assumed that the student has covered the derivative of a function in some detail and has learned how to find the anti-derivative of basic functions. This section is provided as a summary of some of these topics. Knowledge of trigonometric functions, exponential functions, and logarithmic functions is assumed, but will be reviewed as necessary throughout the course for reinforcement.

Definition of derivative. Suppose f is defined on an open interval containing x. The derivative of f at x is defined by

$$D_x\left(f(x)\right) = f'(x) \equiv \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

Tangent line. If $f(x_0) = y_0$ and $f'(x_0) = m$, then an equation for the line tangent to the curve $y = f(x)$ is given by

$$y - y_0 = m(x - x_0)$$

Rules of Differentiation Assume that a and b are real numbers and that $f(x)$ and $g(x)$ are differentiable on an open interval containing x.

Rule 1. The derivative is linear. That is, $D_x\left[af(x) + bg(x)\right] = aD_x(f(x)) + bD_x(g(x)) = af'(x) + bg'(x)$.

Rule 2. Product Rule. $D_x\left[f(x)g(x)\right] = D_x(f(x))g(x) + f(x)D_x(g(x)) = f'(x)g(x) + f(x)g'(x)$.

Rule 3. Quotient Rule. On an interval where $g(x) \neq 0$,

$$D_x\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)D_x\left[f(x)\right] - f(x)D_x\left[g(x)\right]}{\left[g(x)\right]^2} = \frac{g(x)f'(x) - f(x)g'(x)}{\left[g(x)\right]^2}$$

Rule 4. Power function derivative. If r is a real number, then

$$D_x(x^r) = rx^{r-1}$$

Examples: $D_x\left[x^{4/3}\right] = \frac{4}{3}x^{1/3}$, $\quad D_x\left[\frac{1}{x^{4/3}}\right] = -\frac{4}{3}\frac{1}{x^{7/3}}$, $\quad D_x\left[\sqrt{x}\right] = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$

Rule 5. Chain Rule. On an open interval for which $(f \circ g)(x) \equiv f\left[g(x)\right]$ is defined

$$D_x\left[(f \circ g)(x)\right] = f'(g(x))g'(x)$$

Example: $D_x\left[4 + x^3\right]^5 = (5)\left[4 + x^3\right]^4(3x^2)$

Rule 6. Reciprocal Rule. On an interval where $f(x) \neq 0$ we have

$$D_x\left[\frac{1}{f(x)}\right] = \frac{-f'(x)}{\left[f(x)\right]^2}$$

Derivatives of trigonometric functions. First, we list the derivatives of the three basic functions.

$$D_x(\sin x) = \cos x \qquad D_x(\tan x) = \sec^2 x \qquad D_x(\sec x) = (\sec x)(\tan x)$$

Then we use this information to help determine the derivatives of the cofunctions.

Function	Derivative	Cofunction	Derivative
$\sin x$	$\cos x$	$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$	$\cot x$	$-\csc^2 x$
$\sec x$	$\sec x \tan x$	$\csc x$	$-\csc x \cot x$

Note that the first column is simply the three basic functions. The second column is the derivative of the first column. The third column is a listing of the cofunctions of the first column. And to form the fourth column, begin by putting a minus sign in front of the expression to follow. Then, simply put the respective cofunctions of the second column after the minus sign.

Example: $D_x\left[\tan^3(\sqrt{x})\right] = 3\tan^2(\sqrt{x})\sec^2(\sqrt{x})\dfrac{1}{2\sqrt{x}}$.

Derivative of exponential and logarithmic functions. $D_x\left[e^x\right] = e^x$ and $D_x(\ln|x|) = \dfrac{1}{x}$, $x \neq 0$. Also, for $a > 0$, $D_x\left[a^x\right] = (\ln a)a^x$.

We will limit our discussion on inverse trigonometric functions to

$$\sin^{-1} x \equiv \arcsin x \qquad \text{and} \qquad \tan^{-1} x \equiv \arctan x$$

We remind the reader that the exponents refer to inverse functions and **not** to reciprocals. The derivatives and corresponding anti-derivatives we will need are listed below:

$$D_x(\arcsin x) = \frac{1}{\sqrt{1-x^2}} \qquad \int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C$$

$$D_x(\arctan x) = \frac{1}{1+x^2} \qquad \int \frac{1}{1+x^2}\,dx = \arctan x + C$$

Now we turn our attention to anti-derivatives. The easiest place to start is with the power rule.

Anti-derivative of x^r. If $r \neq -1$, then $\int x^r\,dx = \dfrac{1}{r+1}x^{r+1} + C$.

The anti-derivatives of the basic trigonometric functions are found in the following table:

Function	Anti-derivative	Cofunction	Anti-derivative
$\sin x$	$-\cos x + C$	$\cos x$	$\sin x + C$
$\tan x$	$\ln\lvert\sec x\rvert + C$	$\cot x$	$\ln\lvert\sin x\rvert + C$
$\sec x$	$\ln\lvert\sec x + \tan x\rvert + C$	$\csc x$	$\ln\lvert\csc x - \cot x\rvert + C$

There are other basic trigonometric anti-derivatives that are easily found by reversing the differentiation formulas.

Other trigonometric anti-derivatives.

$$\int \sec^2 x\,dx = \tan x + C \qquad \int (\sec x)(\tan x)\,dx = \sec x + C$$

$$\int \csc^2 x\,dx = -\cot x + C \qquad \int (\csc x)(\cot x)\,dx = -\csc x + C$$

Anti-derivatives of exponential functions. It is assumed that $a > 0$.

$$\int e^x\,dx = e^x + C \qquad \int a^x\,dx = \frac{1}{\ln a}a^x + C$$

Review Exercises: Find derivatives of the following: 1. $x^2\cos x$ 2. $\dfrac{\ln x}{x^2+4}$ 3. $e^{3x}\tan(2x)$

4. Find an equation for the line tangent to the curve $y = \dfrac{x+3}{4+x^2}$ at the point where $x = 2$.