

## Limits Involving Infinity

One way of categorizing the involvement of infinity in this discussion is to refer to *unbounded domains* and to *unbounded functions*. Let's deal with unbounded domains first. The basic question here is to see if the values of a function,  $f(x)$ , are getting close to a single value as we let  $x$  get larger and larger without bound. It will be a lot simpler if we recall that *accuracy* and *control* play a role here and then sort out how that happens. The role of accuracy is just like earlier cases, so the conclusion will look like

$$|f(x) - L| < \epsilon$$

whenever we invoke the control. We want this to hold for all "large"  $x$ . Our control here occurs as a number,  $M > 0$ , that we intuitively think of as large. We invoke our control by saying that  $x$  must satisfy

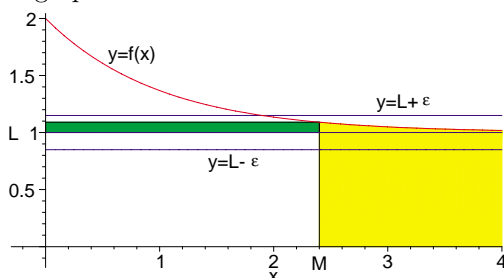
$$x > M \quad \text{or} \quad x \geq M$$

**Definition:** (Limit of a function at  $\infty$ ) We write

$$\lim_{x \rightarrow \infty} f(x) = L$$

if for each  $\epsilon > 0$  (accuracy) there is a number  $M > 0$  (control), so that whenever  $x > M$ , then it follows that  $|f(x) - L| < \epsilon$ .

This is saying geometrically that whenever we draw two horizontal lines,  $y = L + \epsilon$  and  $y = L - \epsilon$ , then from some point ( $x = M$ ) on, the graph will lie between the lines.

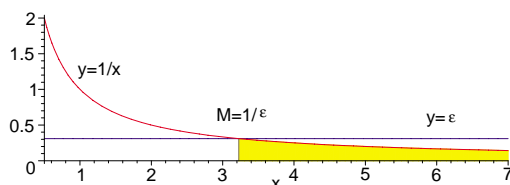


One of the first things that we must do is discuss  $1/x$  as  $x$  gets large.

**Example:** Let's establish this basic fact:

$$\lim_{x \rightarrow \infty} \frac{1}{x} = 0$$

If we are given an accuracy  $\epsilon > 0$ , then we choose  $M = \frac{1}{\epsilon}$  as our control. It isn't too difficult to see that  $M < x \implies \frac{1}{x} < \frac{1}{M} = \epsilon$ .



**Maple Example:** Evaluate  $\sqrt{x^2 + 6x + 4} - x$ .

```
> restart;
> limit(sqrt(x^2+6*x+4)-x,x=infinity);
```

3

Whoa! You might say that this result is not intuitively obvious. Let's see if we can understand what

happened. Follow the development algebraically and then we will take the limit at the end.

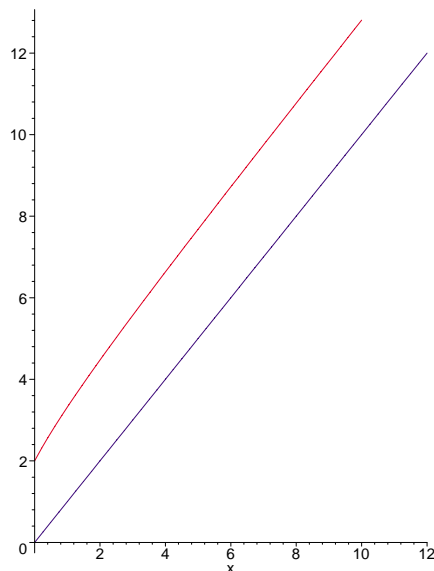
$$\begin{aligned}
 \sqrt{x^2 + 6x + 4} - x &= \left( \sqrt{x^2 + 6x + 4} - x \right) \cdot \left( \frac{\sqrt{x^2 + 6x + 4} + x}{\sqrt{x^2 + 6x + 4} + x} \right) \\
 &= \frac{x^2 + 6x + 4 - x^2}{\sqrt{x^2 + 6x + 4} + x} \\
 &= \frac{6x + 4}{\sqrt{x^2 + 6x + 4} + x} \\
 &= \frac{6 + \frac{4}{x}}{\frac{\sqrt{x^2 + 6x + 4}}{x} + 1} \\
 &= \frac{6 + \frac{4}{x}}{\sqrt{\frac{x^2 + 6x + 4}{x^2}} + 1} \\
 &= \frac{6 + \frac{4}{x}}{\sqrt{1 + \frac{6}{x} + \frac{1}{x^2}} + 1} \longrightarrow \frac{6}{2} = 3
 \end{aligned}$$

The limit is taken as  $x$  goes to infinity, causing several terms to go to 0. Let's consider the graphs of  $\sqrt{x^2 + 6x + 4}$  and  $x$ . You will see that the difference does tend to 3.

```

> H1:=plot(sqrt(x^2+6*x+4),x=0..10):
> H2:=plot(x,x=0..12,color=blue):
> display(H1,H2);

```

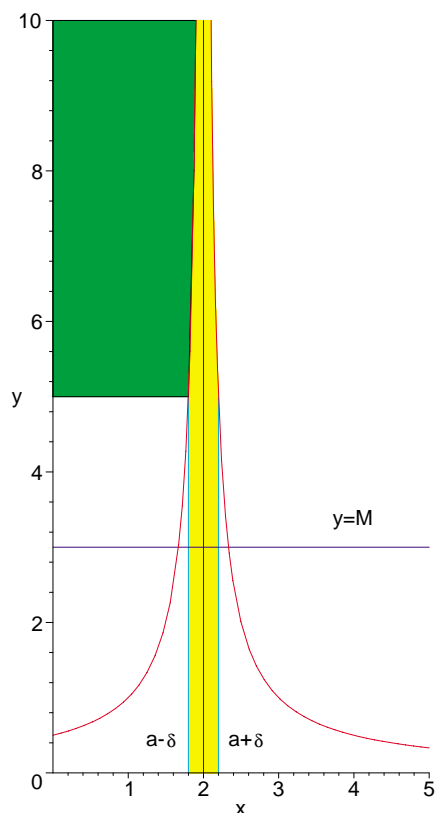


Now let's turn our attention to functions which are unbounded as  $x$  approaches a given value. Here our accuracy is given as a 'large' number  $M$  and we use the usual control,  $\delta > 0$  for  $x$ .

**Definition:** (Limit of an unbounded function) We write

$$\lim_{x \rightarrow a} f(x) = \infty$$

if for each  $M > 0$  (accuracy) there is a number  $\delta > 0$  (control), so that whenever  $x \neq a$  and  $|x - a| < \delta$ , then it follows that  $M < f(x)$ .



We must be very careful here. Although we write  $\lim_{x \rightarrow a} f(x) = \infty$ , in fact, **THIS LIMIT DOES NOT EXIST**. But, saying that the limit is infinity does provide us with useful information. It tells us that as  $x$  gets close to  $a$ , the values of  $f(x)$  are getting larger and larger without any bound on these values.

In the diagram above, as  $x$  approaches  $a$  we see that  $f(x)$  gets large, no matter which side we approach from. In the next example we change that so that we must consider  $-\infty$  as a ‘limit’.

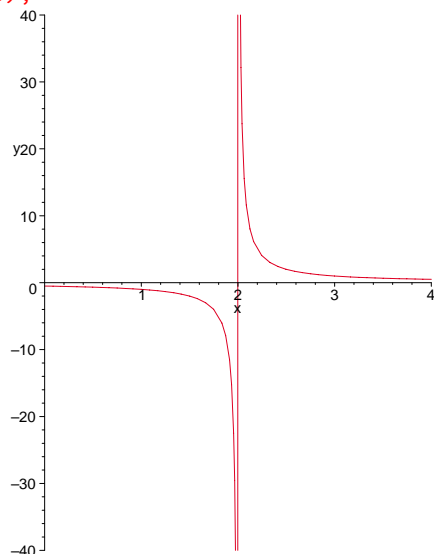
**Example:** Discuss  $\lim_{x \rightarrow 2} \frac{1}{x-2}$ .

This discussion is going to be intuitive rather than precise. Let’s look at the graph of  $y = \frac{1}{x-2}$  first.

> `f:=x->1/(x-2);`

$$f := x \rightarrow \frac{1}{x-2}$$

> `plot(f(x),x=0..4,y=-40..40);`



Note that we inserted a range for  $y$  here so that we have control over the scale. There is a vertical asymptote

at  $x = 2$  and you see that Maple deals with the jump from large negative to large positive values by drawing the vertical line that is not really part of the graph. It is clear that as  $x$  approaches 2 from the left the values are getting larger and larger negatively. From the right, values are getting large without bound. Let's allow Maple to sum things up.

```
> limit(f(x),x=2,left);
                                 $-\infty$ 
> limit(f(x),x=2,right);
                                 $\infty$ 
> limit(f(x),x=2);
                                undefined
```

**C1M8 Problems** Use Maple to plot the graphs and to discuss the limits at the specified points. Where necessary, insert a vertical range into the plot. For situations like the last example, use one-sided limits.

**WARNING!** Spelling is important. Maple recognizes **infinity**, but it ignores creative variations.

1. For  $f(x) = \tan(x/2)$ , discuss  $\lim_{x \rightarrow \pi} f(x)$ .
2. Discuss: (a)  $\lim_{x \rightarrow 0^+} \ln(x)$  and (b)  $\lim_{x \rightarrow 0^+} x \ln(x)$ .
3. Just evaluate: (a)  $\lim_{x \rightarrow \infty} (1 + 1/x)^x$ ; (b)  $\lim_{x \rightarrow \infty} (1 + \sin(\pi/(2x)))^x$
4. Given:  $f(x) = \frac{x^{50}}{e^x}$  Just evaluate: (a)  $f(3)$ ; (b)  $\lim_{x \rightarrow \infty} f(x)$