## C1M7

## Continuity

To paraphrase a judge who said something like "Pornography might be hard to define, but I know it when I see it.", a similar statement about continuity might be, "Continuity might be hard to define, but I know it when I see it." In the two diagrams which follow, when you look at them you will have an immediate sense that one function is continuous and the other is discontinuous at two different points. You might think of the graph of the function as a wire and one graph would allow an electric current to flow through it while the other would not.



As it turns out, we may define continuity of a function at a number a fairly easily, but the application and understanding of the concept might take some effort.
Definition: (Continuity of a function at a number $a$ ) A function $f$ is continuous at a number $a$ if

$$
\lim _{x \rightarrow a} f(x)=f(a)
$$

This says two things: (1) the limit of $f$ at $x=a$ exists; and (2) the limit has the value $f(a)$.
We repeat the definition of limit from a previous module and follow it with a definition of continuity that is equivalent to the one above.
Definition: (Limit of a function) We write

$$
\lim _{x \rightarrow a} f(x)=L
$$

if for each $\epsilon>0$ (accuracy) there is a number $\delta>0$ (control), so that whenever $x \neq a$ and $|x-a|<\delta$, then it follows that $|f(x)-L|<\epsilon$.
Definition: (Continuity of a function at a number $a$ ) We say that a function $f$ is continuous at a number $a$ if for each $\epsilon>0$ (accuracy) there is a number $\delta>0$ (control), so that whenever $|x-a|<\delta$, then it follows that $|f(x)-f(a)|<\epsilon$.

This is just a precise way of saying that the values of $f(x)$ are as close to $f(a)$ as we like, whenever $x$ is close enough to $a$. Here we do not exclude $x=a$ as we would when discussing the limit at $a$.
"What would this look like on a graph?", you may well ask. It will look just like a limit illustration, except that one need not exclude $x=a$ from the discussion.

As you might expect, we may discuss continuity from one side or the other. If the limit taken from the left at $a$ differs from the limit taken from the right, then the limit does not exist at $a$ so continuity there is not possible. But, if the left-hand limit is $f(a)$, then $f$ is continuous from the left at $a$. A similar statement about the right-hand limit is valid. This leads to a simple statement which we will list as a theorem:
Theorem: A function $f$ is continuous at $x=a$ if and only if $f$ is continuous from the left at $a$ and $f$ is continuous from the right at $a$.

It is not fashionable in many calculus courses to actually prove that a function is continuous. And, this author agrees that this should not be over-stressed. But, it is instructive to see how this is done in some simple case at least once, because it reinforces how accuracy and control are involved. So that we may not be accused of focusing on theory we take a

## TIME OUT!

Let's show that the function $f(x)=m x+b$ is continuous at some value $a$ for the case where $m \neq 0$. The main problem here is that at this point we wouldn't know when we have accomplished our task. So, let's begin with what we would like to end up with. We would see something like

$$
|x-a|<\delta \quad \Longrightarrow \quad|(m x+b)-(m a+b)|<\epsilon
$$

where we were given $\epsilon>0$ and found a $\delta>0$ that made this statement valid. So, we go to a piece of scrap paper for some figuring.

## SCRAP PAPER:

Now we can try and work this backwards.

$$
\begin{aligned}
|(m x+b)-(m a+b)| & <\epsilon \\
|(m x-m a)| & <\epsilon \\
|x-a| & <\frac{\epsilon}{|m|} \quad \text { permitted since } m \neq 0
\end{aligned}
$$

Aha! This last inequality looks remarkably like $|x-a|<\delta$. Now we are ready to work forward because each of our inequalities on our scrap paper were reversible.

## Back to the good paper!

Suppose that we are given $\epsilon>0$, which is our requested accuracy. We choose $\delta=\frac{\epsilon}{|m|}$, which is our control. Then, for each $x$ so that $|x-a|<\delta$ we may write

$$
\begin{aligned}
|x-a| & <\delta=\frac{\epsilon}{|m|} \\
|m||x-a| & <\epsilon \\
|m x-m a| & <\epsilon \\
|(m x+b)-(m a+b)| & <\epsilon
\end{aligned}
$$

This means that whenever $|x-a|<\delta$, it follows that $|(m x+b)-(m a+b)|<\epsilon, W^{5 *}$. We have proved that straight line functions are continuous at every point. Maybe the Proof Cops won't catch us.

## TIME IN!

In your text you should find a theorem or statement about building continuous functions from other continuous functions. Suppose that $c$ is a constant and $f$ and $g$ are continuous functions whose domains and ranges align so that the following functions make sense. Then

$$
\begin{array}{ccccccc}
c & f+g & f-g & f g & c f & \frac{f}{g} & f \circ g
\end{array}
$$

are all continuous. It is easy to see why constant functions are continuous. They only take on one value, $c$, so $|f(x)-c|=0<\epsilon$ for every $\epsilon>0$ and all $x$. We know that functions of the form $f(x)=m x+b$ are continuous, so by applying all this we may conclude that:
Theorem: (Polynomials) If $f(x)=a_{n} x^{n}+a_{n-1} x^{n-1}+\ldots a_{2} x^{2}+a_{1} x+a_{0}$ for constants $a_{0}, a_{1}, \ldots, a_{n}$, then $f$ is a continuous function at every point.

Although we won't state it as a theorem, all trigonometric, exponential and logarithmic functions are continuous at every point of their domain. Once we accept all this, we know that a function like

$$
g(x)=3 e^{2 x} \sin ^{2}\left(3 x^{2}+5\right)+\ln \left(x^{2}+1\right)
$$

is continuous everywhere. We have used generously the concept from above that verbalizes as, "A continuous function of a continuous function is itself continuous." Let's state this as a theorem.
Theorem: (Composition of Continuous Functions) If $g$ is continuous at $a$ and $f$ is continuous at $g(a)$, then $f \circ g$ is continuous at $a$, where $(f \circ g)(x)=f(g(x))$.

The concept of continuity is a powerful one. In mathematics, whenever the conclusion of a theorem includes a phrase like "there exists a ...' you have a potentially strong and useful theorem. The reason for this is because it essentially says "there is a solution to ..." and knowing that a problem has a solution

[^0]means that your efforts to find a solution are not doomed from the start. The theorem which follows is one of two such theorems involving continuity on a closed interval that we will state here. Recall that when we say that $f$ is continuous on $[a, b]$, at $x=a$ we mean continuous from the right and at $x=b$ we mean continuous from the left. To keep this straight, remember that you work from the inside of the interval towards the endpoint.
Intermediate Value Theorem: Suppose that $f$ is a continuous function on the closed interval $[a, b]$ and that $C$ is a value strictly between $f(a)$ and $f(b)$ (either $f(a)<C<f(b)$ or $f(b)<C<f(a)$ ). Then there exists a number $c$ so that $f(c)=C$.

This says that if you have two distinct values on the $y$-axis and $f(a)$ is one of them and $f(b)$ is the other, and if you draw a horizontal line between them, then the line must meet the graph at some point between $a$ and $b$. There may be many such points, but you are guaranteed at least one. The obvious diagram follows.


Maple Example: Suppose that $f(x)=e^{x} \cos (x)+(.2)$. For $C=1$, illustrate the Intermediate Value Theorem.

```
> restart:
> f:=x->exp(x)*\operatorname{cos}(\textrm{x})+.2;
    f:= x-> 䬺 cos(x)+.2
> f(0); f(Pi/2); pi/2=evalf(Pi/2);
    1 . 2
    \
> C:=1
> c:=fsolve (f(x)=C,x,0..Pi/2);
    C:= 1
\[
c:=1.365054488
\]
```

The diagram above is actually this example. If you try to use solve you do not get a solution between 0 and $\pi / 2$. However, fsolve provides a floating point answer and it allows you to specify a range from which to seek a solution. Note the omission of $x=$ before the range, which is different from most other cases.
Application of the Intermediate Value Theorem: (optional) Choose any great circle of Earth, $C$, such as the Equator. Suppose that at some instant in time we are able to record the temperature at every point on the circle. Let $R$ be the radius of Earth. Because $P(t)=[R \cos (t), R \sin (t)]$ traces out a circle of radius $R$ for $0 \leq t \leq 2 \pi$, we could (with a little effort) define a function $F$ so that $F(t)$ would produce the temperature of the point $P(t)$ on the circle $C$ for $0 \leq t \leq 2 \pi$. Because there will be no jumps in temperature as we make small changes in our position on $C$, we see that $F$ will be a continuous function of $t$ on the closed interval $[0,2 \pi]$. Note that $P(0)=P(2 \pi)$ because 0 and $2 \pi$ represent the same point on Earth. As a result, $F(0)=F(2 \pi)$. We will show why there must be two points on the circle which are
antipodal and have the same temperature. Points are antipodal if they lie at opposite ends of a diameter. Here this will mean that $F\left(t_{0}\right)=F\left(t_{0}+\pi\right)$ for some value $t_{0}$ of $t$ between 0 and $\pi$. Let's show why this is true.

Begin by defining a function $G$ as $G(t)=F(t)-F(t+\pi)$. As the composition of two continuous functions, $F(t+\pi)$ must be continuous. This means that $G$ is the difference of two continuous functions, so it is also continuous. We seek a solution to the equation $G(t)=0$. Let's see what happens at 0 . Since $G(0)=F(0)-F(\pi)$ there are two possibilities. First, if $G(0)=0$ we see that $F(0)=F(\pi)$ and we have found our solution. If $G(0) \neq 0$ then $G(0)$ and $G(\pi)=F(\pi)-F(2 \pi)=F(\pi)-F(0)$ are opposite in sign. One is positive and the other is negative. By the Intermediate Value Theorem there must be a point $t_{0}$ between 0 and $\pi$ for which $G\left(t_{0}\right)=0$ or $F\left(t_{0}\right)=F\left(t_{0}+\pi\right)$. $W^{5}$

Let's show an example using Maple.

```
> restart: with(plots): with(plottools):
> F:=t->t*(2*Pi-t)*(sin(sqrt(2)*t))^2/5+.5;
    F:=t->\frac{1}{5}t(2\pi-t)\operatorname{sin}(\sqrt{}{2}t\mp@subsup{)}{}{2}+.5
> F(0); F(2*Pi);
```

    . 5
    . 5
$>\mathrm{a}:=\mathrm{fsolve}(\mathrm{F}(\mathrm{t})=\mathrm{F}(\mathrm{t}+\mathrm{Pi}), \mathrm{t}, 0 . \mathrm{Pi})$;
$a:=1.786163402$
$>\operatorname{evalf}(F(\mathrm{a})) ; \quad$ evalf(F(a+Pi));
1.035640467
1.035640466
$>P:=p l o t 3 d([\cos (t), \sin (t), z], t=0 . .2 * \operatorname{Pi}, z=0 \ldots F(t), \operatorname{grid}=[80,30]$,
$>$ orientation=[124,66],style=PATCHNOGRID):
$>$ L: $=\operatorname{line}([\cos (a), \sin (a), F(a)],[\cos (a+P i), \sin (a+P i), F(a+P i)], \operatorname{color=red}$, thickness=2):
$>$ L1:=line([cos(a), sin(a), 0],[cos(a), sin(a),F(a)],color=red,thickness=2):
$>$ L2:=line $([\cos (a+P i), \sin (a+P i), 0],[\cos (a+P i), \sin (a+P i), F(a+P i)], c o l o r=r e d$, thickness=2):
> display(P,L,L1,L2);


DON'T PANIC! Three-dimensional plotting is usually found in Calculus III, or at least in the last part of Calculus II. Since this example is optional, I thought I would toss in the picture just for fun. I tried to color the graph so that it was red when the temperature was high and blue when it was cold, but that seemed to require more trouble than it was worth. Hopefully, you will note that the line across the figure is horizontal, illustrating that these antipodal points have the same temperature. No effort was made to scale the function so that the values were plausible temperatures.

Continuing with this same example, let's look at what is happening two-dimensionally. Because $F(0)=$ $F(2 \pi)$, there must be a point $a$ so that $F(a)=F(a+\pi)$, as we said earlier. What this means is, if we take a line, $L$, that is of length $\pi$ and hold it horizontally and slide the left end along the curve, then there is a value $a$ where the right end will lie on the curve. Here is the picture and the Maple that produced it as we continue in the worksheet.

```
> P2:=plot(F(t),t=0..2*Pi,scaling=constrained):
> M1:=line([a,F(a)],[a+Pi,F(a+Pi)],color=blue,thickness=2):
> M2:=line([0,0],[a,F(a)],color=cyan,thickness=2):
> M3:=line([Pi,0],[a+Pi,F(a+Pi)],color=cyan,thickness=2):
> M4:=line([a,0],[a,F(a)],color=navy):
> V1:=textplot([a-.13,.19,"a"]):
> V2:=textplot([Pi,.19,"p"],font=[SYMBOL,12]):
> V3:=textplot([3.3,f(a)+.2,"L"],font=[HELVETICA,OBLIQUE,14],color=blue)
> display(P2,M1,M2,M3,M4,V1,V2,V3);
```



The other big theorem for continuous functions on closed intervals involves maximization and minimization. Before we discuss this problem, let's look at a quick example of a situation where there is no solution to our question.
Example: Suppose that $f(x)=x$ on the open interval $(0,1)$. Then, there is no point $x_{0}$ in this interval such that $f\left(x_{0}\right) \leq f(x)$ for all $x$ in the interval. Similarly, there is no point $x_{1}$ in $(0,1)$ for which $f(x) \leq f\left(x_{1}\right)$ for all $x$ in this interval. Suppose we think that $x_{0}$ satisfies the condition for being a minimum. Then $\hat{x}=\frac{x_{0}}{2}<x_{0}$ violates this condition because $f(\hat{x})=\frac{x_{0}}{2}<x_{0}=f\left(x_{0}\right)$. Since ( 0,1 ) is not a closed interval, there are no points which serve as a maximum or a minimum for $f$. Certainly, $f$ is continuous on the open interval.
Extreme Value Theorem: If $f$ is a continuous function of a closed interval $[a, b]$, then there are numbers $c$ and $d$ in $[a, b]$ so that $f(c) \leq f(x) \leq f(d)$ for all $x$ in $[a, b]$.

We say that $c$ is a minimum point and $d$ is a maximum point for $f$ on $[a, b]$, while $f(c)$ and $f(d)$ are the (global) minimum and (global) maximum of $f$ respectively.

Later you will deal with these concepts in more depth and how to locate $c$ and $d$. For now, it suffices to say that the power of continuity and dealing with a closed interval guarantees a solution to finding a maximum or a minimum point.

## Maple Example:

variable. We use D1 instead.

```
> c:=fsolve(g(x)=C[1],x,0..2*Pi);
            c:=4.712388980
> evalf(3*Pi/2);
    4.712388981
> D1:=maximize(g(x),x=0..2*Pi,location);
        D1:=\mathbf{e, {[x=\frac{1}{2}\pi,\mathbf{e}]}}}=\mp@code{}}
> d:=fsolve(g(x)=D1[1],x,0..2*Pi);
            d:=1.570796327
> evalf(Pi/2);
    1.570796327
>H:=plot(g(x),x=0..2*Pi,color=red,thickness=2):
> L5:=line([c,0],[c,g(c)],thickness=2,color=blue):
> L6:=line([d,0],[d,g(d)],thickness=2,color=green):
> L7:=line([0,g(d)],[2*Pi,g(d)],thickness=2,color=green):
> L8:=line([0,g(c)],[2*Pi,g(c)],thickness=2,color=blue):
> C1:=textplot([Pi,2.9,"y=g(d) Maximum"],font=[HELVETICA,12],color=green):
> C2:=textplot([2.85,.25,"y=g(c) Minimum"],font=[HELVETICA,12],color=blue):
> C3:=textplot([c,-.2,"c"],font=[HELVETICA,12],color=blue):
> C4:=textplot([d,-.2,"d"],font=[HELVETICA,12],color=green):
> C5:=textplot([3.5,1.5,"y=g(x)"],font=[HELVETICA, 12],color=red):
> display(H,L5,L6,L7,L8, C1,C2,C3,C4,C5);
```



We activated the package plottools because it contains the line command which we needed to connect the points. We could have plotted the horizontal lines as functions, but cutting and pasting seemed a little easier. The minimize and the maximize commands require an expression and accept an optional range. Another convenient option is to include location and actually identify where the extrema occur.

We have provided a lot more Maple detail than is really necessary. On the other hand, if you never see "how" to do something, then you may never improve your skills.
C1M7 Problems: Use Maple to plot the graphs and to find the requested values.

1. If $f(x)=\left\{\begin{array}{ll}\sin (1 / x), & \text { if } x \neq 0 \\ 0, & \text { if } x=0\end{array}\right.$, use the limit definition to see if $f$ is continuous at:
(a) $x=0$;
(b) $x=\frac{6}{\pi}$.
2. If $g(x)=\left\{\begin{array}{ll}x \sin (1 / x), & \text { if } x \neq 0 \\ 0, & \text { if } x=0\end{array}\right.$, use the limit definition to see if $g$ is continuous at:
(a) $x=0$;
(b) $x=\frac{6}{\pi}$.
3. For $K(t)=e^{t} \sin (3 t)$ on $[0, \pi]$ :
(a) determine if there must be a zero for $K$ between 1.5 and 2.5 , and find it if there must be;
(b) determine if there must be a solution to $K(t)=K(t+\pi / 2)$ on $[0, \pi / 2]$, and find it if there must be;
(c) find the maximum and minimum values of $K$ and where they are located.
4. For $G(t)=t(2-t) \sin (\sqrt{5} t)+\frac{1}{2}$ on $[0,2]$ :
(a) determine if there must be a point between 0 and 1 for which $G(t)=1$, and find it if there must be;
(b) determine if there must be a solution to $G(t)=G(t+1)$ on $[0,1]$, and find it if there must be.
(c) find the maximum and minimum values of $G$ and where they are located.

[^0]:    * Which Was What We Wanted

