Extrema

A while back in **C1M10** we discussed the Mean Value Theorem, but only lightly. We used it to determine that a continuous function must be increasing on an interval where the derivative is positive,. Similarly, a negative derivative signals a decreasing function. Let's recall the two hypotheses of the MVT and then add a third.

H1: f is a continuous function on the closed interval $[a, b]$.

H2: f is differentiable on the open interval (a, b).

H3: There is a point c in (a, b) for which $f(x) \leq f(c)$ for all x in some open interval containing c.

The conclusion will be stated below. We are going to approach c from each side and compare the results.

Assume $x < c$. Then $\dfrac{f(x) - f(c)}{x - c} \geq 0$, so $\lim\limits_{x \to c^-} \dfrac{f(x) - f(c)}{x - c} = f'(c) \geq 0$.

Assume $x > c$. Then $\dfrac{f(x) - f(c)}{x - c} \leq 0$, so $\lim\limits_{x \to c^+} \dfrac{f(x) - f(c)}{x - c} = f'(c) \leq 0$.

Taken together, $f'(c) \geq 0$ and $f'(c) \leq 0$ imply $f'(c) = 0$. The one-sided limits existed because $f'(c)$ existed. When there is an open interval containing a point and that point satisfies $f(x) \leq f(c)$ for all x in the interval, we say that f has a *local maximum* or *relative maximum* at c.
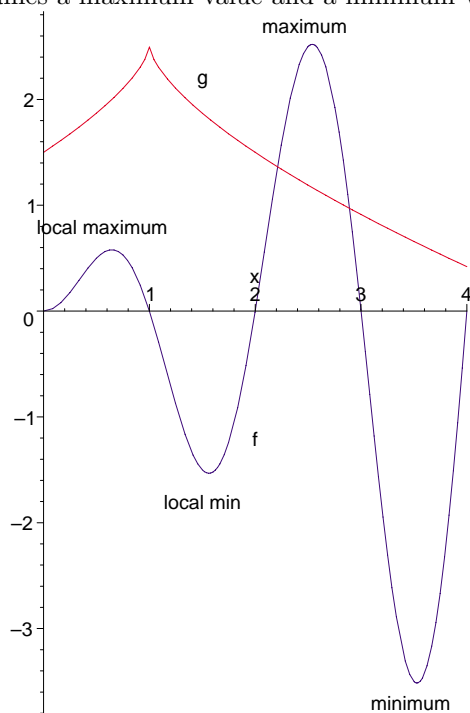
In a similar manner, suppose that we replace **H3** with

H4: There is a point c in (a, b) for which $f(x) \geq f(c)$ for all x in some open interval containing c.

In this case f has a *local minimum* or *relative minimum* at c. By an obvious parallel argument we have the same conclusion. So, with either **H3** or **H4** our conclusion is :

C: $f'(c) = 0$.

This tells us that when our function is differentiable the only way we can have a *local extreme* (local maximum or local minimum) that is not an endpoint, is for the derivative to be zero at the point. For this reason we define a point c to be a *critical point* if either the derivative does not exist at c or $f'(c) = 0$. The Extreme Value Theorem was mentioned in the module on continuity. From it we learned that a continuous function on a closed interval assumes a maximum value and a minimum value.

Here we have the graphs of two continuous functions on $[0, 4]$, f and g. We have labeled the extrema of f, except for the endpoints which are a local minimum $(x = 0)$ and a local maximum $(x = 4)$. It is clear

that at $x = 1$ the function g has a maximum, but the derivative does not exist there. The endpoints serve as a local minimum $(x = 0)$ and a minimum (absolute) $(x = 4)$ for g.

Maple Example: The function f in the diagram above is actually defined by $f(x) = x \sin(\pi x)$. Let's use Maple to locate all extrema of f. We will show all steps, including the plotting of text. After finding our first relative maxima, at c_1, we will test the values of f on both sides of c_1. We will also test the derivative of f on both sides.

```
>  restart:    with(plots):  with(plottools):
>  f:=x->x*sin(Pi*x);   a:=0;   b:=4;
```

$$f := x \to x \sin(\pi x)$$
$$a := 0$$
$$b := 4$$

```
>  fprime:=unapply(diff(f(x),x),x);
```

$$fprime := x \to \sin(\pi x) + x \cos(\pi x)\pi$$

This command takes the derivative of $f(x)$ with respect to x and sets it up as a function.

```
>  f2prime:=unapply(diff(f(x),x,x),x);
```

$$f2prime := x \to 2 \cos(\pi x)\pi - x \sin(\pi x)\pi^2$$

Note how this command takes the *second* derivative of $f(x)$ and sets it up as a function.

```
>  c1:=fsolve(fprime(x)=0,x=0..1);
```

$$c1 := .6457736765$$

```
>  m1:=evalf(f(c1));
```

$$m1 := .5792303274$$

```
>  evalf(f(c1-.02));   evalf(f(c1+.02));
```

$$.5775557266$$
$$.5775085314$$

```
>  evalf(fprime(c1-.02));   evalf(fprime(c1+.02));
```

$$.1662069815$$
$$-.1732844003$$

From the last few lines we see that the derivative of f is positive on the left of c_1 and negative on the right of c_1. This means that as we move from left to right we are going uphill, level, and then downhill − a sure sign that c_1 yields a relative maximum. Later we will state all this as a theorem.
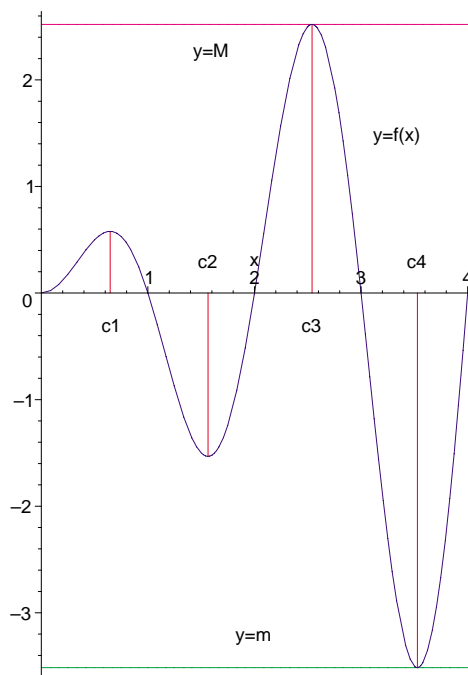
```
>  evalf(f2prime(c1));
```

$$-8.494699804$$

This is overkill, but we may also use the second derivative to see if the first derivative is a decreasing function, and the negative value of `f2prime` at c_1 says that it is, thus signalling that f is concave down at c_1 and there is a relative maximum there.

```
>  c2:=fsolve(fprime(x)=0,x=1..2);
```

$$c2 := 1.563913906$$

```
>  m2:=evalf(f(c2));
```

$$m2 := -1.532493362$$

```
>  evalf(f2prime(c2));
```

$$16.37825282 \qquad \text{Concave up, relative minimum}$$

```
>  c3:=fsolve(fprime(x)=0,x=2..3);
```

$$c3 := 2.539688175$$

```
>  m3:=evalf(f(c3));
```

$$m3 := 2.519972588$$

```
>  evalf(f2prime(c3));
```

$$-25.65251748 \qquad \text{Concave down, relative maximum}$$

```
>  c4:=fsolve(fprime(x)=0,x=3..4);
```

$$c4 := 3.528636468$$

```
>  m4:=evalf(f(c4));
```

$$m4 := -3.514366510$$

```
>  evalf(f2prime(c4));
```

$$35.24990626 \qquad \text{Concave up, relative minimum}$$

```
>  A:=f(a); B:=f(b);
```

$$A := 0$$

$$B := 0$$

```
>  M:=max(m1,m2,m3,m4,A,B);
```

$$M := 2.519972588$$

```
>  m:=min(m1,m2,m3,m4,A,B);
```

$$m := -3.514366510$$

```
>  B1:=plot(f(x),x=0..4,color=blue,scaling=constrained):
>  B2:=line([c1,0],[c1,f(c1)],color=red):
>  B3:=line([c2,0],[c2,f(c2)],color=red):
>  B4:=line([c3,0],[c3,f(c3)],color=red):
>  B5:=line([c4,0],[c4,f(c4)],color=red):
>  B21:=textplot([c1,-.3,"c1"]):
>  B31:=textplot([c2,.3,"c2"]):
>  B41:=textplot([c3,-.3,"c3"]):
>  B51:=textplot([c4,.3,"c4"]):
>  B6:=plot(m,x=0..4,color=green):
>  B61:=textplot([2,m+.3,"y=m"]):
>  B7:=plot(M,x=0..4,color=magenta):
>  B71:=textplot([1.6,M-.24,"y=M"]):
>  B8:=textplot([3.35,1.5,"y=f(x)"]):
>  display(B1,B2,B3,B4,B5,B21,B31,B41,B51,B6,B61,B7,B71,B8);
```


Maple Example: Find the minimum distance from $Q(3/2,1)$ to the graph of $f(x) = x\,\sin(\pi x)$.

   Frequently calculus texts include an exercise to find the minimum distance from a given point to a simple curve, such as a parabola. This is a similar, yet more challenging, problem involving the function in the previous example. We will exploit the power of Maple and have it find numerical solutions for critical points that we would be unlikely to locate easily. The function which determines the distance from the point $Q(3/2,1)$ to the point $(x, f(x))$ on the curve is designated as $R(x)$, its derivative is $S(x)$, and its second derivative is $T(x)$. Because $T(x)$ is a bit cumbersome, its display is suppressed with a colon. Since several points may occur as solutions, we will use the Second Derivative Test to determine whether a critical point is a relative maximum or minimum. The derivative has a denominator that looks as if it could be a problem. However, a close examination reveals that it is simply the distance function $R(x)$. Since Q does not lie on the curve, the denominator won't be 0, but we will check it anyway. We chose to make the derivatives of f functions instead of expressions.

```
>  restart:    with(plots):    with(plottools):
```

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```
> f:=x->x*sin(Pi*x);
```

$$f := x \rightarrow x \, \sin(\pi x)$$

```
> Q:=[3/2,1];
```

$$Q := \left[\frac{3}{2}, 1\right]$$

Set up our distance formula.

```
> R:=x->sqrt((x-Q[1])^2+(f(x)-Q[2])^2);
```

$$R := x \rightarrow \sqrt{(x - Q_1)^2 + (f(x) - Q_2)^2}$$

Set up the first derivative as a function.

```
> S:=unapply(diff(R(x),x),x);
```

$$S := x \rightarrow \frac{1}{2} \frac{(2x - 3 + 2(x \, \sin(\pi x) - 1)(\sin(\pi x) + x \, \cos(\pi x) \, \pi))}{\sqrt{(x - 3/2)^2 + (x \, \sin(\pi x) - 1)^2}}$$

Let's take a look at that denominator. If it has a zero in the interval, then we have a problem. We will "eyeball" it rather than solving the equation.

```
> De:=denom(S(x));
```

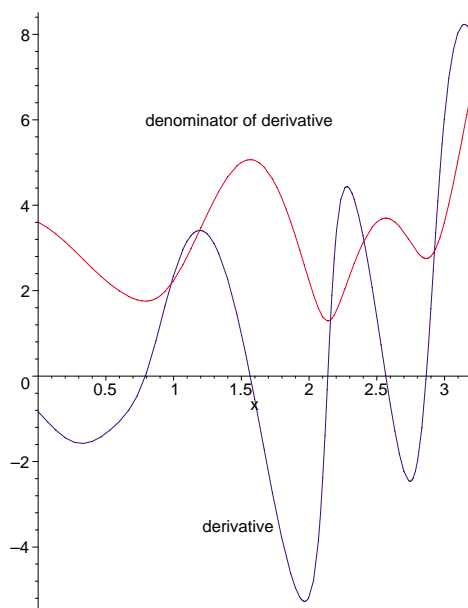$$2(\sqrt{(x - 3/2)^2 + (x \, \sin(\pi x) - 1)^2})$$

```
> D1:=plot(De,x=0..3.2,color=red):
> D2:=plot(S(x),x=0..3.2,color=blue):
> D3:=textplot([1.55,6,"denominator of derivative"]):
> D4:=textplot([1.5,-3.5,"derivative"]):
> display(D1,D2,D3,D4);
```


Set up the second derivative of the function as a function.

```
> T:=unapply(diff(R(x),x,x),x):
> P0:=plot(f(x),x=0..3.2,color=blue):
> p1:=fsolve(S(x)=0,x,1/2..5/2);
```

$$p1 := 1.565491407$$

```
> evalf(T(p1));
```

$$-16.00806205$$

$T(p1) = R \ (p1) < 0 \implies$ relative maximum

```
> P1:=pointplot([p1,f(p1)],symbol=circle):
> p2:=fsolve(S(x)=0,x,1.6..5/2);
```

$$p2 := 2.138585926$$

```
> evalf(T(p2));
```

$$67.70258030$$

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$T(p2) = R\ (p2) > 0\ \Rightarrow$ relative minimum

```
>  P2:=pointplot([p2,f(p2)],symbol=circle):
>  p3:=fsolve(S(x)=0,x,.5..1.4);
```
$$p3 := .7918557211$$

```
>  evalf(T(p3));
```
$$9.019878695$$

$T(p3) = R\ (p3) > 0\ \Rightarrow$ relative minimum

```
>  P3:=pointplot([p3,f(p3)],symbol=circle):
>  A:={evalf(R(p1)),evalf(R(p2)),evalf(R(p3))};
```
$$A := \{2.533319655, .6460679294, .8775682312\}$$

```
>  mindistance:=min(op(A));
```
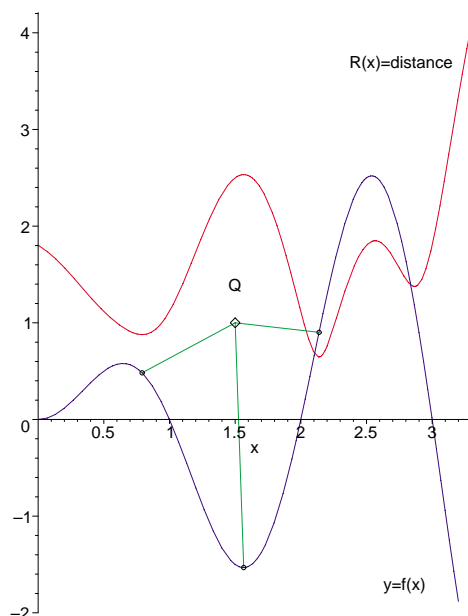$$mindistance := .6460679294$$

We used `op(A)` in order to access the components of the set A. We have our answer and the point at which it occurs, $(p2, f(p2))$. Now we will display a graph that illustrates the items which we found.

```
>  P4:=pointplot(Q,symbol=diamond,symbolsize=15):
>  P5:=line(Q,[p1,f(p1)],color=green):
>  P6:=line(Q,[p2,f(p2)],color=green):
>  P7:=line([p2,f(p2)],[p2,0],color=magenta):
>  P8:=line(Q,[p3,f(p3)],color=green):
>  P9:=plot(R(x),x=0..3.3,color=red):
>  T1:=textplot([2.8,-1.7,"y=f(x)"]):
>  T2:=textplot([2.8,3.7,"R(x)=distance"]):
>  T3:=textplot([1.5,1.4,"Q"]):
>  T4:=textplot([p2,-.35,"p2"]):
>  display(P0,P1,P2,P3,P4,P5,P6,P7,P8,P9,T1,T2,T3,T4);
```


Procedure to Locate Extrema: It is time to summarize our approach to extrema. Suppose that we have a continuous function f on a closed interval. We locate all points at which the first derivative of f is zero or for which the derivative does not exist, calling such points *critical points*. We evaluate f at the critical points and the endpoints and know that our global maximum and global minimum will be found among these values. To determine whether a critical point for which f is differentiable yields a relative maxima or minima, we apply the First Derivative Test.

First Derivative Test: Assume that f is differentiable on an open interval containing c and that $f(c) = 0$.

(a) If the derivative $f(x)$ changes sign from positive to negative as x moves through c from left to right, then f has a relative maximum at c.

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(a) If the derivative $f'(x)$ changes sign from negative to positive as x moves through c from left to right, then f has a relative minimum at c.

Noting that in case (a) the derivative is decreasing and that in case (b) the derivative is increasing, we state the Second Derivative Test for determining the status of critical points where the derivative is zero.

Second Derivative Test: Assume that f is differentiable on an open interval containing c and that $f'(c) = 0$. Assume that the second derivative, f'', also exists on the open interval.

(a) If $f''(c) < 0$, then f has a relative maximum at c.

(b) If $f''(c) > 0$, then f has a relative minimum at c.

There is another concept to consider that we have not yet mentioned. What happens when $f''(c) = 0$? The answer is inconclusive without additional information. A *point of inflection* occurs when a function stops being concave one way and starts being concave the other way. This happens when the derivative stops increasing and starts decreasing, or vice versa.

Point of Inflection: Suppose that $f''(x)$ exists on an open interval containing c and that $f''(c) = 0$. Then c is a point of inflection of f if $f''(x)$ changes sign as x passes through c.

Examples: (a) If $f(x) = x^4$, then $f'(x) = 4x^3$ and $f''(x) = 12x^2$. If $c = 0$ then $f''(c) = 0$, but c is **not** a point of inflection for f because f'' does not change sign as x passes through 0.
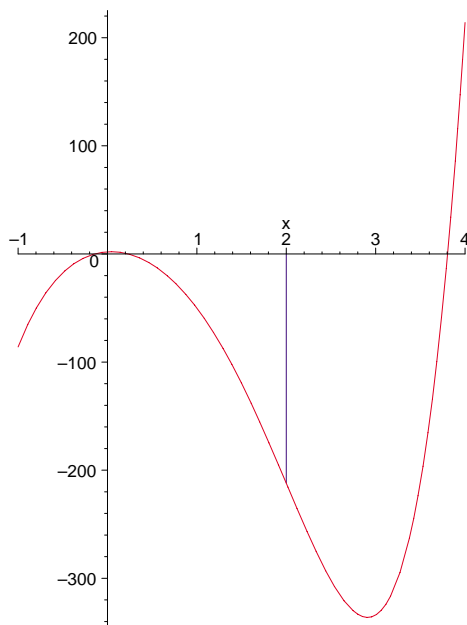
(b) Suppose that $f''(x) = (x-1)^2(x-2)$. Then f'' is zero at 1 and 2. But f'' changes sign only at 2, so 2 produces the only inflection point for f.

Maple Example: Find all points of inflection of $f(x) = 3x^5 - 10x^4 + 10x^3 - 60x^2 + 5x + 2$ and plot the graph of f.

```
> restart:   with(plots):   with(plottools):
> f:=x->3*x^5-10*x^4+10*x^3-60*x^2+5*x+2;
```
$$f := x \to 3x^5 - 10x^4 + 10x^3 - 60x^2 + 5x + 2$$
```
> f2prime:=unapply(diff(f(x),x,x),x);
```
$$f2prime := x \to 60\,x^3 - 120\,x^2 + 60\,x - 120$$
```
> factor(f2prime(x));
```
$$60\,(x-2)\,(x^2+1)$$
```
> f2prime(1.98);   f2prime(2.02);
```
$$-5.904480$$
$$6.096480$$

At 2, f stops being concave down and starts being concave up.
```
> U1:=plot(f(x),x=-1..4):
> U2:=line([2,0],[2,f(2)],color=blue):
> display(U1,U2);
```

C1M12 Problems:

1. Use Maple to find the critical points, points of inflection, and to display the graphs of the given functions, their derivatives and second derivatives.

a. $f(x) = x^4 + 6x^3 - 24x^2 + 24$

b. $g(x) = 3x^5 - 5x^3$

2. Given: $f(x) = x\,\cos(\pi x)$ on $[1, 5]$.

a. Use Maple to find the relative maxima and minima and the global maximum and minimum as was done in the Maple Example. Display vertical lines at the extrema as in the example.

b. Use Maple to locate the point $P(x, y)$ on the graph of $f(x)$ that is closest to the point $Q(3, 5)$. Display lines connecting the possible solutions to Q as was done in the Maple Example.