## C1M11

## Differentiation: Rules and Review

This begins with a modification of part of the review we do for those students who begin with Calculus II their first semester. We remind you of the definition and then list the rules. The notation we use for the derivative of $f(x)$ with respect to $x$ is $\quad D_{x}(f(x))=f^{\prime}(x)$.
Definition of derivative. Suppose $f$ is defined on an open interval containing $x$. The derivative of $f$ at $x$ is defined by

$$
D_{x}(f(x))=f^{\prime}(x) \equiv \lim _{h \rightarrow 0} \frac{f(x+h)-f(x)}{h}
$$

Tangent line. If $f\left(x_{0}\right)=y_{0}$ and $f^{\prime}\left(x_{0}\right)=m$, then an equation for the line tangent to the curve $y=f(x)$ is given by

$$
y-y_{0}=m\left(x-x_{0}\right)
$$

Rules of Differentiation Assume that $a$ and $b$ are real numbers and that $f(x)$ and $g(x)$ are differentiable on an open interval containing $x$.
Rule 1. The derivative is linear. That is, $D_{x}(a f(x)+b g(x))=a D_{x}(f(x))+b D_{x}(g(x))=a f^{\prime}(x)+b g^{\prime}(x)$.
Rule 2. Product Rule. $D_{x}(f(x) g(x))=D_{x}(f(x)) g(x)+f(x) D_{x}(g(x))=f^{\prime}(x) g(x)+f(x) g^{\prime}(x)$.
Rule 3. Quotient Rule. On an interval where $g(x) \neq 0$,

$$
D_{x}\left(\frac{f(x)}{g(x)}\right)=\frac{g(x) D_{x}(f(x))-f(x) D_{x}(g(x))}{(g(x))^{2}}=\frac{g(x) f^{\prime}(x)-f(x) g^{\prime}(x)}{(g(x))^{2}}
$$

Rule 4. Power function derivative. If $r$ is a real number, then

$$
D_{x}\left(x^{r}\right)=r x^{r-1}
$$

Examples: $\quad D_{x}\left(x^{4 / 3}\right)=\frac{4}{3} x^{1 / 3}, \quad D_{x}\left(\frac{1}{x^{4 / 3}}\right)=-\frac{4}{3} \frac{1}{x^{7 / 3}}, \quad D_{x}(\sqrt{x})=\frac{1}{2} x^{-\frac{1}{2}}=\frac{1}{2 \sqrt{x}}$
Rule 5. Chain Rule. On an open interval for which $(f \circ g)(x) \equiv f(g(x))$ is defined

$$
D_{x}((f \circ g)(x))=D_{x}(f(g(x)))=f^{\prime}(g(x)) \cdot g^{\prime}(x)
$$

Example: $D_{x}\left(\left(4+x^{3}\right)^{5}\right)=(5)\left(4+x^{3}\right)^{4}\left(3 x^{2}\right)$
Rule 6. Reciprocal Rule. On an interval where $f(x) \neq 0$ we have

$$
D_{x}\left(\frac{1}{f(x)}\right)=\frac{-f^{\prime}(x)}{(f(x))^{2}}
$$

Derivatives of trigonometric functions.
Using Maple earlier we showed that $D_{x}(\sin (x))=\cos (x)$ and $D_{x}(\cos (x))=-\sin (x)$. Let's use these facts and the rules to find the other trig derivatives.

1. Using the Reciprocal Rule,

$$
D_{x}(\sec (x))=D_{x}\left(\frac{1}{\cos (x)}\right)=\frac{-D_{x}(\cos (x))}{(\cos (x))^{2}}=\frac{-(-\sin (x))}{\cos ^{2}(x)}=\left(\frac{1}{\cos (x)}\right)\left(\frac{\sin (x)}{\cos (x)}\right)=\sec (x) \tan (x)
$$

2. Using the Quotient Rule,

$$
\begin{aligned}
D_{x}(\tan (x))=D_{x}\left(\frac{\sin (x)}{\cos (x)}\right) & =\frac{\cos (x) \cdot D_{x}(\sin (x))-\sin (x) \cdot D_{x}(\cos (x))}{(\cos (x))^{2}} \\
& =\frac{\cos (x) \cos (x)-\sin (x)(-\sin (x))}{\cos ^{2}(x)}=\frac{1}{\cos ^{2}(x)}=\sec ^{2}(x)
\end{aligned}
$$

In a similar manner we obtain

$$
D_{x}(\csc (x))=-\csc (x) \cot (x)
$$

$$
D_{x}(\cot (x))=-\csc ^{2}(x)
$$

Let's develop a means of remembering this information in a systematic manner. We are going to form a four-column table by listing the basic functions in the first column, their derivatives in the second column, and their cofunctions in the third column. The fourth column will eventually contain the derivatives of the cofunctions, but for now we leave it blank.

| Function | Derivative | Cofunction | Derivative |
| :--- | :--- | :--- | :--- |
| $\sin (x)$ | $\cos (x)$ | $\cos (x)$ |  |
| $\tan (x)$ | $\sec ^{2}(x)$ | $\cot (x)$ |  |
| $\sec (x)$ | $\sec (x) \tan (x)$ | $\csc (x)$ |  |

Then, put a minus sign in each of the remaining boxes. Remember, the derivative of a CO-function always gets a minus sign!

| Function | Derivative | Cofunction | Derivative |
| :--- | :--- | :--- | :--- |
| $\sin (x)$ | $\cos (x)$ | $\cos (x)$ | - |
| $\tan (x)$ | $\sec ^{2}(x)$ | $\cot (x)$ | - |
| $\sec (x)$ | $\sec (x) \tan (x)$ | $\csc (x)$ | - |

Complete the table by putting the cofunctions of the entries in the second column into the fourth column.

| Function | Derivative | Cofunction | Derivative |
| :--- | :--- | :--- | :--- |
| $\sin (x)$ | $\cos (x)$ | $\cos x$ | $-\sin (x)$ |
| $\tan (x)$ | $\sec ^{2}(x)$ | $\cot (x)$ | $-\csc ^{2}(x)$ |
| $\sec (x)$ | $\sec (x) \tan (x)$ | $\csc (x)$ | $-\csc (x) \cot (x)$ |

This is an excellent memory device. However, you must know the derivatives of the basic trig functions $\sin (x), \tan (x)$, and $\sec (x)$, in order to complete the table.

As a very simple consequence of the Chain Rule, it is useful to list:

$$
D_{x}(\sin (a x))=a \cos (a x) \quad D_{x}(\cos (a x))=-a \sin (a x)
$$

© It is important to remember when applying the Chain Rule to trigonometric, and to other functions:

## THE ARGUMENT NEVER CHANGES!

© When differentiating a composite function with respect to $x$, the last step in the Chain Rule is to differentiate some element with respect to $x$.
Example: Using the Chain Rule,

$$
\begin{aligned}
D_{x}\left(\tan ^{3}(\sqrt{x})\right) & =3 \tan ^{2}(\sqrt{x}) \cdot D_{x}(\tan (\sqrt{x}) \\
& =3 \tan ^{2}(\sqrt{x}) \cdot \sec ^{2}(\sqrt{x}) \cdot D_{x}(\sqrt{x}) \\
& =3 \tan ^{2}(\sqrt{x}) \cdot \sec ^{2}(\sqrt{x}) \cdot\left(\frac{1}{2 \sqrt{x}}\right)
\end{aligned}
$$

Basically, this is $u^{3}$ with $u=\tan (\sqrt{x})$, resulting in $3 u^{2} \cdot D_{x}(u)$. The argument of the tangent function is $\sqrt{x}$, so when the tangent is differentiated, the result is the secant squared of the same argument. This produced a factor of $\sec ^{2}(\sqrt{x})$. Finally, we differentiate $\sqrt{x}$ with respect to $x$.

Derivative of exponential and logarithmic functions. You may recall that earlier we used Maple to see what the derivatives of these functions would be. We found

$$
D_{x}\left(e^{x}\right)=e^{x} \quad D_{x}(\ln x)=\frac{1}{x}, x>0 \quad D_{x}\left(a^{x}\right)=(\ln a) a^{x}, a>0
$$

If $x<0$, then we will use the Chain Rule to look at

$$
D_{x}(\ln (-x))=\left(\frac{1}{-x}\right) D_{x}(-x)=\left(\frac{1}{-x}\right)(-1)=\frac{1}{x}
$$

This allows us to include the derivative of $\ln |x|$.

$$
D_{x}(\ln |x|)=\frac{1}{x}, x \neq 0
$$

There is an identity that I have found to be extremely useful when dealing with exponential functions. Namely,

$$
a^{b}=e^{b \ln (a)}
$$

Using this, start with $a^{x}$. Then, $a^{x}=e^{\ln (a) x}$, and taking the derivative (we know $D_{x}\left(e^{c x}\right)=c e^{c x}$ by the Chain Rule) we get

$$
D_{x}\left(a^{x}\right)=D_{x}\left(e^{\ln (a) x}\right)=\ln (a) e^{\ln (a) x}=\ln (a) a^{x}
$$

Because $a^{0}=1$ for all positive $a$, every function of the form $y=a^{x}$ passes through $(0,1)$ and the slope of the tangent line there is always $\ln (a)$. If we draw a line from $(a-1,0)$ to $(a, \ln (a))$, then this line must be parallel to the tangent line we just discussed. The figure which follows illustrates this concept for $a=3$.


We will limit our discussion on inverse trigonometric functions to

$$
\sin ^{-1} x \equiv \arcsin x \quad \text { and } \quad \tan ^{-1} x \equiv \arctan x
$$

We remind the reader that the exponents refer to inverse functions and not to reciprocals. The derivatives are listed below:

$$
D_{x}(\arcsin x)=\frac{1}{\sqrt{1-x^{2}}} \quad D_{x}(\arctan x)=\frac{1}{1+x^{2}}
$$

Examples: (a) Find $D_{x}\left(\arcsin (3 x)-\sqrt{1-9 x^{2}}\right)$.
This is a straightforward application of linearity and the Chain Rule.

$$
\begin{aligned}
& D_{x}(\arcsin (3 x))=\frac{1}{\sqrt{1-(3 x)^{2}}} \cdot D_{x}(3 x)=\frac{1}{\sqrt{1-9 x^{2}}} \cdot 3 \\
& D_{x}\left(\sqrt{1-9 x^{2}}\right)=\left(\frac{1}{2}\right) \cdot\left(1-9 x^{2}\right)^{-1 / 2} \cdot D_{x}\left(1-9 x^{2}\right)=\left(\frac{1}{2}\right) \cdot \frac{-18 x}{\sqrt{1-9 x^{2}}}=\frac{-9 x}{\sqrt{1-9 x^{2}}}
\end{aligned}
$$

Combining both parts, we have
$D_{x}\left(\arcsin (3 x)-\sqrt{1-9 x^{2}}\right)=\frac{3}{\sqrt{1-9 x^{2}}}-\frac{-9 x}{\sqrt{1-9 x^{2}}}=\frac{3+9 x}{\sqrt{1-9 x^{2}}}$
(b) Find $D_{x}\left(3 \arctan \left(\frac{x}{3}\right)+4 \ln \left(x^{2}+9\right)\right)$

Taking the derivative we get

$$
3 \cdot \frac{1}{1+\left(\frac{x}{3}\right)^{2}} \cdot\left(\frac{1}{3}\right)+4 \cdot\left(\frac{1}{x^{2}+9}\right) \cdot(2 x)=\frac{9}{9+9 \cdot \frac{x^{2}}{9}}+\frac{4 \cdot 2 x}{x^{2}+9}=\frac{9}{9+x^{2}}+\frac{8 x}{x^{2}+9}=\frac{8 x+9}{x^{2}+9}
$$

## C1M11 Problems:

1. Find the derivative with respect to $x$ of the following and check your answers using Maple:
a. $\frac{1}{\ln (x)}$
b. $\ln (\cos (x))$
c. $\frac{1}{\sqrt{1-x^{2}}} \arcsin (x)$
2. Find the derivative with respect to $x$ of the following and check your answers using Maple:
a. $e^{\cos (2 x)}$
b. $\sin \left(e^{3 x}\right)$
c. $\frac{\sin (2 x)}{\cos (3 x)}$
3. Use Maple to plot $y=\arcsin (x)$ and to evaluate $\lim _{x \rightarrow-1^{+}} \arcsin (x)$ and $\lim _{x \rightarrow 1^{-}} \arcsin (x)$
4. Use Maple to plot $y=\arctan (x)$ and to evaluate $\lim _{x \rightarrow \infty} \arctan (x)$ and $\lim _{x \rightarrow-\infty} \arctan (x)$
