## Second Review Sheet Math 456

In addition to the topics on the first review sheet:

(1) Know the Period 3 Theorem and Sarkovskii's Theorem

Sample exercise: Let $F$ be the piecewise linear function defined from $[1, 5]$ to itself that sends $F(1) = 5, F(2) = 4, F(3) = 2, F(4) = 1, F(5) = 3$. Does this have a period 5 cycle? A period 3 cycle? Justify your answers.

Sample exercise: Can a continuous function on $\mathbf{R}$ have a periodic point of period 224 but not one of period 800? Why?

(2) Know the definition (formula) for the Schwarzian derivative of a function $SF$. Know the chain rule for the Schwarzian.

Know the definition of the basins of attraction of an attracting or neutral periodic point of $F$.

Know the definition of a critical point of $F$.

Sample exercises: Compute Schwarzian derivatives of some functions, for example Ex 1 p 161, Ex 10 p. 163.

(3) Know the theorem:

Theorem: Suppose $SF < 0$. If $x_0$ is an attracting periodic point for $F$, then either the immediate basin of $x_0$ extends to $\pm\infty$, or it contains a critical point of $F$ and the orbit of the critical point is attracted to $x_0$.

Sample exercise: Is there a critical point attracted to the fixed point of $F(x) = \cos^2 x$?

(4) Know the Newton iteration function and Newton fixed point theorem.

Sample exercise: Compute $\sqrt{9}$ using Newton's method.

Sample exercise: Use calculus to sketch the graph of the Newton iteration for $F(x) = x(x^2 + 1)$. For which $x$-values does this iteration converge to a root?

Sample exercise: What happens when Newton's method is applied to $(x - 1)^{1/3}$

(5) Know how complex arithmetic works, and how to express a complex number $z = x + iy$ in polar coordinates as $z = re^{it}$.

Sample exercise: Draw the images of a circle not containing the origin under the square root map.

(6) Know the dynamics of the complex linear function $L_\alpha = \alpha z$.

Sample exercise: Draw the orbit of the point $z_0 = (1 + i)$ if $\alpha = i/2$. Try other values of $z_0$ and $\alpha$.

(7) Know the definition of the derivative of a complex function.

Sample exercise: Does the following function have a complex derivative: $f(x + iy) = x^3 + iy^3$?

(8) Know the definitions of attracting, neutral and repelling periodic points for complex functions.

Sample exercise: Find all fixed points of $F(z) = z^2 + 3z + 1$ and decide whether they are attracting, repelling or neutral.

(9) List the properties of a chaotic dynamical system. Know the definition of a dense subset of a given set. Know the statement of the theorem that density is preserved by a continuous map. Know the theorem

**Theorem. 1.** *If two dynamical systems are conjugate, they are either both chaotic or neither is.*

Sample exercise: Show that the squaring function acting on $|z| = 1$ satisfies the definition of a chaotic set.

Sample exercise: Show that that $Q_{-2}$ is chaotic on $[-2, 2]$.

(10) Define the filled Julia set and the Julia set for a function $Q_c$.
(11) Define what it means for the orbit of $z$ to be "super-sensitive".
(12) State and prove the Escape criterion.

Sample exercise: Show how to use the escape criterion to find the Julia set of $Q_c$ for $|c| > 2$.

(13) Describe an algorithm to compute the filled Julia set of $Q_c$.
(14) Describe an algorithm to compute the Julia set of $Q_c$.
(15) Describe the fundamental dichotomy for quadratic polynomials.
(16) Define the Mandelbrot set.
(17) What is the main cardioid of the Mandebrot set? What are its characteristic properties?