Completeness Axioms — Real Numbers

The **Real Numbers** $\mathbb{R}$ are defined by **Completing** the rational numbers. This means we add *limits* of *sequences* of rational numbers to the field. We should then check that all the field axioms hold and that the ordering properties persist. The Real Numbers are characterized by the properties of Complete Ordered Fields. In these notes we give definitions of these terms.

Definition 0.1 *A* sequence *of real numbers is an assignment of the set of counting numbers of a set* $\{a_n\}, a_n \in \mathbb{R}$ *of real numbers,* $n \mapsto a_n$.

Definition 0.2 *A sequence* a_n *of real numbers has a* limit a *if, for every positive number* $\epsilon > 0$*, there is an integer* $N = N(\epsilon)$ *such that* $|a_n - a| < \epsilon$ *for all* a_n *with* $n > N$.

Example 1: The sequence $a_n = 1/n$ has limit 0 since we can take $N = [1/\epsilon] + 1$. This says

$$1/n < 1/(1/[epsilon] + 1) < \epsilon, \text{ for all } n > N$$

Example 2: The sequence $a_n = 1/2^n$ has limit 0 since we can take $N > \frac{\log(1/\epsilon)}{\log 2}$. This says

$$1/2^n < \epsilon, \text{ for all } n > N$$

Here is a list of equivalent statements. We can choose any one of them as an axiom for completeness. Choosing one, we can prove that all the other properties hold:

1. A field is complete if every infinite continued fraction has a limit. (Nested Intervals) If $I_n = [a_n, b_n]$ is an infinite sequence of closed intervals with a_n, b_n in the field such that $I_{n+1} \subset I_n$ then the field is complete if the infinite intersection of the intervals is non-empty; that is, $\cap_{n=1}^{\infty} I_n \neq \emptyset$.

2. (Dedekind Cuts) A subset of a field is called a *cut* if: 1) It is non-empty, but is not the whole set of rationals 20 every rational number of the set is smaller than every rational number not in the set; 3) it does no contain a number that is greater than any other number of the set. A field is complete if it contains cuts.

3. (Greatest lower bound or Least upper bound) A lower bound for a set is a number less than every number in the set; that is, if $B \leq x$ for all x in the set, B is a lower bound. G is a *greatest lower bound* for the set if it is a lower bound and every lower bound B satisfies $B \leq G$. Least upper bounds are defined similarly. For a complete field every set that has a lower bound (upper bound) has a greatest lower bound (least upper bound). The bound may or may not be in the set.

4. A field is complete if every bounded monotonic sequence has a limit.

5. If a_n is a sequence with the following property: given $\epsilon > 0$ there exists an $N = N(\epsilon)$ such that for all $m, n > N$ we have $|a_n - a_m| < \epsilon$ then the sequence is called a *Cauchy Sequence*. A field is complete if every Cauchy sequence has a limit.

Example 3: Let a_n be the finite decimal whose n entries are the first n digits of the infinite decimal for $\sqrt{2}$. Then a_n is monotonic increasing, bounded by 2, and has limit $\sqrt{2}$ (the least upper bound). Example 4: Consider the sequence $a_n = [k_1, k_2, \ldots, k_n]$ of continued fractions. Let $p_n/q_n = a_n$. Then it is not hard to show that the intervals a_{2n-1}, a_{2n} are nested. The diameters of the intervals goes to zero so the infinite continued fraction is the unique point in the intersection. One way to prove this uses the cutting sequence technique. p_n/q_n and p_{n+1}/q_{n+1} have cutting sequences that agree until the last set of $L's$ (or $R's$). If the sequence ends in $L's$ it is to the left of the number and if it ends in $R's$ it is to the right. Note that p_n/q_n and p_{n+1}/q_{n+1} must be neighbors.

Example 5: Consider the sequence $a_n = [1, 1, \ldots, 1]$ where there are n $1's$ in the continued fraction. Let $p_n/q_n = a_n$. It is easy to check that $p_n = f_n$ and $q_n = f_{n+1}$ where f_n is the n^{th} Fibonacci number. Then the sequence a_n is a Cauchy sequence. The limit is $(1 + \sqrt{5})/2$.

For real numbers we can talk about continuous functions and define derivatives. Consider the polynomials $x^2 - r = 0$ for $r > 0$. We can graph them. Because of the completeness of the reals, the graph is continuous and must cross the x axis in two places. These are the *roots* of the equation. For all odd degree polynomials we can see that they must cross the x axis in at least one place so they have at least one real root. Note though that not all polynomials have real roots. For example $x^2 + 1 = 0$.

Complex Numbers

If we want to solve for roots of all algebraic equations, $f(x) = 0$, we need to introduce complex numbers. We look at pairs of real numbers (x, y) and define the following operations for them:

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$$
$$(x_1, y_1) * (x_2, y_2) = (x_1y_1 - x_2y_2, x_1y_2 + x_2y_1)$$

Taking the additive identity as $(0, 0)$ and the multiplicative identity as $(1, 0)$ it is possible to check that the pairs of numbers with this operation define a field. Note that it is NOT an ordered field! We set $z = (x, y)$. We can plot the complex numbers as points in a plane using the first and second coordinate as the horizontal and vertical coordinates. Using the standard distance in the plane we can define the distance between two points

$$|z_1 - z_2| = sqrt(x_2 - x_1)^2 + (y_2 - y_1)^2$$

Using this distance we can define Cauchy sequences. One can prove that all Cauchy sequences have limits using the fact that the coordinates are real numbers. It follows that the complex numbers are a COMPLETE field.

The distance from a point z to the origin (additive identity), $(0, 0)$ is $|z| = sqrtx^2 + y^2$. We can specify a point z by its distance from the origin and the angle a line joining the point to the origin makes with the horizontal axis. Call this angle θ.

Exercise: Show that if $r = |z|$ and θ is the angle defined above,

$$z = (x, y) = (r \cos \theta, r \sin \theta)$$

Addition of complex numbers can be thought of as addition of vectors. To find the sum $z_1 + z_2$, draw the line from the origin to z_1; then translate the line from the origin to z_2 so that it starts at the end of the first line. The final endpoint is the sum. Multiplication of complex numbers also has a geometric interpretation: if $r_1 = |z_1|, r_2 = |z_2|$ and the corresponding angles are θ_1, θ_2, then $|z_1z_2| = r_1r_2$ and the angle that z_1z_2 makes with the horizontal is $\theta_1 + \theta_2$. This leads us to define

$$e^{i\theta} = (\cos \theta, \sin \theta)$$

to describe points on the unit circle. It is clear that $|e^{i\theta}| = 1$ since $\cos^2\theta + \sin^2\theta = 1$. We see from the multiplication law that we have DeMoivre's theorem:

$$e^{in\theta} = (\cos n\theta, \sin n\theta)$$

Note that there is another notation that we often use:

$$(x, y) = x + iy$$

We think of $(0, 1) = i$ as the *imaginary* number that solves $x^2 + 1 = 0$. That is, $i^2 = -1$. We have the following interesting formula: $e^{i\pi} = -1$.