Problems from the Problem Sessions at the Snowbird Conference on the 25th Birthday of the Mandelbrot Set

1. (Devaney) Consider the family of rational maps $z \mapsto z^n + \frac{\lambda}{z^n}$. These are perturbations of polynomials because they fix the point at infinity. For small λ the boundary of the immediate basin of infinity $\partial B(\infty)$ is a simple closed curve. Is this true for all λ in the connectedness locus? The connectedness locus is $\{\lambda | J(\lambda)\}$ is connected.

2. (Devaney) For the same family: the attracting basin of infinity contains a component containing the origin called the "trap door". If $J(\lambda)$ is a Sierpinski curve and if the critical points take the same number of iterates to get to the trap door — but not necessarily with the same itineraries —are the dynamics conjugate or not?

3. (Rogers) Suppose G is a Siegel disk for a quadratic polynomial such that ∂G contains a critical point. Can the orbit of the critical point be dense in the Julia set? Equivalently can $\partial G = J$.

4. (Rogers) Suppose G is a Siegel disk for a rational function and set $B = \partial G$. Can B contain a Cremer point? The answer is no for a polynomial (Poirier).

5. (Lyubich) Let P be a polynomial with Julia set J. The *hyperbolic Hausdorff dimension of* J is

   $$\mathcal{H}(P) = \sup\{HD(X) | X \text{ is invariant and hyperbolic and } X \subset J\}$$

   where $HD(X)$ is the usual Hausdorff dimension of X.

   The *critical dimension of* J is defined as

   $$\delta_P = \sup_{\delta} | \sum_n |P^{-n}(z)|^{\delta} \text{diverges}$$

   where the sum is taken over all inverse branches and $z \notin J$ is an arbitrarily chosen point. The sum converges for all $\delta > \delta_P$.

   The question is: $\delta_P = \mathcal{H}(P)$? This is known to be true for hyperbolic and parabolic polynomials and for the Feigenbaum and Collet-Eckmann polynomials. A similar statement is true for Kleinian groups. It is known that $\delta_P \leq \mathcal{H}(P)$.

6. (Douady) Persistence of the Fatou Coordinate:
   First consider the polynomial $P_\epsilon(z) = z^2 + \frac{1}{4} + \epsilon$ for a real small ϵ. This polynomial has two repelling fixed points z_1, z_2 such that $z_1 = \bar{z}_2$. Let γ be the vertical line connecting them. $P_\epsilon(\gamma)$ is a curve connecting the fixed points to the right of γ.

We define a *Fatou coordinate* on a neighborhood U_ϵ that contains γ, $P_\epsilon(\gamma)$, the region between them, but NOT the endpoints z_1, z_2 as the map satisfying

$$\Phi_\epsilon(P_\epsilon(z)) = \Phi(z) + 1$$

We can vary ϵ into the complex plane. As its argument increases we can follow $U_\epsilon, \Phi_\epsilon$. The deformed curves γ and $P_\epsilon(\gamma)$ spiral into the fixed points. As $\frac{1}{4} + \epsilon$ crosses into the main cardioid of the Mandelbrot set, one of the fixed points, say z_1, becomes attracting. Φ_ϵ is still defined. The curves spiral in at the attracting fixed point; $P_\epsilon(\gamma)$ is attracted "inside" γ. When we reach the boundary of the cardioid again (with argument of $\epsilon > \pi$), the spiral breaks. That is, because the fixed point becomes indifferent, the spiral is no longer defined; $P_\epsilon(\gamma)$ cannot be attracted "inside" γ. Thus the continuation of the Fatou coordinate breaks down at the second crossing (below the real axis) of the cardioid.

The construction can be made exactly the same way by making the argument of ϵ negative. Thus there are two overlapping regions of the parameter plane in which a Fatou coordinate is defined. Each can only be defined for a single crossing of the cardioid.

The Fatou coordinate can be defined near a parabolic point for an arbitrary rational or entire map. The question is to understand the obstruction to extending the coordinate in this setting. Is the obstruction local or global?

7. (Petersen) Shrinking of dyadic decorations:
   Choose a small copy M' of M inside M. Then there is a homeomorphism $\phi : M' \to M$.

   Consider a dyadic external ray $R_{p/2^n}$ to M and its pullback $\phi^{-1}(R_{p/2^n})$. Since a dyadic ray has a landing point on M, its pullback has a landing point p on M'. This point is a separation point of M'; that is, there is another external ray landing at p that separates off a piece of M' called the dyadic decoration at the point. Experimentally the Euclidean diameters of the dyadic decorations shrink uniformly in n. The question is whether this is indeed true. If MLC were true, it would imply this uniform shrinking.

8. (Wolf) Uniqueness of the ergodic measure of maximal dimension:
   Let P be a hyperbolic polynomial automorphism of $\mathbb{C}^2$. Let

   $$M = \{ \text{ all probability measures invariant under } P\}.$$

   For $\mu \in M$ set

   $$HD(\mu) = \inf\{HD(S) | \mu(S) = 1\}$$

   where HD is Hausdorff measure. Let

   $$\delta = \sup\{HD(\mu) | \mu \in M\}$$

   We say μ has *maximal dimension* if $HD(\mu) = \delta$.

There exist at least one and at most finitely many ergodic measures of maximal dimension. Is there only one? This can be proved if $|Jac(P)|$ is either close to 1 or 0.

9. (Devaney) Consider the parameter plane for $z^n + \lambda/z^n$.

   (a) Is the boundary of the connectedness locus for this family a simple closed curve? This would imply all external rays land.

   (b) Are the boundaries of the Sierpinski holes simple closed curves. It is known that the boundary of the "internal" ones are.

   (c) There exist infinitely many unburied Baby Mandelbrot sets — ones whose mouth touches the outside of the Sierpinski curve. How are the internal Baby Mandelbrot sets arranged? What do they touch?

10. (Mayer) In the above family consider the Connectedness locus minus the Sierpinski holes. Is the remainder simply connected?

11. (Mayer) Let R be a rational map of degree $d \geq 2$, and such that $J(R) \neq \hat{\mathbb{C}}^2$. Suppose there is no Fatou component U such that $R^2(U) = U$. Does it follow that there are buried points in $J(R)$?

    Equivalently, there exists no Fatou component V such that $\partial V = J(R)$. This is known as the Makienko conjecture.

    The corresponding statement for Kleinian groups is true and was proved by Abikoff.

12. (Douady) Branner-Hubbard compression of Cantor sets:
    Let c be a point outside the Mandelbrot set M. Then the filled Julia set K_c is a Cantor set. There is an external ray passing through c. Let c_t be the parametrization of this ray such that $c = c_1$ and c_t approaches M as $t \to 0$. The sets K_{c_t} are all Cantor sets. The following is known:
    Almost surely, $c_t \to c_0 \in M$ and $K_{c_t} \to K_{c_0}$ such that K_{c_0} is a dendrite; that is, a compact, connected, locally connected set with more than one point whose complement is also connected. This is the compression for Cantor sets.

    The complement $\mathbb{C} \setminus K_c$ has a metric defined by dG_c where G_c is the Green's function for $z^2 + c$. With respect to this metric, $\mathbb{C} \setminus K_c$ is divided into a binary tree of cylinders, each with the same modulus, but different scale $s_{n+1} = 1/2(s_n)$, as follows. Each cylinder C_n is marked by three points. One end of the cylinder has one point p^n and the other end has two symmetric points q_1^n, q_2^n. The points q_1^n and q_2^n are pinched together. Two cylinders C_{n+1} at scale s_{n+1} are attached to a cylinder C_n of scale s_n so that the point p^{n+1} is attached to the pinch point $q_1^n = q_2^n$. Thus at level n there are 2^n cylinders of the same size. The Cantor set is the boundary of the tree. During the compression, for each t, all of the cylinders have the same moduli, but their heights are multiplied by t.

The question involves a new construction. For a given c, form the tree of cylinders of equal modulus. Now modify this tree as follows. Choose a random sequence $\{\theta_k\} \in \mathbb{R} \bmod 2\pi$. Enumerate the cylinders in the tree and perform a partial Dehn twist by θ_k on the k^{th} cylinder. Now reattach the cylinders as before, attaching the p point at level n to the pinched points at level $n+1$. The boundary of this set is again a Cantor set — but it has no dynamics attached to it. Next perform a compression by changing the heights of the cylinders in the tree by a factor of t but keeping the moduli equal. One obtains a deformation of the Cantor set. The question is what the limit of the Cantor set is. That is, is it true that for almost every choice of $\{\theta_k\}$, the Cantor sets tend to a limit and is this limit a dendrite.

13. (Rogers) Suppose G is a Siegel disk for a rational function R and set $B = \partial G$. Is B a Jordan curve? Some steps to proving this would be:
    (a) Must B contain an arc?
    (b) Does B have exactly 2 complementary domains (one of the would be G)?
    (c) Is B an irreducible separator of $\hat{\mathbb{C}}$? That is, any closed subset of B does not separate.
    (d) Can it be a pseudo-circle? That is, a "circle-like" continuum which is a set that separates the plane, and is heriditarily indecomposable. This means that for any $\epsilon > 0$ there is a map $f : B \to S^1$ such that for every $p \in S^1$, diam $f^{-1}(p) < \epsilon$.
    (e) Can B be the whole Julia set? This is not known even for a quadratic polynomial.
    (f) Is the Julia set an indecomposable continuum. Yes to 13e would imply this.

14. (Milnor) Given a rational map R of $\mathbb{P}^2$ and an invariant elliptic curve E for R. Can there be an open set U that is a subset of the Fatou set and is also in the attracting basin of E? The conjecture is that this cannot happen.

15. (Lyubich) What is the connectedness locus in the Hénon family?
    (a) Does there exist a Hénon map H with connected Julia set that is not in the closure of a hyperbolic component. A candidate would be an infinitely renormalizable Hénon map that is a perturbation of the Feigenbaum polynomial
    (b) $J \supset J^* = \overline{\{\text{saddle points}\}}$. Are they equal for all complex Hénon maps. Equality holds for hyperbolic maps or if J is totally disconnected.
    (c) (du Jardin) Are homoclinic tangencies dense in the bifurcation locus, at least if $|Jac| < 1$?

16. (Mayer) Is there a rational function R with $J(R)$ connected, not locally connected, and the boundary of every component of the Fatou set is a Jordan curve (or locally connected)?

17. (Pilgrim) Let J be the Julia set of $z^2 - 1$. Is

$$\inf\{HD(h(J))|h : \mathbb{C} \to \mathbb{C} \text{ is quasiconformal } \} > 1$$

For the Sierpinski Gasket $HD = \frac{\log 3}{\log 2}$ but under qc deformation the infinimum is 1.

18. (Mayer) Let J be a connected quadratic Julia set with a Siegel disk. How big is the ω-limit set of the critical point, $\omega(c)$, and what is it? We know $\omega(c)$ contains the boundary of the Siegel disk. Sometimes it is strictly larger. Is it always a continuum? Is $\omega(c)$ intersect its preimage a continuum?

19. (Bedford) Let $p(z)$ be expanding and let $q(z, w)$ be a polynomial in $\mathbb{C}^2$. Let $f(z, w) = (p(z), q(z, w))$ be a skew product with Julia set $J \subset \mathbb{C}^2$. $\forall z \in J_p$, let f_z be the fiber over z: $J_p = J \cap (z, \mathbb{C})$. Is there a generic fiber? That is, is there a compact X homeomorphic to f_z for almost every z? Can you find such assuming f is hyperbolic?