

# The Geometry of Two Generator Groups: Hyperelliptic Handlebodies

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**Abstract.** A Kleinian group naturally stabilizes certain subdomains and closed subsets of the closure of hyperbolic three space and yields a number of different quotient surfaces and manifolds. Some of these quotients have conformal structures and others hyperbolic structures. For two generator free Fuchsian groups, the quotient three manifold is a genus two solid handlebody and its boundary is a hyperelliptic Riemann surface. The convex core is also a hyperelliptic Riemann surface. We find the Weierstrass points of both of these surfaces. We then generalize the notion of a hyperelliptic Riemann surface to a “hyperelliptic” three manifold. We show that the handlebody has a unique order two isometry fixing six unique geodesic line segments, which we call the *Weierstrass lines* of the handlebody. The Weierstrass lines are, of course, the analogue of the Weierstrass points on the boundary surface. Further, we show that the manifold is foliated by surfaces equidistant from the convex core, each fixed by the isometry of order two. The restriction of this involution to the equidistant surface fixes six *generalized Weierstrass points* on the surface. In addition, on each of these equidistant surfaces we find an orientation reversing involution that fixes curves through the *generalized Weierstrass points*.

**Keywords:** Fuchsian groups, Kleinian groups, Schottky groups, Riemann surfaces, Hyperelliptic surfaces

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## 1. Introduction

We begin with a Kleinian group,  $G$ , a discrete group of Möbius transformations. We can think of this group as acting on the closure of hyperbolic three space. That is, on the union of  $\mathbb{H}^3$  hyperbolic three space and its boundary,  $\hat{\mathbb{C}}$  the complex sphere. In addition to stabilizing each of these, the group naturally stabilizes certain subdomains

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and closed subsets, yielding a number of different quotient surfaces and manifolds. Some of these quotients have conformal structures and others hyperbolic structures.

If the group is a Fuchsian group, it fixes both the interior and the exterior of a disc in the complex plane, and a corresponding hyperbolic plane lying over the disc. The limit set of the group is contained in the circle separating the discs in the plane. In this case there are additional quotient surfaces to consider and certain symmetries within the quotients or among the quotients.

Kleinian groups whose limit sets are proper subsets of  $\hat{\mathbb{C}}$  and Fuchsian groups whose limit sets are proper subsets of the boundary of their invariant disc are known as “groups of the second kind”. In this paper, we consider only finitely generated groups of the second kind that are also free groups. For such groups, we find the relations between all of these quotients and the relations between the various hyperbolic and conformal metrics that can be placed on these spaces.

The quotient surfaces include the Nielsen kernel, the Fuchsian quotient and its Schottky double. There is also a Nielsen double. The Schottky double is the boundary of the quotient three manifold. For a two generator Fuchsian group (of the second kind) we prove that the boundary of the convex core of the three manifold is the image under a pleating map of the Nielsen double. That is, we find the Fuchsian group that uniformizes the Nielsen double and use it to obtain an isometric map from the Nielsen double to the boundary of the convex core of the manifold. This map is a pleating map.

We are interested primarily in the case where the groups are two-generator groups. For these groups (when they are free and discrete) the quotient three manifold is a genus two solid handlebody and its boundary is a hyperelliptic Riemann surface. In this case we generalize the notion of a hyperelliptic Riemann surface to a “hyperelliptic” three manifold. We show that the handlebody has a unique order two isometry fixing six unique geodesic line segments, which we call the *Weierstrass lines* of the handlebody. The Weierstrass lines are, of course, the analogue of the Weierstrass points on the boundary surface. Further, we show that the manifold is foliated by surfaces equidistant from the convex core, each fixed by the isometry of order two. The restriction of this involution to the equidistant surface fixes six *generalized Weierstrass points* on the surface. In addition, on each of these equidistant surfaces we find an orientation reversing involution that fixes curves through the *generalized Weierstrass points*.

The organization of the paper is outlined in the table of contents below. We begin with basic definitions and notation. We then define

the various quotient spaces of the Kleinian group; that is, the Fuchsian surfaces, the Schottky and Nielsen doubles, the handlebody and the convex core. It turns out that there is a natural dichotomy based on whether the axes of the generators are either disjoint or intersect. We carry out our discussion in two parallel parts; in part I, we state and prove all our results in the disjoint axes case and in part II we do the same for the intersecting axes case.

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## 2. Setup and Notation

We let  $G$  be a finitely generated discrete group of Möbius transformations.

### 2.1. MÖBIUS TRANSFORMATIONS AND METRICS

A fractional linear transformation or a Möbius transformation is a conformal homeomorphism of  $\hat{\mathbb{C}}$  of the form  $z \rightarrow \frac{az+b}{cz+d}$  where  $a, b, c, d \in \mathbb{C}$ ,  $ad - bc = 1$ . We let  $\mathbb{M}$  be the group of Möbius transformations. We also consider elements of  $\overline{\mathbb{M}}$ , the group of anti-conformal Möbius transformations. These act on  $\hat{\mathbb{C}}$  as fractional linear transformations sending  $z \rightarrow \frac{a\bar{z}+b}{c\bar{z}+d}$ .

We use the upper-half-space model for  $\mathbb{H}^3$ . If the coordinates for Euclidean three-space are  $(x, y, t)$ , then  $\mathbb{H}^3 = \{(x, y, t) \mid t > 0\}$ ; the plane with  $t = 0$  is  $\partial\mathbb{H}^3 = \hat{\mathbb{C}}$ .

A Euclidean sphere whose center is on  $\hat{\mathbb{C}}$ , intersects  $\hat{\mathbb{C}}$  in a circle and  $\mathbb{H}^3$  in Euclidean hemisphere whose *horizon* is that circle. Inversion in such a Euclidean sphere fixes  $\mathbb{H}^3$ . It also fixes  $\hat{\mathbb{C}}$  and its restriction to  $\hat{\mathbb{C}}$  has the same action as inversion in the horizon. We include Euclidean planes perpendicular to  $\hat{\mathbb{C}}$  as spheres, thinking of them as “spheres” through  $\infty$ ; the horizon of such a plane is a straight line or “circle” through  $\infty$  on  $\hat{\mathbb{C}}$ . With this convention the terms *inversion* and *reflection* are used interchangeably.

The action of elements of  $\mathbb{M}$  and  $\overline{\mathbb{M}}$  on  $\mathbb{H}^3$  is related to the action on  $\hat{\mathbb{C}}$  as follows: each element of  $\mathbb{M}$  can be factored as a product of reflections in circles lying in  $\hat{\mathbb{C}}$  and its action on  $\hat{\mathbb{C}}$  is the restriction to  $\hat{\mathbb{C}}$  of the product of reflections in the Euclidean hemispheres whose horizons are these circles.

There is a natural metric on  $\mathbb{H}^3$ , the hyperbolic metric, preserved by these reflections. Products of even numbers of reflections are the orientation preserving isometries for this metric and the hyperbolic metric is the unique metric for which  $\mathbb{M}$  is the full group of orientation preserving isometries. Geodesic lines in this metric are circles (and straight lines) orthogonal to  $\hat{\mathbb{C}}$ . The Euclidean hemispheres are geodesic surfaces called *hyperbolic planes*. The boundary of a hyperbolic plane in  $\hat{\mathbb{C}}$  is called the horizon of the plane since it is at infinite distance from every point in the plane.

We set

$$\Delta_{int} = \{z : |z| < 1\}, \quad \Delta_{ext} = \{z : |z| > 1\} \quad \text{and} \quad \partial\Delta = \{z : |z| = 1\}.$$

For readability we use  $\Delta$  for  $\Delta_{int}$  when no confusion is apt to arise.

There is also a natural hyperbolic metric on  $\Delta$  viewed as the hyperbolic plane. The isometries are elements of  $\mathbb{M}$  that fix  $\Delta$  and geodesics are circles orthogonal to  $\partial\Delta$ .

We let  $\mathbb{P}$  denote the plane in  $\mathbb{H}^3$  whose horizon is the unit circle. We can think of  $\mathbb{P}$  as consisting of  $\mathbb{P}_{ext}$  and  $\mathbb{P}_{int}$ , the hyperplane boundaries of hyperbolic half-spaces lying over  $\Delta_{ext}$  and  $\Delta_{int}$  respectively. The hyperbolic metric in  $\mathbb{H}^3$  restricts to a metric on  $\mathbb{P}$  and orthogonal projection from  $\mathbb{P}_{int}$  (or  $\mathbb{P}_{ext}$ ) with this metric to  $\Delta_{int}$  (or  $\Delta_{ext}$ ) with the usual hyperbolic metric  $\rho_\Delta$  (or  $\rho_{\Delta_{ext}}$ ) is an isometry.

We use the notation  $R_{\partial\Delta}$  to denote the reflection in  $\partial\Delta$  mapping  $\hat{\mathbb{C}}$  to itself and  $R_{\mathbb{P}}$  to denote the reflection in  $\mathbb{P}$  mapping  $\mathbb{H}^3$  to itself.

We remind the reader that elements of  $\mathbb{M}$  have both an algebraic classification by their traces, and corresponding geometric classification according to their action on  $\mathbb{H}^3$  or  $\hat{\mathbb{C}}$ .

In this paper, we are concerned with finitely generated free groups that are subgroups of  $\mathbb{M}$ . They contain no elliptic elements. We shall also make the simplifying assumption that there are no parabolic elements. Every element  $A \in \mathbb{M}$  therefore has two fixed points in  $\hat{\mathbb{C}}$ . The line  $Ax_A$  in  $\mathbb{H}^3$  joining the fixed points is invariant under  $A$  and is called its axis. Similarly, if the fixed points of  $A$  lie on  $\partial\Delta$ , the geodesic in  $\Delta$  joining them is also called the axis. We will also denote the axis by  $Ax_A$ . The context will make clear which axis we mean.

We will also have occasion to consider groups generated by elliptic elements of order 2. These groups contain free groups as subgroups of index 2 as well as subgroups containing certain orientation reversing elements of order two.

## 2.2. HYPERELLIPTIC SURFACES AND THE HYPERELLIPTIC INVOLUTION

If a compact Riemann surface of genus  $g$  admits a conformal involution with  $2g + 2$  fixed points, the involution is unique and the surface is called a *hyperelliptic surface*. The fixed points of the hyperelliptic involution are called the *Weierstrass points* ([18] p. 275 and [12]) of the surface. In addition to this geometric definition, these points also have an analytic definition in terms of the possible zeros and poles of meromorphic functions on the surface. Every Riemann surface of genus two is hyperelliptic. It admits a unique conformal involution with six fixed points, the *hyperelliptic involution*.

### 3. Subspaces and Quotient Spaces

#### 3.1. LIMIT SETS, REGULAR SETS, CONVEX REGIONS AND THEIR QUOTIENTS

We assume that  $G$  is a finitely generated Kleinian group. Since it is discrete, it acts discontinuously everywhere in  $\mathbb{H}^3$ . We denote the subset of  $\hat{\mathbb{C}}$  where it acts discontinuously by  $\Omega(G)$  and denote its complement in  $\hat{\mathbb{C}}$ , which is called the *limit set*, by  $\Lambda(G) = \hat{\mathbb{C}} \setminus \Omega(G)$ .

A *fundamental domain*  $F \subset \mathbb{S}$  for  $G$  acting discontinuously on a space  $\mathbb{S} = \Delta, \mathbb{P}, \Omega, \dots$  is an open set such that if  $z_1, z_2$  are a pair of points in the interior of  $F$  and if  $W(z_1) = W(z_2)$  for some  $W \in G$ , then  $W = Id$  and  $\cup_{W \in G} \overline{W(F)} = \mathbb{S}$ .

If  $W$  is any closed subset of  $\hat{\mathbb{C}}$ , we call the smallest closed hyperbolically convex subset of  $\mathbb{H}^3$  containing it, the *convex hull of  $W$  in  $\mathbb{H}^3$* . If  $W$  is the limit set of  $G$  we denote it by  $\mathcal{C}(G)$  or simply  $\mathcal{C}$ . Analogously, if  $W$  is any closed subset of  $\partial\Delta$ , we call the smallest closed hyperbolically convex subset of  $\Delta$  the *convex hull of  $W$  in  $\Delta$* .

Since the set  $W$  we use in this paper is the limit set  $\Lambda$  of a Fuchsian group, we need to distinguish between its convex hull in  $\mathbb{H}^3$  and its convex hull in  $\Delta$ . Because the convex hull in  $\Delta$  of the limit set of a Fuchsian group historically was called the *Nielsen (convex) region for  $G$* , we continue that tradition and denote it by  $K(G)$  or  $K$ .

#### The Quotients for free Kleinian groups of the second kind

**The surface  $S$**  As  $G$  is a group of the second kind,  $\Omega$  is connected. The holomorphic projection  $\Omega(G) \rightarrow \Omega(G)/G = S$  determines a Riemann surface  $S$ . Since  $G$  is finitely generated, free, and contains no parabolics,  $S$  is a compact surface of finite genus.

**The solid handlebody  $\bar{H}$**  The projection  $\mathbb{H}^3 \rightarrow \mathbb{H}^3/G = H$  determines  $H$  as a complete hyperbolic 3-manifold. Its boundary is the Riemann surface  $S = \Omega(G)/G$ . That is,  $\partial H = S$ . Therefore  $\bar{H}$  is a solid handlebody.

**The Convex Core  $\mathcal{C}/G$**  The projection  $\mathcal{C}(G) \rightarrow \mathcal{C}(G)/G = \mathcal{C}/G$  is a convex closed submanifold of  $H$ . Its boundary,  $\partial\mathcal{C}/G$  is homeomorphic to the surface  $S$ .

#### The Quotients for free Fuchsian groups of the second kind

If  $G$  is a Fuchsian group, replacing  $G$  by a conjugate if necessary we may assume that  $G$  acts invariantly on the unit disc.

**The Fuchsian surface  $\Delta/G$ .** The projection  $\Delta \rightarrow \Delta/G$  is a complete Riemann surface that we call the *Fuchsian surface of  $G$* . Since  $G$  is free and contains no parabolics,  $\Delta/G$  is a surface of finite genus with finitely many removed discs.

**The Nielsen kernel  $K(G)/G$**  The *Nielsen kernel of  $G$*  is the quotient of the Nielsen region; that is, the surface  $K(G)/G$ . It is also called the Nielsen kernel of the surface  $\Delta/G$ . It is homeomorphic to  $\Delta/G$ .

If  $G$  is a free group of rank two, then as we will see below,  $S$  is a compact surface of genus two. Thus it is the boundary of a handlebody  $H$  of genus two.

### 3.2. SCHOTTKY DOUBLES, FUNNELS AND NIELSEN DOUBLES

In this section we use the exposition of [2] for constructing the Schottky double and the Nielsen double for Fuchsian groups of the second kind. We also define metrics for them.

For any compact Riemann surface of genus  $g$  with  $k$  holes, one can always form the conformal double. We denote the conformal double of  $\Delta/G$  by  $S$  and the conformal double of  $K(G)/G$  by  $S_K$ . Bers termed  $S$  the *Schottky double* and clarified the relationship between  $S$  and  $S_K$ . (See theorem 3.1 below.) To distinguish between these doubles we call  $S_K$  the *Nielsen double*.

The anti-conformal reflection in  $\partial\Delta$  induces a canonical anti-conformal involution on  $S$  which we denote by  $J$ .

As a compact Riemann surface,  $S$  admits a uniformization by a Fuchsian group  $\hat{G}$  and the projection  $\Delta \rightarrow \Delta/\hat{G} = S$  defines a hyperbolic metric on  $S$ . The restriction of this metric to the Fuchsian surface  $\Delta/G \subset S$  is called the *intrinsic metric* on  $\Delta/G$ .

The Nielsen kernel of  $S$ ,  $K/G$ , is a Riemann surface homeomorphic to  $S$ . One can construct its conformal double  $S_K$  which we call the *Nielsen double*. Again the restriction of the hyperbolic structure on  $S_K$  defines the intrinsic metric on the Nielsen kernel. The intrinsic metric agrees with the restriction of the hyperbolic metric on the Fuchsian surface  $\Delta/G$ .

The intersection of  $\Omega(G)$  with  $\partial\Delta$  consists of an infinite set  $\beta$  of open intervals  $I$ , called the *intervals of discontinuity of  $G$* . The images of the intervals of discontinuity project to (ideal) boundary curves of  $\Delta/G$ .

Each  $I \in \beta$  corresponds to the axis of some element in  $A_I \in G$ . Let  $\text{Circ}_{A_I}$  be the circle determined by the axis of  $A_I$  (that is, the circle in  $\hat{\mathbb{C}}$  orthogonal to  $\partial\Delta$  through the fixed points of  $A_I$ ), and let  $D(A_I)$  be



the disc it bounds containing  $I$ . The quotient  $F(A_I) = D(A_I)/G \subset S$  is called a *funnel* in  $S$ . The curves  $\text{Circ}_{A_I} \cap \Delta_{ext}$ ,  $\text{Circ}_{A_I} \cap \Delta_{int}$  and  $I$  project onto homotopic curves on  $S$ . The first two are the *boundary curves of the funnel* and the last is the *central curve of the funnel*.

The following theorem follows from results in [2]

**Theorem 3.1.** *We have the following relationship between the Nielsen kernel and the Schottky double:*

$$S = \Omega(G)/G = K(G)/G \cup J(K(G)/G) \cup_{I \in \beta} F(A_I)$$

Moreover, the central curves of the funnels are geodesics on  $S$ . The involution  $J$  interchanges the boundary curves and fixes the central curve.

### 3.3. THE THREE-MANIFOLD

We continue assuming  $G$  a Fuchsian group of the second kind with invariant disc  $\Delta$ . As a group acting on  $\mathbb{H}^3$  it acts invariantly on a hyperbolic plane  $\mathbb{P}$  whose horizon is  $\partial\Delta$ . In this section, we consider it as acting on  $\mathbb{P}$ .

**The Nielsen double,  $S_{K_{\mathbb{P}}}$**  The Nielsen region for  $G$  acting on  $\mathbb{P}$ ,  $K_{\mathbb{P}}$ , is the convex hull in  $\mathbb{P}$  of  $\Lambda(G)$ . If we think of  $\mathbb{P}$  as having two sides,  $\mathbb{P}_{int}$  and  $\mathbb{P}_{ext}$  we form the doubled kernel by taking one copy on each side:  $S_{K_{\mathbb{P}}} = K_{\mathbb{P}_{int}}/G \cup K_{\mathbb{P}_{ext}}/G$ . This is isometrically equivalent to the Nielsen double  $S_K$  defined in the previous section.

**The convex hull,  $\mathcal{C}$**  The hyperbolic convex hull of  $\Lambda(G)$  in  $\mathbb{H}^3$  is  $\mathcal{C} = \mathcal{C}(G)$ . Because  $G$  is Fuchsian,  $\mathcal{C}$  is contained in  $\mathbb{P}$  and has no interior so that  $\partial\mathcal{C} = \mathcal{C}$ . It is, therefore, sometimes natural to think of  $\partial\mathcal{C}$  as two-sided with  $\mathcal{C}_{int}$ , the interior side facing  $\Delta_{int}$  and  $\mathcal{C}_{ext}$ , the exterior side facing  $\Delta_{ext}$ . Note we have the obvious identifications  $\mathcal{C} = K_{\mathbb{P}}$ ,  $\mathcal{C}_{int} = K_{\mathbb{P}_{int}}$  and  $\mathcal{C}_{ext} = K_{\mathbb{P}_{ext}}$ .

**The convex core of  $H$ ,  $N = N_{int} \cup N_{ext}$**  Since  $\mathcal{C} = \partial\mathcal{C}$  the convex core  $N = \mathcal{C}/G$  can be thought of as consisting of two hyperbolic surfaces  $N_{int} = \mathcal{C}_{int}/G$  and  $N_{ext} = \mathcal{C}_{ext}/G$  joined along their common boundary curves. Again we have the obvious identifications  $N_{int} = K_{\mathbb{P}_{int}}/G$ ,  $N_{ext} = K_{\mathbb{P}_{ext}}/G$ . Note that  $N$  is isometric to  $S_{K_{\mathbb{P}}}$ .

We conclude this section with a concept that relates surfaces and three manifolds.

**The Pleated surface**  $(\tilde{S}, pl)$  A pair,  $(\tilde{S}, pl)$ , is a *pleated surface*, if  $\tilde{S}$  is a complete hyperbolic surface and  $pl$  is an immersion and a hyperbolic isometry of  $\tilde{S}$  into a hyperbolic three manifold  $H$ , such that every point in  $\tilde{S}$  is in the interior of some geodesic arc which is mapped to a geodesic in  $H$ . The *pleating locus* is the set of points in  $\tilde{S}$  that are contained in the interior of exactly one geodesic arc that is mapped by  $pl$  to a geodesic arc.

Our goal is to explain the connection between the various quotient Riemann surfaces and related quotient hyperbolic surfaces and quotient three manifolds we have now defined. We will restrict our discussion to the situation where  $G$  is a free two generator Fuchsian group.

We will show that if  $G$  is a two generator Fuchsian group, then  $N$ , with its intrinsic metric, is the image of the Nielsen double  $S_K$ , with its intrinsic metric, under a pleating map,  $pl$ . We will see that the image of the pleating locus consists of the identified geodesic boundary curves of  $N_{int}$  and  $N_{ext}$ . Further we will show that  $\bar{H} = \mathcal{S} \times I$  where  $\mathcal{S}$  is a surface of genus 2,  $I = [0, \infty]$  and  $\mathcal{S} \times 0 = N$ ,  $\mathcal{S} \times \infty = S$ , and the surfaces  $\mathcal{S} \times s = S(s)$  are *equidistant surfaces* from the convex core  $N$ .

We begin with the discrete free Fuchsian group on two generators,  $G = \langle C, D \rangle$ . We want to find a set of generators  $\langle A, B \rangle$  for  $G$  that have two special properties: first, they are *geometric*, that is, they are the side pairings for a fundamental polygon ([16]) and second, the traces of  $A$  and  $B$  are less than the traces for any other pair of generators. Although we know  $G$  is discrete, we apply what is known as the discreteness algorithm to find the special generators. They are the pair of generators at which the algorithm stops. We thus call  $\langle A, B \rangle$  the *stopping generators* for  $G$ .

The first step of the algorithm determines whether the axes of the generators intersect or not. This is determined by computing the trace of the commutator,  $[C, D] = CDC^{-1}D^{-1}$  (see [4] or [17]). If  $Tr [C, D] < -2$ , the axes intersect; if  $-2 < Tr [C, D] < 2$ , the group contains an elliptic element; and if  $Tr [C, D] > 2$ , the axes are disjoint. Either the axes intersect for every Nielsen equivalent pair of generators or they are disjoint for every pair. Since  $G$  is free, it contains no elliptics and since we have assumed it contains no parabolics, we have  $|Tr [C, D]| > 2$ .

Our constructions are somewhat different depending on whether the axes of the generators are disjoint or not. We separate these constructions into two parts.

**Part 1. DISJOINT AXES -  $Tr [C, D] > 2$** 

Throughout part I we assume the axes of the generators of  $G$  are disjoint. We begin by finding fundamental domains for  $G$  acting on  $\Delta$  and on  $K(G)$ . Next, we find the Weierstrass points for  $S$ . We then turn to  $\mathbb{H}^3$  and find fundamental domains for  $G$  acting on  $\mathbb{P}$  and on  $\mathcal{C}$ . We construct the Fuchsian group uniformizing  $S_K$  and the pleating map sending  $K(G)$  onto  $\partial\mathcal{C}$ . We then use the pleating map to find the Weierstrass points of  $S_K$ . Finally, we work with the handlebody  $H$  and construct the Weierstrass lines and the equidistant surfaces. We conclude with a discussion of anti-conformal involutions on the various quotients.

**4. Subspaces of  $\Delta$  and quotients by  $G$** **4.1. FUNDAMENTAL DOMAINS IN  $\Delta$  AND  $K(G)$** 

When the axes of the generators of  $G$  are disjoint, the Gilman-Maskit discreteness algorithm [7] proceeds by successively replacing the generators with a Nielsen equivalent. After finitely many steps, it decides whether the group generated by the original generators is discrete or not. If the group is discrete, the final set of generators, the stopping generators, are both geometric and have minimal traces.

The reflection lines  $(L, L_A, L_B)$  defined as follows for any pair of generators, play a major role in the algorithm:

- $L$  is the common perpendicular in  $\Delta$  to the axes of  $A$  and  $B$
- $L_A$  is the geodesic in  $\Delta$  such that  $A$  factors as the product  $A = R_L R_{L_A}$  where  $R_K$  denotes reflection in the geodesic  $K$ .
- $L_B$  is the geodesic in  $\Delta$  such that  $B = R_L R_{L_B}$

The stopping condition for the algorithm is that the lines  $(L, L_A, L_B)$  bound a domain in  $\Delta$ ; that is, no one separates the other two. This is equivalent to  $R_L, R_{L_A}, R_{L_B}$  being geometric generators for the index two extension of  $G$ ,  $\langle G, R_L \rangle$ ; the domain bounded by  $(L, L_A, L_B)$  is the fundamental domain.

Note that  $A^{-1}B = R_{L_A} R_{L_B}$  so that its axis is the common perpendicular to  $L_A$  and  $L_B$ . In fact, the axes  $(Ax_A, Ax_B, Ax_{A^{-1}B})$  bound a domain if and only if the lines  $(L, L_A, L_B)$  do. We orient the lines so that  $L$  is oriented from  $Ax_A$  to  $Ax_B$ ,  $L_B$  is oriented from  $Ax_B$  to  $Ax_{A^{-1}B}$  and  $L_A$  is oriented from  $Ax_{A^{-1}B}$  to  $Ax_A$ .

Taking the union of the domain bounded by the lines  $(L, L_A, L_B)$  together with its reflection in  $L$ , we obtain a domain  $F$  in  $\Delta$  bounded by the lines  $L_A, L_B, L_{\bar{A}} = R_L(L_A) = A(L_A)$  and  $L_{\bar{B}} = R_L(L_B) = B(L_B)$ .

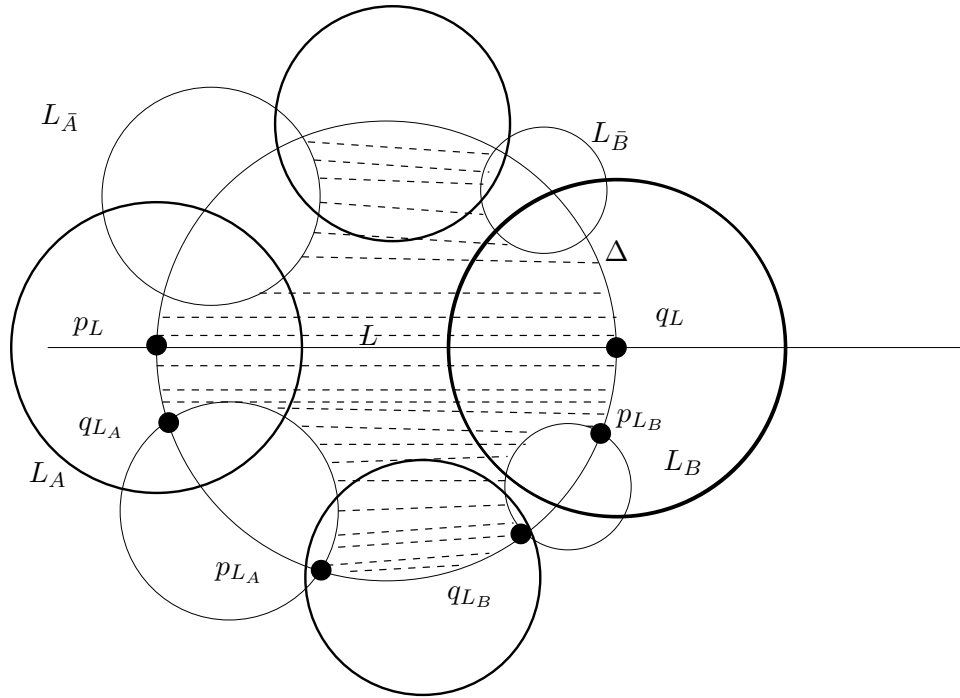


Figure 1. The domain  $F \subset \Delta$  is shaded. The axes, drawn with their reflections in  $\partial\Delta$  are the darker circles; the lines  $L, L_A, L_B, L_{\bar{A}}, L_{\bar{B}}$  drawn with their reflections are the lighter circles.

**Proposition 4.1.** *If  $\langle A, B \rangle$  are the stopping generators for  $G$ , then  $F$  is a fundamental domain for  $G$  acting in  $\Delta$ . If  $F_K$  is the domain obtained from  $F$  by truncating along the axes of  $A, B, A^{-1}B$  and  $AB^{-1}$  then  $F_K$  is a fundamental domain for the action of  $G$  on the Nielsen region of  $G$  in  $\Delta$ .*

*Proof.* By construction, the algorithm stops at the pair of generators  $\langle A, B \rangle$  such that  $F$  is a fundamental domain for  $G$  acting on  $\Delta$  ([7]).

Let  $D(I_A)$  be the half plane bounded by  $Ax_A$  that does not contain the segment of  $L$  between the axes of  $A$  and  $B$ . Note that it is invariant under  $A$ . Tiling  $\Delta$  with copies of  $F$ , we see that the only images of  $F$  that intersect  $D(I_A)$  are of the form  $A^n(F)$ , for some integer  $n$ . It follows that  $D(I_A)$  is stabilized by the cyclic group  $\langle A \rangle$  and the interval  $I_A$  joining the fixed points of  $Ax_A$  is an interval of discontinuity for  $G$ . The same is true for the other three intervals  $I_B, I_{A^{-1}B}$  and  $I_{AB^{-1}}$  corresponding to the other three axes that intersect  $F$ . We conclude that these axes all lie on the boundary of the Nielsen convex region for  $G$ .

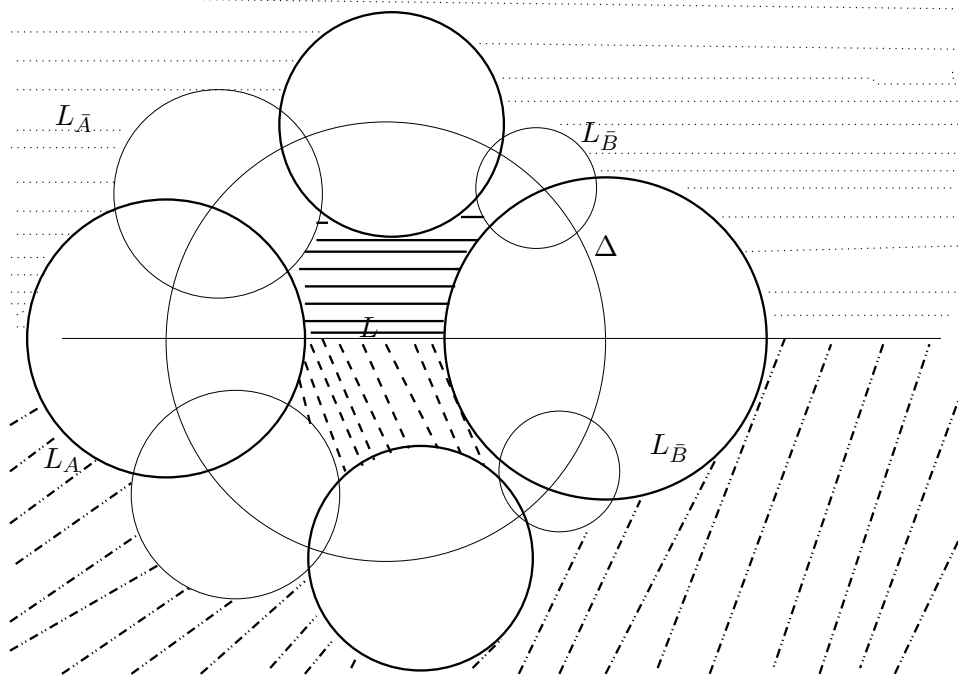


Figure 2. The domain  $F_K \subset \Delta$  and its reflection  $R_{\partial\Delta}(F_K)$  are shaded.

Identifying the sides of  $F$  we easily see that  $\Delta/G$  is a sphere with three holes. The three ideal boundary curves are the projections of the intervals  $I_A, I_B$  and  $I_{A^{-1}B}$ ; note that  $A^{-1}B$  and  $AB^{-1}$  are conjugate.

Since the Nielsen kernel  $K(G)/G$  is homeomorphic to  $\Delta/G$ , it has only three boundary curves. Looking at  $F$ , these must be the projections of the axes of  $A, B$  and  $A^{-1}B$  and all their conjugates. Thus the boundary of the Nielsen region consists of the axes of the generators and the axes of all their conjugates and  $F_K$  is a fundamental domain as claimed.  $\square$

#### 4.2. WEIERSTRASS POINTS OF THE SCHOTTKY DOUBLE $S$

We now want to reflect the lines in  $\partial\Delta$  so we introduce subscripts to distinguish between the lines and their reflections as oriented line segments in  $\hat{\mathbb{C}}$ . Set  $L_{int} = L$ ,  $L_{ext} = R_{\partial\Delta}(L)$ , etc. The lines and their reflections in  $\partial\Delta$  form triple of circles which we denote by  $(\text{Circ}_L, \text{Circ}_{L_A}, \text{Circ}_{L_B})$ . The circles are oriented so that  $\text{Circ}_L = L_{int} \cup (L_{ext})^{-1}$ . Note further that these three circles also bound a domain in  $\hat{\mathbb{C}}$  and in fact, a domain in  $\Omega(G)$ .

**Theorem 4.2.** *Suppose  $G = \langle C, D \rangle$  is a discrete free Fuchsian group such that the axes of  $C$  and  $D$  do not intersect. Let  $A, B$  be the stopping generators determined by the Gilman-Maskit algorithm and let the points  $p_L, q_L, p_{L_B}, q_{L_B}, p_{L_A}, q_{L_A}$  be the respective intersections of  $L, L_A, L_B$  with  $\partial\Delta$  (labelled in clockwise order beginning with  $p_L$  and ending with  $q_{L_B}$ ) and so that  $(q_L, p_{L_B}), (q_{L_B}, p_{L_A}), (q_{L_A}, p_L)$  are disjoint segments of  $\partial\Delta$  respectively in the intervals of discontinuity bounded by the axes  $Ax_B, Ax_{A^{-1}B}$  and  $Ax_A$ . Then the Schottky double  $S$  is a compact Riemann surface of genus two and the projections of these points onto  $S$  under the projection  $\Omega(G) \rightarrow \Omega(G)/G = S$  are the Weierstrass points of  $S$ .*

*Proof.* To simplify notation, we denote the reflections  $R_{\text{Circ}_L}$  by  $R_L$ ,  $R_{\text{Circ}_{L_A}}$  by  $R_{L_A}$  and  $R_{\text{Circ}_{L_B}}$  by  $R_{L_B}$ . Consider the circles  $\text{Circ}_{L_{\bar{A}}} = R_L(\text{Circ}_{L_A}) = A(\text{Circ}_{L_A})$  and  $\text{Circ}_{L_{\bar{B}}} = \text{Circ}_{R_L(L_B)} = B(\text{Circ}_{L_B})$ .

Since  $(L, L_A, L_B)$  bound a domain in  $\Delta$ , the four circles  $\text{Circ}_{L_A}$ ,  $\text{Circ}_{L_{\bar{A}}}$ ,  $\text{Circ}_{L_B}$ , and  $\text{Circ}_{L_{\bar{B}}}$  bound a domain  $F$  in  $\Omega$ .

This domain is the union of the domain  $F$  of figure 4.1 and its reflection in  $\partial\Delta$ . Moreover,  $A$  maps the exterior of  $\text{Circ}_{L_A}$  to the interior of  $\text{Circ}_{L_{\bar{A}}}$  and  $B$  maps the exterior of  $\text{Circ}_{L_B}$  to the interior of  $\text{Circ}_{L_{\bar{B}}}$ . One can easily verify that  $F$  is a fundamental domain for  $G = \langle A, B \rangle$  and that  $\Omega(G)/G$  is a compact Riemann surface of genus two.

Let  $R_{\partial\Delta}$  denote reflection in  $\partial\Delta$  so that  $R_{\partial\Delta} : \Delta_{\text{int}} \rightarrow \Delta_{\text{ext}}$ . Set  $E = R_{\partial\Delta}R_L$ . The product of reflections in a pair of intersecting circles is a rotation (an elliptic element of  $\mathbb{M}$ ) about the intersection points of the circles (the fixed points) with rotation angle equal to twice the angle between the circles. As  $E$  is the elliptic of order two with fixed points  $p_L, q_L$ , it takes any circle through  $p_L$  and  $q_L$  into itself and interchanges the segments between the endpoints. In particular, it sends  $\text{Circ}_L$  to itself and sends  $L_{\text{int}}$  to  $(L_{\text{ext}})^{-1}$ .

One easily verifies that  $EAE^{-1} = A^{-1}$  and  $EBE^{-1} = B^{-1}$  so that  $E$  is an orientation preserving conformal involution from  $\Omega(G)$  to itself that conjugates  $G$  to itself. It therefore induces a conformal involution  $j : S \rightarrow S$ .

Similarly,  $E_A = R_{\partial\Delta}R_{L_A} = EA$  and  $E_B = R_{\partial\Delta}R_{L_B} = EB$  are elliptic elements of order two with fixed points  $\{p_{L_A}, q_{L_A}\}$  and  $\{p_{L_B}, q_{L_B}\}$  respectively. Therefore,  $E(p_{L_A}) = A(p_{L_A}), E(q_{L_A}) = A(q_{L_A})$  and  $E(p_{L_B}) = B(p_{L_B}), E(q_{L_B}) = B(q_{L_B})$ . Since  $E_A E^{-1} \in G$  and  $E_B E^{-1} \in G$ , we conclude that  $E_A, E_B$  and  $E$  induce the same conformal involution  $j$ . Moreover, the projections of  $p_L, q_L, p_{L_A}, q_{L_A}, p_{L_B}, q_{L_B}$  are distinct points each, fixed by  $j$ . Since  $j$  is a conformal involution fixing six distinct points, it is the hyperelliptic involution on  $S$ .  $\square$

The reflection  $R_{\partial\Delta}$  also normalizes  $G$  and thus descends to a homeomorphism of  $S$  which is the anti-conformal involution  $J$  (see section 3.1). As we saw in section 3.2, the clockwise arc of  $\partial\Delta$  joining  $q_{L_A}$  to  $A(q_{L_A}) = R_L(q_{L_A})$  projects to the central curve of the funnel on  $S$  corresponding to the projection of interval of discontinuity bounded by the axis of  $A$ . Similarly, the clockwise arc from  $R_L(p_{L_B})$  to  $p_{L_B}$  projects to the central curve of the funnel on  $S$  corresponding to the projection of interval of discontinuity bounded by the axis of  $B$ . In addition, the pair of segments joining (counterclockwise)  $q_{L_B}$  to  $p_{L_A}$  and  $R_L(p_{L_A})$  to  $R_L(q_{L_B})$  together project the central curve of the third funnel corresponding to the projection of the interval of discontinuity of  $A^{-1}B$ . These three simple curves are fixed point-wise by  $J$  and they partition  $S$  into two spheres with three holes (pairs of pants).

The product  $Jj$  is again an anti-conformal self-map of  $S$ . Its invariant curves are the projections of the circles  $\text{Circ}_L, \text{Circ}_{L_A}, \text{Circ}_{L_B}$ . The homotopy classes of these projections also partition  $S$  into two spheres with three holes.

We remind the reader that  $J$  is an anti-conformal map on  $S$  that fixes its three fixed curves point-wise and  $j$  is a conformal map that sends each fixed curve into itself as a point-set, but only fixes two points on each curve and maps the curve into its inverse.

## 5. The convex hull and the convex core

In this section we consider the group  $G$  acting on  $\mathbb{P}, \mathbb{P}_{int}$  and  $\mathbb{P}_{ext}$  which contain respectively the convex hull  $\mathcal{C}(G)$  in  $\mathbb{H}^3$  and its two boundaries  $\mathcal{C}_{int}$  and  $\mathcal{C}_{ext}$ . All the axes of elements of  $G$  lie in  $\mathbb{P}$ .

Note that the reflection  $R_{\mathbb{P}}$  interchanges  $\mathbb{P}_{int}$  and  $\mathbb{P}_{ext}$ . Since  $\mathbb{P} \subset \mathbb{H}^3$ , in addition to factoring elements of  $G$  as products of reflections in hyperbolic planes we can also factor them products of half turns about hyperbolic lines. (See [3]). If  $M$  is a hyperbolic line in  $\mathbb{H}^3$ , we denote the half turn about  $M$  by  $H_M$ .

### 5.1. FUNDAMENTAL DOMAINS IN $\mathbb{P}$ AND $\mathcal{C}$

Since the axes of the generators  $C$  and  $D$  do not intersect and these axes lie in the plane  $\mathbb{P}$  (which is another model for  $\Delta$ ), we can apply the Gilman-Maskit algorithm to obtain stopping generators  $A, B$  as above. We again have a triple of lines  $L, L_A, L_B$  defined as follows:  $L$  is the hyperbolic line mutually orthogonal to the axes of  $A$  and  $B$ ; it now lies in  $\mathbb{P}$ . The hyperbolic line  $L_A$  in  $\mathbb{P}$  is determined by the factorization  $A = H_L H_{L_A}$ . Similarly  $L_B$  is the line in  $\mathbb{P}$  such that  $B = H_L H_{L_B}$ . The

condition for  $A, B$  to be stopping generators is that the lines  $L, L_A, L_B$  bound a domain in  $\mathbb{P}$ . Set  $L_{\bar{A}} = H_L(L_A)$ ,  $L_{\bar{B}} = H_L(L_B)$ . It is easy to check that  $L_A$  is the mutual perpendicular to the axes of  $A$  and  $A^{-1}B$  while  $L_B$  is the mutual perpendicular to the axes of  $B$  and  $A^{-1}B$

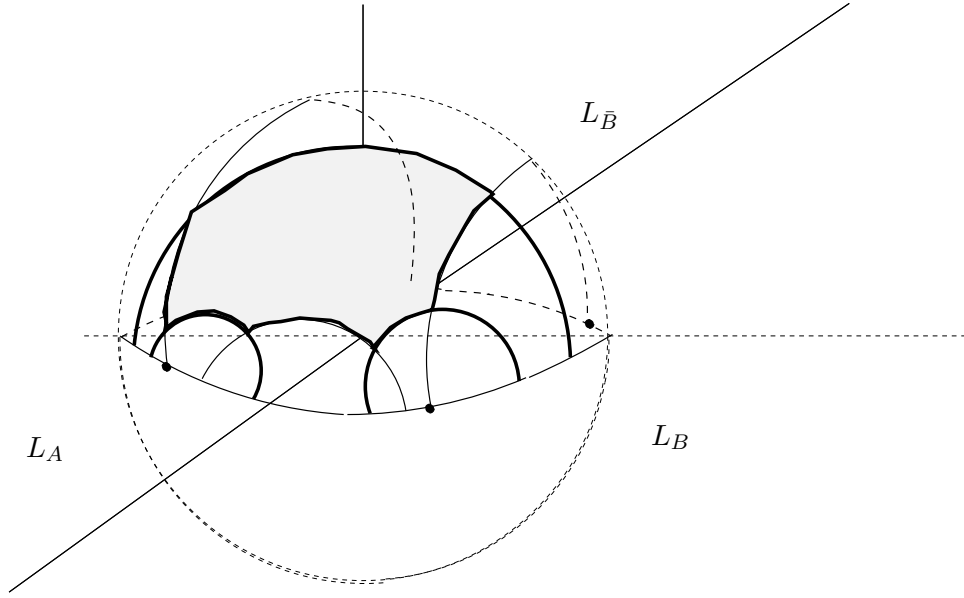


Figure 3. The domain  $\mathcal{H} \subset \mathbb{P} \subset \mathbb{H}^3$ .

Let  $\mathcal{H}$  be the closed right-angled hexagon formed by the six hyperbolic lines,  $L, L_A, L_B$  and  $Ax_A, Ax_B, Ax_{A^{-1}B}$  and shown in figure 5.1.

**Proposition 5.1.** *The interior of the domain  $F_{\mathcal{C}} = \text{int}(\mathcal{H} \cup H_L(\mathcal{H}))$  is a fundamental domain for  $G$  acting on the convex hull  $\mathcal{C}(G)$ .*

*Proof.* Since  $\mathcal{C}(G) \subset \mathbb{P}$ , it is same as the Nielsen region  $K(G_{\mathbb{P}})$ , and the proposition follows directly from proposition 4.1.  $\square$

We define the domains by  $F_{int} = F_{\mathcal{C}} \cap \mathbb{P}_{int}$  and  $F_{ext} = F_{\mathcal{C}} \cap \mathbb{P}_{ext}$  respectively. Note that  $F_{int} \subset \mathcal{C}_{int} \subset \mathbb{P}_{int}$  and  $F_{ext} \subset \mathcal{C}_{ext} \subset \mathbb{P}_{ext}$ . An immediate corollary of the above proposition is

**Corollary 5.2.** *The domain  $F_{int} \cup F_{ext}$  is a fundamental domain for  $G$  acting on  $\mathcal{C}_{int} \cup \mathcal{C}_{ext} = \partial\mathcal{C}$ .*

If we identify the appropriate sides of  $F_{\mathcal{C}}$  under the action of  $A$  and  $B$ , we obtain a sphere with three boundary curves that are geodesic in the intrinsic metric. Therefore, identifying the sides of  $F_{int} \cup F_{ext}$  we obtain two such spheres. Gluing their boundary curves yields a surface



of genus two as the boundary of the convex core that is isometric with the Nielsen double of  $G$ .

## 6. The Pleated Surface

In this section we explicitly construct the pleated surface  $(\tilde{S}, pl)$  where the hyperbolic surface  $\tilde{S} = S_K$ . As usual we assume we have simple stopping generators with disjoint axes.

To define the pleated surface we need first to construct a hyperbolic surface of genus two. We do this by constructing a Fuchsian group and forming the quotient. The hyperbolic surface we obtain is the Nielsen double,  $S_K$ .

### 6.1. THE FUCHSIAN GROUP FOR THE NIELSEN DOUBLE

We begin with the fundamental domain  $F_K = \mathcal{H} \cup R_L(\mathcal{H}) \subset \Delta$  defined in section 4 for the Nielsen kernel. Set  $F_\Gamma = F_K \cup R_{Ax_A}(F_K) \subset \Delta$ . Note that  $F_\Gamma$  contains four copies of the hexagon  $\mathcal{H}$ . Define a new group acting on  $\Delta$  by  $\Gamma = \langle A, A', B, B' \rangle$ , where  $A' = R_{Ax_{A^{-1}B}} R_{Ax_A}$  and  $B' = R_{Ax_B} R_{Ax_A}$ .

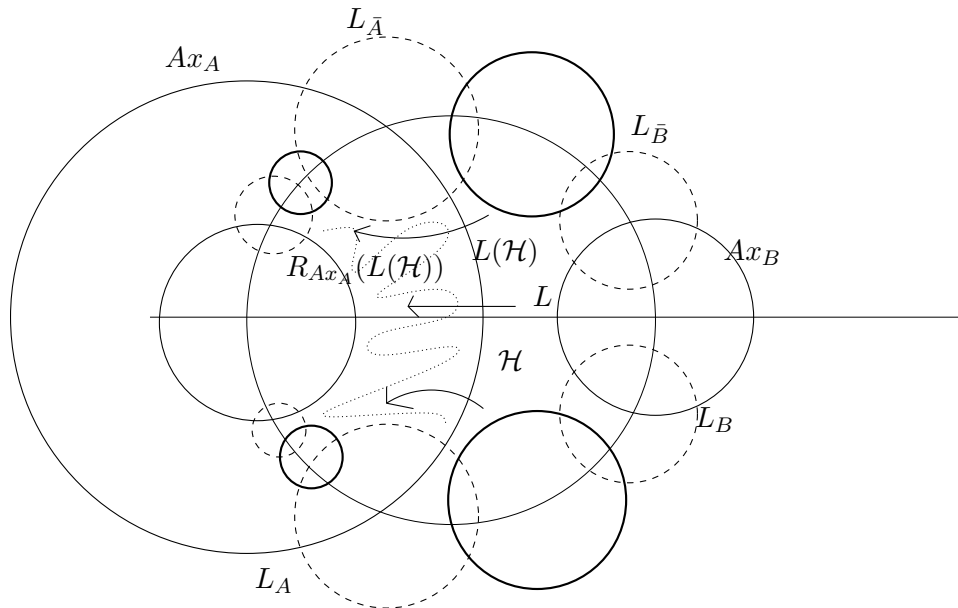


Figure 4. The fundamental domain for  $\Gamma$  is a union of four copies of  $\mathcal{H}$ . The circle sides,  $L_A, L_{\bar{A}}, L_B, L_{\bar{B}}$  and their reflections in  $Ax_A$  are drawn dotted. The axis sides are drawn solid.

**Proposition 6.1.** *The domain  $F_\Gamma$  is a fundamental domain for the group  $\Gamma = \langle A, A', B, B' \rangle$  acting on  $\Delta$ . The Fuchsian surface  $\tilde{S} = \Delta/\Gamma$  is a Riemann surface of genus two isometric to the Nielsen double  $S_K$ .*

*Proof.* It suffices to show that the sides are identified in pairs by elements of  $\Gamma$  and the angles at the vertices satisfy the conditions of the Poincaré polygon theorem ([13]).

In  $F_K$  the pair of sides that are segments of  $L_B$  and  $L_{\bar{B}}$  are identified by  $B$ ; similarly the sides of  $L_A$  and  $L_{\bar{A}}$  in  $F_\Gamma$  are identified by  $A$ . In  $F_\Gamma$  the pair of sides that are segments  $Ax_B$  and  $R_{Ax_A}(Ax_B)$  are identified by  $(B')^{-1}$  since  $R_{Ax_A} : R_{Ax_A}(Ax_B) \rightarrow Ax_B$  and  $R_{Ax_B} : Ax_B \rightarrow Ax_B$ . Then the sides that are segments of  $R_{Ax_A}(L_B)$  and  $R_{Ax_A}(L_{\bar{B}})$  are identified by  $B'B(B')^{-1}$ . Similarly, The sides that are segments of  $Ax_{A^{-1}B}$  and  $R_{Ax_A}(Ax_{A^{-1}B})$  are identified by  $(A')^{-1}$  and the sides that are segments of  $Ax_{AB^{-1}}$  and  $R_{Ax_A}(Ax_{AB^{-1}})$  are identified by  $AA'A^{-1}$ . (See figure 6.1.)

It is easy to check that the twelve vertices of  $F_\Gamma$  are partitioned into three cycles of four vertices. Denote the three vertices of  $\mathcal{H}$  in  $F_\Gamma$  as follows: let  $p_A$  be the intersection of  $Ax_{A^{-1}B}$  with  $L_A$ , let  $p_B$  be the intersection of  $Ax_{A^{-1}B}$  with  $L_B$  and let  $q_B$  be the intersection of  $Ax_B$  and  $L_B$ . The cycles are then

$$\begin{aligned} &\{p_A, A'(p_A), AA'(p_A), A(p_A)\}, \\ &\{p_B, B(p_B), B'B(p_B), B'(p_B)\}, \\ &\{q_B, A'(q_B), B(q_B), B'B(B')^{-1}A'(q_B)\} \end{aligned}$$

As each angle in  $F_\Gamma$  is a right angle, the sum of the angles at each cycle of vertices is  $2\pi$ .

We conclude that  $F_\Gamma$  is a fundamental domain for the group generated by the side pairing transformations. Since these are generated by the generators of  $\Gamma$ ,  $F_\Gamma$  is a fundamental domain for  $\Gamma$ .

Sides of  $F_\Gamma$  either are termed axis sides or reflection sides depending upon whether they are segments of an axis of an element of  $G$  or segments of a reflection line of  $G$ .

Now  $F_K$  is composed of two copies of  $\mathcal{H}$ , and, as we saw in section 4, identifying non-axis sides (segments of  $L_A, L_{\bar{A}}, L_B, L_{\bar{B}}$ ) yields a sphere with three holes. The boundaries of the holes are the identified axes  $Ax_A, Ax_B, Ax_{AB^{-1}}$ . Similarly  $R_{Ax_A}(F_K)$  with non-axis sides identified is another three holed sphere. The axis sides ( $Ax_A, Ax_B, Ax_{AB^{-1}}$ ) of  $F_\Gamma$  are identified as indicated above, and the two three holed spheres join up to form a compact surface  $\tilde{S}$  of genus two. Since  $F_\Gamma$  is just  $F_K$  doubled, the surface  $\tilde{S}$  is isometric to the Nielsen double  $S_K$ .  $\square$

## 6.2. THE PLEATING MAP

To construct the pleating map  $pl : \Delta/\Gamma \rightarrow \mathbb{H}^3/G$ , we begin by defining a pleating map  $PL : \Delta \rightarrow \mathbb{H}^3$ . We first define  $PL$  on  $F_\Gamma$  and then extend by the groups  $\Gamma$  and  $G$ . Recall that the domain  $F_\Gamma \subset \Delta$  is a union of two copies of  $F_K$ , and  $F_{int} \subset \mathbb{P}_{int}$  and  $F_{ext} \subset \mathbb{P}_{ext}$  as defined in section 5.1 are each isometric to  $F_K$ . Set  $PL : F_K \rightarrow F_{ext}$ . Next set  $PL : R_{Ax_A}(F_K) \rightarrow F_{int}$ . Define the group homomorphism  $\phi : \Gamma \rightarrow G$  by first defining it on the generators:

$$A \mapsto A, B \mapsto B, A' \mapsto id, B' \mapsto id.$$

To show that this map on generators gives a group homomorphism from  $\Gamma$  to  $G$  we must show that  $\phi$  preserves the defining relation(s) of  $\Gamma$ . We know from the Poincaré polygon theorem that the reflection relations and the cycle relations form a complete set of relations for  $\Gamma$ . There are no reflection relations and it is easy to calculate that substituting  $\phi(A)$ ,  $\phi(A')$ ,  $\phi(B)$ , and  $\phi(B')$  into the cycle relation gives the identity.

We now extend  $PL$  to a map from  $\Delta$  into  $\mathbb{H}^3$ . That is, for  $x \in \Delta$  let  $g \in \Gamma$  be chosen so that  $g(x) \in F_\Gamma$  and set  $PL(x) = \phi(g^{-1}) \circ PL \circ g(x)$ . This is well defined when  $g(x)$  lies interior to  $F_\Gamma$  and when  $g(x)$  lies on a boundary curve of  $F_\Gamma$  or on  $Ax_A$  it is defined by continuity. Thus,  $(\Delta, PL)$  is a pleated surface with image in  $\mathbb{H}^3$  whose pleating locus is the image of the axis sides of  $F_\Gamma$  and all their images under  $\Gamma$ .

We now define the map  $pl : \tilde{S} \rightarrow \mathbb{H}^3/G$  by taking quotients. The map  $pl$  is clearly a hyperbolic isometry since  $PL$  is and its pleating locus consists of the images of  $Ax_A$ ,  $Ax_B$  and  $Ax_{A^{-1}B}$ .

By corollary 5.2, we can apply the group  $G$  to identify  $PL(F_\Gamma)$  with  $\partial\mathcal{C}$ , and taking quotients, identify  $pl(S_K)$  with the convex core boundary  $\partial N = \mathcal{C}_{int}/G \cup \mathcal{C}_{ext}/G$ .

## 6.3. THE WEIERSTRASS POINTS OF THE NIELSEN DOUBLE $S_K$

We now determine the Weierstrass points of  $S_K$ . This is very reminiscent of our earlier discussion of the Weierstrass points of the Schottky double, except that now, instead of using the simple reflection in  $\partial\Delta$  we need to use reflections in the axis sides of  $\mathcal{H}$  that correspond to the generating triple with simple axes.

Let  $E_0 = R_L R_{Ax_A}$ . Let  $p$  be the fixed point of  $E_0$ .

Let  $E_1 = R_L R_{Ax_B}$ . Let  $q$  be the fixed point of  $E_1$ .

Let  $E_2 = R_{L_A} R_{Ax_A}$ . Let  $q_A$  be the fixed point of  $E_2$ .

Note that  $E_1 = B'E_0$  and  $E_2 = A^{-1}E_0$ . The vertices of  $F_\Gamma$  are also fixed points of order two elliptics:  $p_A$  is the fixed point of  $E_3 =$

$E_2A'^{-1} = A^{-1}E_0A'^{-1}$ ,  $q_B$  is the fixed point of  $E_4 = A^{-1}E_0A'B^{-1}A = E_3B^{-1}A$  and  $p_B$  is the fixed point of  $E_5 = E_1B = B'E_0B$ .

**Theorem 6.2.** *The Weierstrass points of  $S_K$  are the projections of the six points  $p, q, p_A, q_A, p_B, q_B$ .*

*Proof.* Since  $E_0 = R_L R_{Ax_A}$ , we see that  $E_0(F_\Gamma) = F_\Gamma$  and verify that  $E_0$  normalizes  $\Gamma$ . It follows that  $E_0$  induces a conformal involution  $\tilde{j}$  on  $\tilde{S}$ .

Similarly,  $E_i$ ,  $i = 1, \dots, 5$  preserves the tiling by images of  $F_\Gamma$  and thus normalizes  $\Gamma$  so that  $E_i$  projects a conformal involution. Since the products of pairs of the  $E_i$ 's are elements of  $\Gamma$ , they all project to the same conformal involution  $\tilde{j}$  on  $\tilde{S}$ . As the six points are inequivalent under  $\Gamma$ , they project to six distinct fixed points of the involution  $\tilde{j}$ , characterizing it as the hyperelliptic involution.  $\square$

## 7. Weierstrass points and lines in the handlebody

In this section we give an explicit description of the handlebody  $H$  as  $\mathcal{S} \times I$  where  $\mathcal{S}$  is a compact surface of genus two and  $I$  is the interval  $[0, \infty]$ . We show that  $H$  admits a unique order two isometry  $j$  that we call the *hyperelliptic isometry of  $H$* . It fixes six unique geodesic line segments, which we call the *Weierstrass lines of  $H$* . The Weierstrass lines are, of course, the analogue of the Weierstrass points on the boundary surface.

In addition, we show that  $H$  is foliated by surfaces  $\mathcal{S}(s) = (\mathcal{S}, s)$  that are at hyperbolic distance  $s$  from the convex core, and that each  $\mathcal{S}(s)$  is fixed by  $j$ . The restriction of  $j$  to  $\mathcal{S}(s)$  fixes six *generalized Weierstrass points* on  $\mathcal{S}(s)$ . On each  $\mathcal{S}(s)$  we also find an orientation reversing involution  $J$  that fixes curves through the generalized Weierstrass points.

### 7.1. CONSTRUCTION OF THE FOLIATION

We construct the *hyperelliptic isometry for  $H$*  as follows. We saw, in section 5.1, that  $G$  is generated by products of even numbers of half-turns in the lines  $(L, L_A, L_B)$ . It follows that  $G$  is a normal subgroup of index two in the group  $\langle H_L, H_{L_A}, H_{L_B} \rangle$ . Since  $H_L$ ,  $H_{L_A}$  and  $H_{L_B}$  all differ by elements of  $G$ , their actions by conjugation on  $G$  all induce the same order two automorphism  $j$  under the projection  $\pi : \mathbb{H}^3 \rightarrow \mathbb{H}^3/G = H$ . The fixed points of  $j$  in  $H$  are precisely the points on the images of the lines  $L, L_A, L_B$ ; denote the projection of the lines by  $(\pi(L), \pi(L_A), \pi(L_B))$ .

We work with the lines  $L$  and  $\pi(L)$  but we have analogous statements for the lines  $L_A$  and  $L_B$  and their projections. As we saw in theorem 4.2, the endpoints  $(p, q)$  of  $L$  on  $\partial\Delta$  project to Weierstrass points on the Schottky double  $S = \partial H = \mathcal{S}(\infty)$ .

In theorem 6.2 we found the Weierstrass points of the Nielsen double  $S_K$ . In the fundamental domain  $F_\Gamma \subset \Delta$  for  $\Gamma$  we found points  $p \in F_\Gamma$  and  $q \in \partial F_\Gamma$  such that  $PL(p)$  and  $PL(q)$  both lie on the line  $L \in \mathbb{P}$ . For the moment, call the hyperelliptic involution on  $S_K$ ,  $j_{S_K}$  and the hyperelliptic involution on  $H$ ,  $j_H$ . By construction, the pleating map  $pl$  commutes with these involutions:  $pl \circ j_{S_K} = j_H \circ pl$ . It follows that the projections  $\tilde{p} = \pi(PL(p))$  and  $\tilde{q} = \pi(PL(q))$  are fixed under  $j_H$ . By the same argument, we find two other pairs of points  $\pi(PL(p_A)), \pi(PL(q_A))$  on  $\pi(L_A)$  and  $\pi(PL(p_B)), \pi(PL(q_B))$  on  $\pi(L_B)$  fixed by  $j_H$ . They lie on the genus two surface  $\partial N = \mathcal{S}(0)$  and are its *generalized Weierstrass points*.

The segment of the line  $\pi(L)$  from  $\tilde{p}$  to  $\tilde{q}$  lies inside  $N$  and its complement consists of two segments: one from  $\tilde{p}$  to  $\pi(p) \in \mathcal{S}(\infty)$  which we denote  $L_p$ , and the other from  $\tilde{q}$  to  $\pi(q) \in \mathcal{S}(\infty)$  which we denote by  $L_q$ . Each of these is point-wise fixed under  $j_H$ . We call the two segments outside  $N$ , *the generalized Weierstrass lines of  $H$* . Similarly, we have two other pairs of generalized Weierstrass lines  $(\pi(L_{A_p}), \pi(L_{A_q}))$  and  $(\pi(L_{B_p}), \pi(L_{B_q}))$ .

We can parameterize these lines by hyperbolic length. We next construct the family of surfaces  $\mathcal{S}(s)$ . We will show that  $j_H$ , which we denote again simply by  $j$  since it will not cause confusion, is an order two isometry of  $\mathcal{S}(s)$  that has as its set of fixed points, the intersection points of these six lines with the surface.

For a point  $p \in \mathbb{P}$ , let  $V(t)$ ,  $t \in (-\infty, \infty)$  be the line in  $\mathbb{H}^3$  perpendicular to  $\mathbb{P}$  at  $p$  and parameterized so that  $s = |t|$  is hyperbolic arc length and oriented such that  $V(s)$  is exterior to  $\mathbb{P}$  and  $V(-s)$  is interior to  $\mathbb{P}$ . There is a family of surfaces  $\Pi(\pm s)$  with boundary  $\partial\Delta$ , passing through the point  $V(\pm s)$  such that the distance from any point on  $\Pi(\pm s)$  to  $\mathbb{P}$  is  $s$ . We call them *equidistant surfaces*. We can think of  $\Pi(s) \cup \Pi(-s)$  as the (full) equidistant surface from  $\mathbb{P}$  at a distance  $s$ , with the points lying above  $\mathbb{P}$  having directed distance  $s$  and the points below  $\mathbb{P}$  having directed distance  $-s$ . (See [3] chapters III.4 and IV.5). Note that since  $G$  acts on  $\mathbb{H}^3$  by isometries, it leaves the hyper-surfaces  $\Pi(\pm s)$  invariant.

Using orthogonal projection, we can project  $\mathcal{C}_{ext}$  onto  $\Pi(s)$  to obtain  $\mathcal{C}_{ext}(s)$  and project  $\mathcal{C}_{int}(s)$  onto  $\Pi(-s)$  to obtain  $\mathcal{C}_{int}(s)$ .

Each quotient,  $N_{ext}(s) = \mathcal{C}_{ext}/G$  and  $N_{int}(s) = \mathcal{C}_{int}(s)/G$ , is topologically a sphere with three boundary curves. In analogy with our

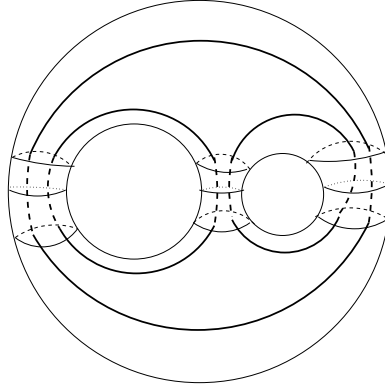


Figure 5. The convex core inside the handlebody

construction of the Schottky double, we want to join these boundary curves, in pairs, by funnels to form the surface  $\mathcal{S}(s)$ .

Let  $Eq_A(s)$  be the equidistant cylinder in  $\mathbb{H}^3$  about  $Ax_A$ . Note that, for each  $s > 0$ ,  $A$  maps  $Eq_A(s)$  to itself. Because  $Ax_A$  lies in  $\mathbb{P}$ ,  $A(\pm s) = Eq_A(s) \cap \Pi(\pm s)$  is the orthogonal projection of  $Ax_A$  onto  $\Pi(\pm s)$ ; notice that  $Eq_A(s)$  and  $\Pi(\pm s)$  are tangent along  $A(\pm s)$ . The curves  $A(\pm s)$  intersect  $\mathcal{C}_{ext}$  and  $\mathcal{C}_{int}$  in boundary curves. The curves  $A(\pm s)$  divide  $Eq_A(s)$  into two pieces, each a doubly infinite topological strip. Both of these strips intersect  $\mathbb{P}$ , one on each side of the axis of  $A$ . Thus one strip intersects  $\mathcal{C}$  and one does not; we denote the strip disjoint from  $\mathcal{C}$  by  $Rec_A(s)$ .

Similarly, we form  $Rec_B(s)$ ,  $Rec_{A^{-1}B}(s)$ ,  $Rec_{AB^{-1}}(s)$ ,  $\dots$ , for each of the boundary curves of  $\mathcal{C}$ . The quotient of each of these strips under the action of  $G$  is a funnel; there are three distinct funnels and we denote them  $F_A(s)$ ,  $F_B(s)$  and  $F_{A^{-1}B}(s)$ .

The surface  $\mathcal{S}(s)$  is defined as

$$\mathcal{S}(s) = N_{ext}(s) \cup N_{int}(s) \cup F_A(s) \cup F_B(s) \cup F_{A^{-1}B}(s)$$

We have

**Proposition 7.1.** *The surface  $\mathcal{S}(s)$  is invariant under the involution  $j$ .*

*Proof.* By construction, the distance from each point of  $\mathcal{S}(s)$  to the convex core  $N$  is  $s$ . As the involution  $j : H \rightarrow H$  is an isometry and it preserves  $N$ , it leaves  $\mathcal{S}(s)$  invariant.  $\square$

We want to find the fixed points of  $j$  acting on  $\mathcal{S}(s)$ . To this end consider the intersection points of the line segments  $L_p(s)$  and  $L_{A_p}(s)$  with  $Rec_A(s)$ ; denote them respectively by  $p(s)$  and  $p_A(s)$ . Similarly,

define the points  $q(s)$  and  $q_B(s)$  as the intersection points of the line segments  $L_q(s)$  and  $L_{B_q}(s)$  with  $Rec_B(s)$  and define the points  $q_A(s)$  and  $p_B(s)$  as the intersection points of the line segments  $L_{A_q}(s)$  and  $L_{B_p}(s)$  with  $Rec_{A^{-1}B}(s)$ .

**Proposition 7.2.** *The projections of the points  $p(s), q(s), p_A(s), q_A(s)$  and  $p_B(s), q_B(s)$  lie on the surface  $\mathcal{S}(s) \subset H$  and comprise the set of fixed points of  $j$  restricted to  $\mathcal{S}(s)$ .*

*Proof.* Each of these points lies on one of the six generalized Weierstrass lines in  $H$  at distance  $s$  from the convex core. These lines are distinct so the points are distinct. Since these lines are fixed pointwise by  $j$ , and since  $j$  leaves  $\mathcal{S}(s)$  invariant, these are fixed points of  $j$  on  $\mathcal{S}(s)$  as claimed.  $\square$

We call these projected points the *generalized Weierstrass points* of  $\mathcal{S}(s)$ .

We summarize these results as

**Theorem 7.3.** *The projections of the lines  $L, L_A, L_B$  in  $\mathbb{P} \subset H$  are point-wise fixed by the involution  $j$ . The projection of each line consists of three segments. The endpoints of the internal segment, lying in the convex core, are generalized Weierstrass points of  $(\mathcal{S}, 0)$ . Each line has two external segments, parameterized by hyperbolic arc length and called generalized Weierstrass lines. The points at distance  $s$  on the generalized Weierstrass lines are the generalized Weierstrass points of the surface  $(\mathcal{S}, s)$ . The endpoints of the external segments lie on the boundary of the handlebody and are the Weierstrass points of the boundary surface  $S = (\mathcal{S}, \infty)$ .*

## 7.2. ANTICONFORMAL INVOLUTIONS

We can also construct the anti-conformal involution  $J$  acting on  $H$ , by defining it to be the self-map of  $H$  induced by the anti-conformal reflection in the plane  $\mathbb{P}$ ,  $R_{\mathbb{P}}$ . Since the axes of the generators  $A$  and  $B$  of  $G$  lie in  $\mathbb{P}$ ,  $R_{\mathbb{P}}$  fixes these axes point-wise and since  $R_{\mathbb{P}}$  is an orientation reversing map, it induces an orientation reversing map on  $H$ .

We want to see how  $J$  acts on

$$\mathcal{S}(s) = N_{ext}(s) \cup N_{int}(s) \cup F_A(s) \cup F_B(s) \cup F_{A^{-1}B}(s)$$

Since  $R_{\mathbb{P}}$  is an isometry,  $R_{\mathbb{P}}(C_{ext}(s)) = C_{int}(s)$  and  $J(N_{ext}) = N_{int}$ . The involution maps  $Rec_A(s)$  to itself, fixing the curve that is the intersection of the plane  $\mathbb{P}$  with  $Rec_A(s)$ . We call the projection of this

curve the central curve of the funnel. Since  $R_{\mathbb{P}}$  interchanges the part of  $Rec_A(s)$  on one side of  $\mathbb{P}$  with the part on the other, we deduce that  $J$  maps the funnel  $F_A(s)$  to itself, interchanging its boundary curves and fixing the central curve.

There is another anti-conformal self-map of  $H$  induced by  $R_{\mathbb{P}} \circ H_L$ ;  $\hat{J} = J \circ j$ . To see how  $\hat{J}$  acts on  $\mathcal{S}(s)$  we first look at the fundamental domains  $F_{ext}(s)$  and  $F_{int}(s)$ . Both  $j$  and  $J$  interchange these so  $\hat{J}$  leaves each of them invariant. We look at  $F_{ext}(s)$ : Let  $\mathbb{P}_L$  be the plane orthogonal to  $\mathbb{P}$  through the line  $L$  and let  $\mathbb{P}_L(s)$  be the intersection of  $\mathbb{P}_L$  and  $\Pi(s)$ ; equivalently,  $\mathbb{P}_L(s)$  is the orthogonal projection of  $L$  onto  $\Pi(s)$ . Looking back at the construction of  $F_{ext}$ , we see that it is symmetric about  $L$  and the symmetry is given by  $R_{\mathbb{P}} \circ H_L$ ; thus  $F_{ext}(s)$  is symmetric about  $\mathbb{P}_L(s)$  by the same map. It follows that  $\hat{J}$  maps  $N_{ext}(s)$  to itself with fixed curve, the projection of  $\mathbb{P}_L(s)$ . We have an analogous symmetry for  $N_{int}(s)$ .

The plane  $\mathbb{P}_L$  also bisects the region  $Rec_A(s)$ . The map  $R_{\mathbb{P}} \circ H_L$  interchanges the two halves of  $Rec_A(s)$ . Thus  $\hat{J}$  maps each funnel to itself; it leaves each boundary curve and the central curve invariant, but changes its orientation. On the central curve, it leaves the Weierstrass points fixed but interchanges the segments they divide the curve into.

## Part 2. INTERSECTING AXES - $Tr [C, D] < -2$

In this part we always assume the axes of the generators of  $G$  intersect. The organization parallels part I. We begin by finding fundamental domains for  $G$  acting on  $\Delta$  and on  $K(G)$ . Next, we find the Weierstrass points for  $S$ . We then turn to  $\mathbb{H}^3$  and find fundamental domains for  $G$  acting on  $\mathbb{P}$  and on  $\mathcal{C}$ . We construct the Fuchsian group uniformizing  $S_K$  and the pleating map sending  $K(G)$  onto  $\partial\mathcal{C}$ . We then use the pleating map to find the Weierstrass points of  $S_K$ . Finally, we work with the handlebody  $H$  and construct the Weierstrass lines and the equidistant surfaces. We conclude with a discussion of anti-conformal involutions on the various quotients.

### 8. Subspaces of $\Delta$ and quotients by $G$

Any pair of generators in the intersecting axes case are geometric (see [8], [9]). To unify our discussion with the non-intersecting case and to simplify our constructions and proofs below, however, we again replace the generators by some Nielsen equivalent generators with special properties.



Let  $E_x$  denote the elliptic element of order two that leaves the disc  $\Delta$  invariant and fixes the point  $x$ . We let  $p = Ax_C \cap Ax_D$  and choose  $p_C$  and  $p_D$  so that  $C = E_p \circ E_{p_C}$  and  $D = E_p \circ E_{p_D}$ . Note that  $p$  and  $p_C$  both lie on  $Ax_C$  and that  $\rho_\Delta(p, p_C)$  is half the translation length of  $C$ , where  $\rho_\Delta$  is the usual hyperbolic metric in  $\Delta$ .

In a manner analogous to the steps of the algorithm in the case when the commutator is elliptic ([5]), one can replace the original generators by generators  $A, B$  so that the distance between the three points  $p, p_A, p_B$  is minimal (among all Nielsen equivalent pairs of generators). That is,  $\rho_\Delta(p, p_A) \leq \rho_\Delta(p, p_B) \leq \rho_\Delta(p_A, p_B)$  and the traces of  $A$  and  $B$  are minimal among all pairs of generators. Geometrically, this condition says that the triangle with vertices  $p, p_A, p_B$  is acute. This acute triangle is the analog of the domain bounded by the lines  $L, L_A, L_B$ .

By analogy with the disjoint axes case, we will call the minimal trace generators *stopping generators*. In this part we will always assume that the generators  $A$  and  $B$  are the stopping generators of  $G$ .

### 8.1. FUNDAMENTAL DOMAINS IN $\Delta$ AND $K(G)$

We begin by constructing a fundamental domain for  $G$  acting on  $\Delta$  whose side pairings are the stopping generators.

We observe that  $Ax_{[A,B]}$  is disjoint from the axes  $Ax_A$  and  $Ax_B$  as follows. Following [14] construct  $h$ , the perpendicular from  $p_B$  to  $Ax_A$ . By the trace minimality of the stopping generators,  $h$  intersects  $Ax_A$  between  $p_A$  and  $A^{-1}(p)$  (or vice-versa) [5]. Let  $\delta$  be a geodesic perpendicular to  $h$  passing through  $p_B$ . The transformation  $E_p \circ E_{p_A} \circ E_{p_B}$  identifies  $\delta$  and  $A(\delta)$ . It is easily seen to be the square root of the commutator  $[A, B^{-1}]$  so that both have the same axis, the common perpendicular to  $\delta$  and  $A(\delta)$ . By construction, this axis must be disjoint from  $Ax_A$  and  $Ax_B$ . The construction also assures that the axes of the commutators  $[B^{-1}, A^{-1}], [A, B^{-1}], [B, A], [A^{-1}, B]$  are disjoint from the axes of  $A$  and  $B$ .

**Lemma 8.1.** *Let  $M_A \in \Delta$  be the mutual orthogonal to  $Ax_{[B^{-1}, A^{-1}]}$  and  $Ax_{[A^{-1}, B]}$  and  $M_B$  the mutual orthogonal to  $Ax_{[B^{-1}, A^{-1}]}$  and  $Ax_{[A, B^{-1}]}$ . Then  $M_A$  passes through  $p_A$  and  $M_B$  passes through  $p_B$ . Moreover, if  $p_{\bar{A}} = A(p_A)$ , and  $p_{\bar{B}} = B(p_B)$  and if  $M_{\bar{A}}$  is the mutual orthogonal to  $Ax_{[B, A]}$  and  $Ax_{[A, B^{-1}]}$  and  $M_{\bar{B}}$  is the mutual orthogonal to  $Ax_{[B, A]}$  and  $Ax_{[A^{-1}, B]}$  then  $M_{\bar{A}}$  passes through  $p_{\bar{A}}$  and  $M_{\bar{B}}$  passes through  $p_{\bar{B}}$ .*

*Proof.* (See figure 8.1) We give the proof for  $M_A$ . The others follow in the same way. We factor  $A$  and  $B$  as  $A = E_p E_{p_A} = E_{p_{\bar{A}}} E_p$  and  $B = E_p E_{p_B} = E_{p_{\bar{B}}} E_p$ . Then we easily compute

$$[B^{-1}, A^{-1}] = E_{p_A} [A^{-1}, B] E_{p_A}$$

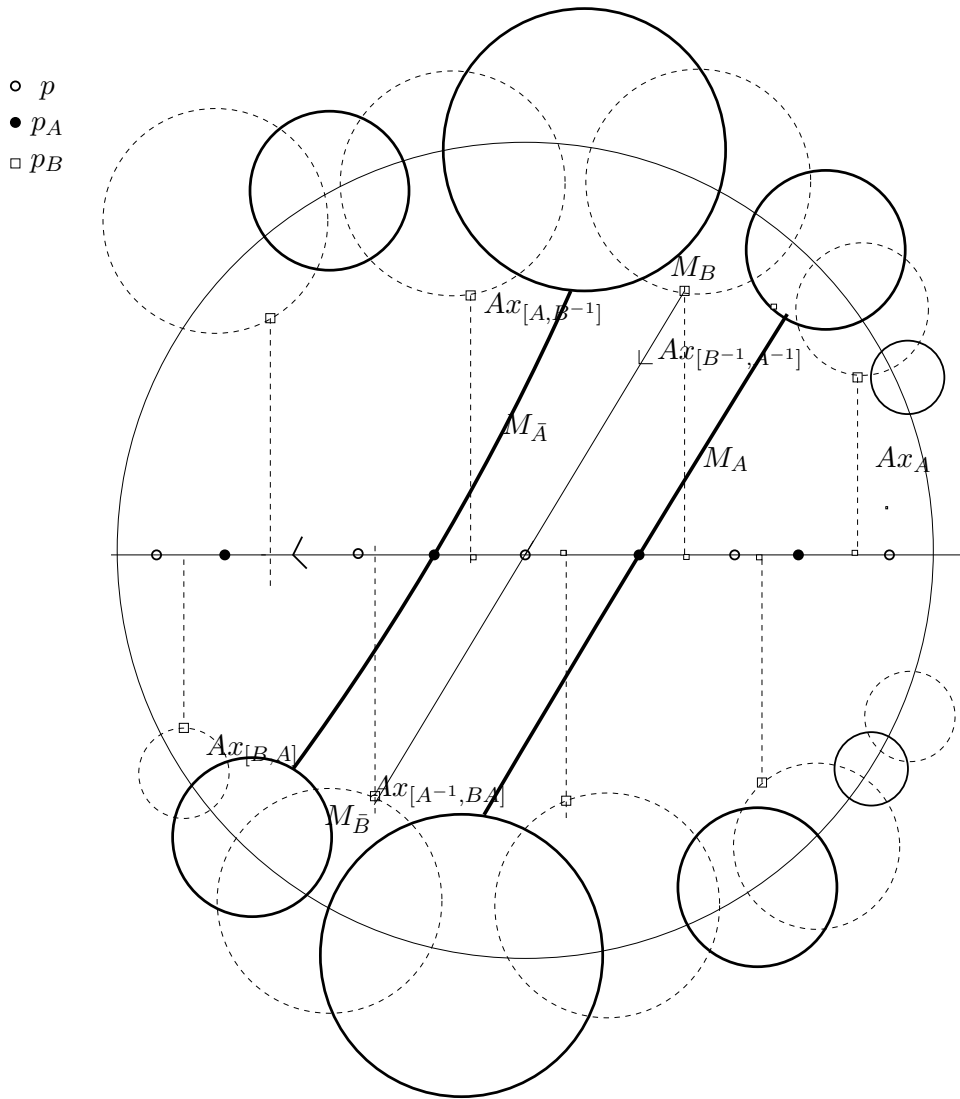


Figure 6. The construction of the stopping generators: The commutator axes are dark circles,  $Ax_A$  and  $Ax_B$  intersect at the origin. The fundamental domain  $F_K$  is bounded by the dark lines  $M_A, M_{\bar{A}}$  and the dotted circles  $M_B, M_{\bar{B}}$ .

so that  $E_{p_A}$  interchanges the axis of  $[B^{-1}, A^{-1}]$  and the axis of  $[A^{-1}, B]$ . The elliptic sends any geodesic lying on  $\Delta$  and passing through  $p_A$  to its inverse and moves any geodesic not passing through  $p_A$ . Since it interchanges the axes, it must send their mutual orthogonal to itself. Thus  $p_A$  is on this orthogonal as claimed. In the case of  $M_B$ , note that  $M_B$  is the same as  $\delta$ .  $\square$

This construction yields a new description of a fundamental domain for the intersecting acting case, one not previously used in the literature.

**Proposition 8.2.** *The domain  $F$  bounded by the geodesics  $M_A, M_B, M_{\bar{A}}$  and  $M_{\bar{B}}$  is a fundamental domain for the action of  $G$  on  $\Delta$  and  $\Delta/G$  is a torus with a hole. The domain  $F_K$  obtained by truncating  $F$  along the commutator axes is a fundamental domain for  $G$  acting on the Nielsen region.*

*Proof.* Note that the lines  $M_A, M_B, M_{\bar{A}}$  and  $M_{\bar{B}}$  are mutually disjoint since if two of them intersected, they would form, together with one of the commutator axes, a triangle with two right angles.

Next, by definition,  $M_A$  and  $M_{\bar{A}}$  are identified by  $A$  and  $M_B$  and  $M_{\bar{B}}$  are identified by  $B$  and so, by Poincaré's theorem, they bound a fundamental domain for  $G$  acting on  $\Delta$ .

Let  $D(I_{[B,A]})$  be the half plane with boundary  $Ax_{[B,A]}$  that does not contain the point  $p$ . It is invariant under  $[B, A]$ . Tiling  $\Delta$  with copies of  $F$ , we see that the only images of  $F$  that intersect  $D(I_{[B,A]})$  are of the form  $[B, A]^n(F)$ , for some integer  $n$ . It follows that  $D(I_{[B,A]})$  is stabilized by the cyclic group  $\langle [B, A] \rangle$  and the interval  $I_{[B,A]}$  joining the fixed points of  $Ax_{[B,A]}$  is an interval of discontinuity for  $G$ . The same is true for the other three intervals  $I_{[A,B^{-1}]}$ ,  $I_{[A^{-1},B]}$  and  $I_{[A^{-1},B^{-1}]}$  corresponding to the other three commutators whose axes intersect  $F$ .

Identifying the sides of  $F$  we obtain a torus with one hole. (See figure 8.1). The ideal boundary of the hole is made up the four segments of  $\bar{F} \cap I_{[B,A]}$ ,  $\bar{F} \cap I_{[A,B^{-1}]}$ ,  $\bar{F} \cap I_{[A^{-1},B]}$  and  $\bar{F} \cap I_{[A^{-1},B^{-1}]}$ . There is one funnel, the quotient  $D(I_{[B,A]})/\langle [B, A] \rangle$ .

Truncating  $F$  along the four commutator axes intersecting it we have a fundamental domain for the Nielsen region of  $G$  as claimed.  $\square$

## 8.2. WEIERSTRASS POINTS OF THE SCHOTTKY DOUBLE

We note that the Schottky double  $S$  is a surface of genus two with one funnel whose central curve is the projection of  $I_{[B,A]}$ . We now characterize its Weierstrass points.

Let  $p, p_A, p_B \in \Delta$  be the intersection points of axes as in the previous section and let  $q, q_A, q_B$  be their respective reflections in  $\partial\Delta$ . Let  $E$  be the elliptic of order two with fixed point  $p$ , and  $E$  leaves  $\partial\Delta$  invariant, its other fixed point is  $q$ . We can factor  $A$  as  $A = E \circ E_A$  where  $E_A$  is the elliptic of order two with fixed point  $p_A$  and second fixed point  $q_A$ .

Similarly factor  $B = E \circ E_B$  where  $E_B$  is a third elliptic of order two with fixed points  $p_B$  and  $q_B$ .

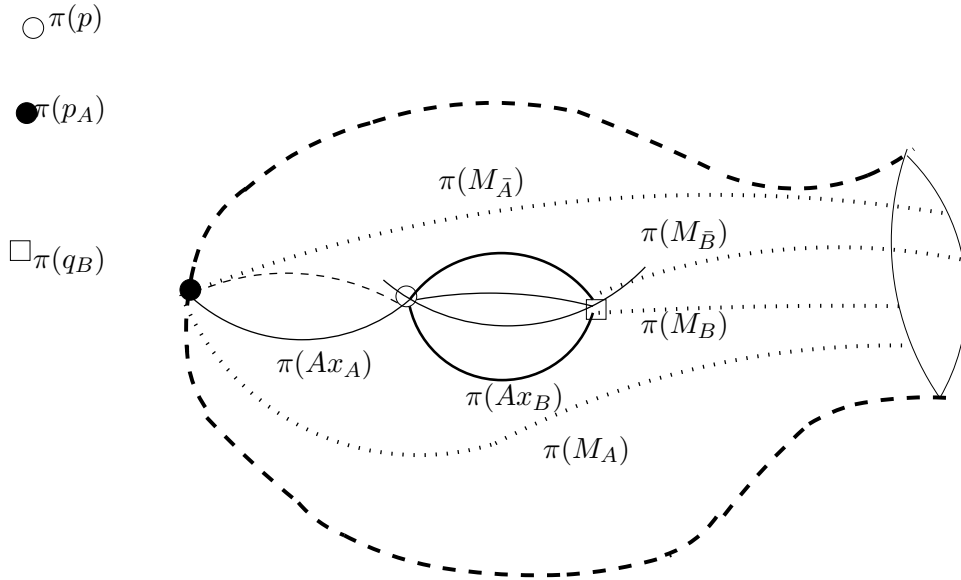


Figure 7. Intersecting axes: the quotient  $\Delta/G$

**Theorem 8.3.** *Let  $G = \langle A, B \rangle$  be a discrete free Fuchsian group such that the axes of  $A$  and  $B$  intersect. Let  $(p, q)$ ,  $(p_A, q_A)$ ,  $(p_B, q_B)$  be the respective fixed points of the elliptics  $E, E_A, E_B$  where  $A = E \circ E_A$  and  $B = E \circ E_B$ . Then the projections of these points onto the surface  $S$  of genus 2 under the projection  $\Omega(G) \rightarrow \Omega(G)/G = S$ , are the Weierstrass points of  $S$ .*

*Proof.* We saw above that  $\Delta/G$  is a torus with a hole so that its double  $S$  is a surface of genus two with one funnel.

Conjugation of the group  $G$  by  $E$  sends  $A$  to  $A^{-1}$  and  $B$  to  $B^{-1}$  since  $p$  lies on both their axes. It follows that  $E$  induces a conformal involution  $j$  of  $S$ . Moreover, since  $E_A = E \circ A$  and  $E_B = E \circ B$ , these elliptics induce the same involution on  $S$ . The projection of each of the six points is fixed by  $j$ , characterizing  $j$  as the hyperelliptic involution.  $\square$

There is again an anti-conformal involution  $J$  of  $S$  induced by the symmetry of the group  $G$  with respect to  $\partial\Delta$ , that is by  $R_{\partial\Delta}$ . As a Schottky double,  $S$  has a single funnel and the central geodesic of this funnel is the projection,  $\gamma$ , of  $\Omega \cap \partial\Delta$ ; it is point-wise invariant under  $J$ . Note that  $\gamma$  does not contain any of the Weierstrass points.

Both of the involutions  $j$  and  $J$  of  $S$  fix  $\gamma$ . The first is conformal and fixes no point on  $\gamma$ . The other involution  $J$  is anti-conformal and fixes the entire curve point wise.

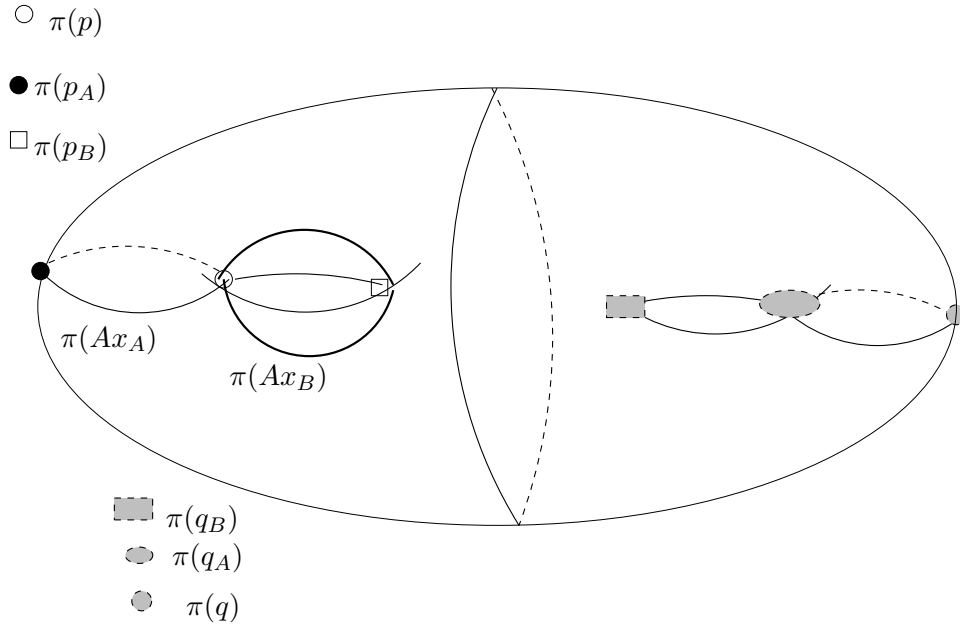


Figure 8. Intersecting axes, one view of  $\Omega(G)/G$ : on the left we have the projection of  $\Delta_{int}/G$  with the projections of the axes of  $A$  and  $B$  and the Weierstrass points  $\pi(p), \pi(p_A), \pi(p_B)$  indicated and on the right we have the projection of  $\Delta_{ext}/G$  with the projections of the axes of  $A$  and  $B$  and the Weierstrass points  $\pi(q), \pi(q_A), \pi(q_B)$  indicated.

There is a third involution  $\hat{J} = Jj$ . This is the anti-conformal map on  $S$  induced by the product  $R_{\partial\Delta} \circ E$ . Since  $j$  fixes the Weierstrass points, both  $J$  and  $\hat{J}$  have the same action on the Weierstrass points. Namely, they both interchange the projections of  $p$  and  $q$ ,  $p_A$  and  $q_A$  and  $p_B$  and  $q_B$  respectively.

### 9. The convex hull and the convex core

As in section 5, we again consider the group  $G$  acting on  $\mathbb{P}$ ,  $\mathbb{P}_{int}$  and  $\mathbb{P}_{ext}$  which contain respectively the convex hull  $\mathcal{C}(G)$  in  $\mathbb{H}^3$  and its two boundaries  $\mathcal{C}_{int}$  and  $\mathcal{C}_{ext}$ .

#### 9.1. FUNDAMENTAL DOMAINS IN $\mathbb{P}$ AND $\mathcal{C}$

We now work in  $\mathbb{P}$  and draw in the axes  $Ax_A, Ax_B, Ax_{AB}, Ax_{BA}, Ax_{A^{-1}B}, Ax_{AB^{-1}}$ ; again we denote the intersection point of  $Ax_A$  and  $Ax_B$  by  $p$ , the intersection point of  $Ax_A$  and  $Ax_{A^{-1}B}$  by  $p_A$  and the intersection point of  $Ax_B$  and  $Ax_{A^{-1}B}$  by  $p_B$ .

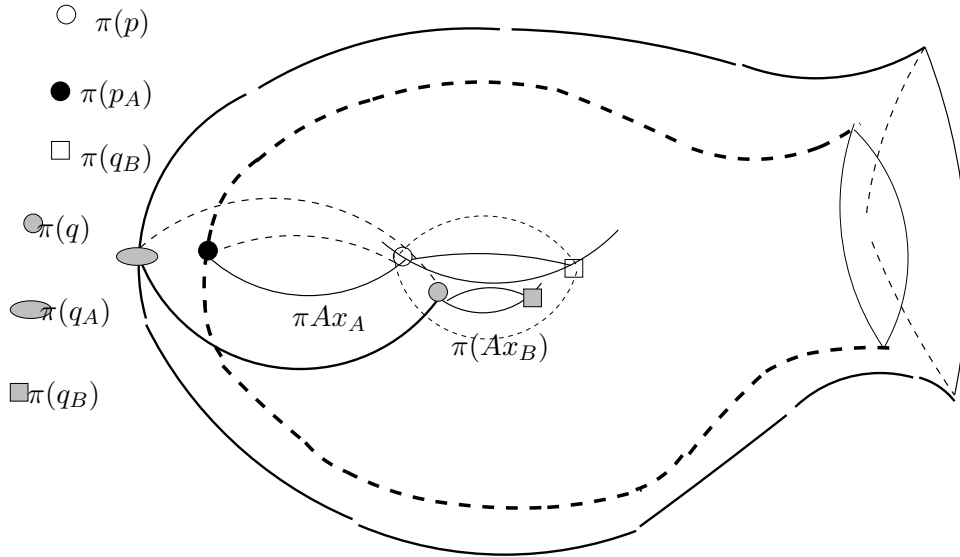


Figure 9. Intersecting axes: alternate view of  $\Omega(G)/G$ . Here  $\Delta/G$  is viewed as lying inside  $J(\Delta/G)$

Denote the half turns about lines orthogonal to  $\mathbb{P}$  that pass through  $p, p_A, p_B$  by  $H_p, H_{p_A}, H_{p_B}$  respectively. When restricted to  $\mathbb{P}$  these are the same as the rotations of order two we denoted by  $E_p, E_{p_A}$  and  $E_{p_B}$ .

We define the lines  $M_A, M_{\bar{A}}, M_B, M_{\bar{B}}$  just as we did in  $\Delta$  and obtain a domain  $F_{\mathbb{P}}$  in  $\mathbb{P}$ . The arguments of the proof of lemma 8.2 applied to  $\mathbb{P}$  instead of  $\Delta$  give an immediate proof of

**Proposition 9.1.** *The domain  $F \subset \mathbb{P}$  bounded by the lines  $M_A, M_{\bar{A}}, M_B, M_{\bar{B}}$  is a fundamental domain for  $G$  acting on  $\mathbb{P}$ . The truncation  $F_{\mathcal{C}}$  of  $F$  along the axes of the commutators is a fundamental domain for the convex hull  $\mathcal{C}(G)$  in  $\mathbb{H}^3$ .*

*Proof.* Since  $\mathcal{C}(G) \subset \mathbb{P}$ , it is precisely the Nielsen convex region for  $G$  acting on  $\mathbb{P}$  and lemma 8.2 applies.  $\square$

Again we define  $F_{int} = F_{\mathcal{C}} \cap \mathbb{P}_{int}$  and  $F_{ext} = F_{\mathcal{C}} \cap \mathbb{P}_{ext}$  as domains in  $\mathbb{P}_{int}$  and  $\mathbb{P}_{ext}$  respectively. As an immediate corollary we have

**Corollary 9.2.** *The domain  $F_{int} \cup F_{ext}$  is a fundamental domain for  $G$  acting on  $\mathcal{C}_{int} \cup \mathcal{C}_{ext} = \partial\mathcal{C}$ .*

## 10. The Pleated Surface

In this section we explicitly construct the pleated surface  $(\tilde{S}, pl)$  for the intersecting axis case. Again, we do this by first constructing a Fuchsian group  $\Gamma$  that uniformizes the Nielsen double  $S_K = \Delta/\Gamma$ . We then construct a pleating map  $PL : \Delta \rightarrow \mathbb{H}^3$  that intertwines the actions of  $\Gamma$  on  $\Delta$  and  $G$  on  $\mathcal{C} \in \mathbb{H}^3$ . Finally, we take quotients.

### 10.1. THE FUCHSIAN GROUP FOR THE NIELSEN DOUBLE

We begin with the domain  $F_K \subset \Delta$  defined in proposition 8.2 for the Nielsen region  $K$ . For readability, denote the reflection  $R_{Ax_{[B^{-1}, A^{-1}]}}$  by  $R_0$ . Define the domain  $F_\Gamma = F_K \cup R_0(F_K) \subset \Delta$ . Let  $A' = R_0 A R_0$  and  $B' = R_0 B R_0$  and let  $\Gamma = \langle A, B, A', B' \rangle$ .

**Proposition 10.1.** *The domain  $F_\Gamma$  is a fundamental domain for  $\Gamma$  acting on  $\Delta$ . The Fuchsian surface  $S_K = \Delta/G$  is a Riemann surface of genus two, the Nielsen double.*

*Proof.* (See figure 10) The sides of the domain  $F_K$  of proposition 8.2 are  $M_A, M_{\bar{A}}, M_B$  and  $M_{\bar{B}}$ . Set  $M'_A = R_0(M_A)$ ,  $M'_B = R_0(M_B)$ ,  $M'_{\bar{A}} = R_0(M_{\bar{A}})$  and  $M'_{\bar{B}} = R_0(M_{\bar{B}})$ . The axis  $Ax_{[B^{-1}, A^{-1}]}$  divides  $M_A = M'_{\bar{A}}$  into two segments; without causing confusion we will say  $M_A$  is the segment in  $F_K$  and  $M'_{\bar{A}}$  is the segment in  $R_0(F_K)$  and similarly for  $M_B$  and  $M'_{\bar{B}}$ .

Using the side identifications

$$A : M_A \rightarrow M_{\bar{A}}, B : M_B \rightarrow M_{\bar{B}}$$

we see that if we set  $A' = R_0 A R_0$  and  $B' = R_0 B R_0$  we obtain the identifications

$$A' : M'_A \rightarrow M'_{\bar{A}}, B' : M'_B \rightarrow M'_{\bar{B}}$$

Now denote the reflection in the axis  $Ax_{[A^{-1}, B]}$  by  $R_1$  and note that  $R_1 = B R_0 B^{-1}$ . The axes  $Ax_{[A^{-1}, B]}$  and  $Ax_{[B^{-1}, A^{-1}]}$  are both orthogonal to  $M_A$ . Thus we see that

$$R_1 R_0 = (B')^{-1} B : Ax_{[A, B^{-1}]} \rightarrow R_0(Ax_{[A, B^{-1}]})$$

Similarly

$$(A')^{-1} A : Ax_{[A^{-1}, B]} \rightarrow R_0(Ax_{[A^{-1}, B]})$$

$$A' B' B^{-1} A^{-1} : Ax_{[B, A]} \rightarrow R_0(Ax_{[B, A]})$$

Including the two vertices where  $M_A$  meets  $M'_{\bar{A}}$  and where  $M_B$  meets  $M'_{\bar{B}}$  there are 14 vertices in all. They fall into two cycles of

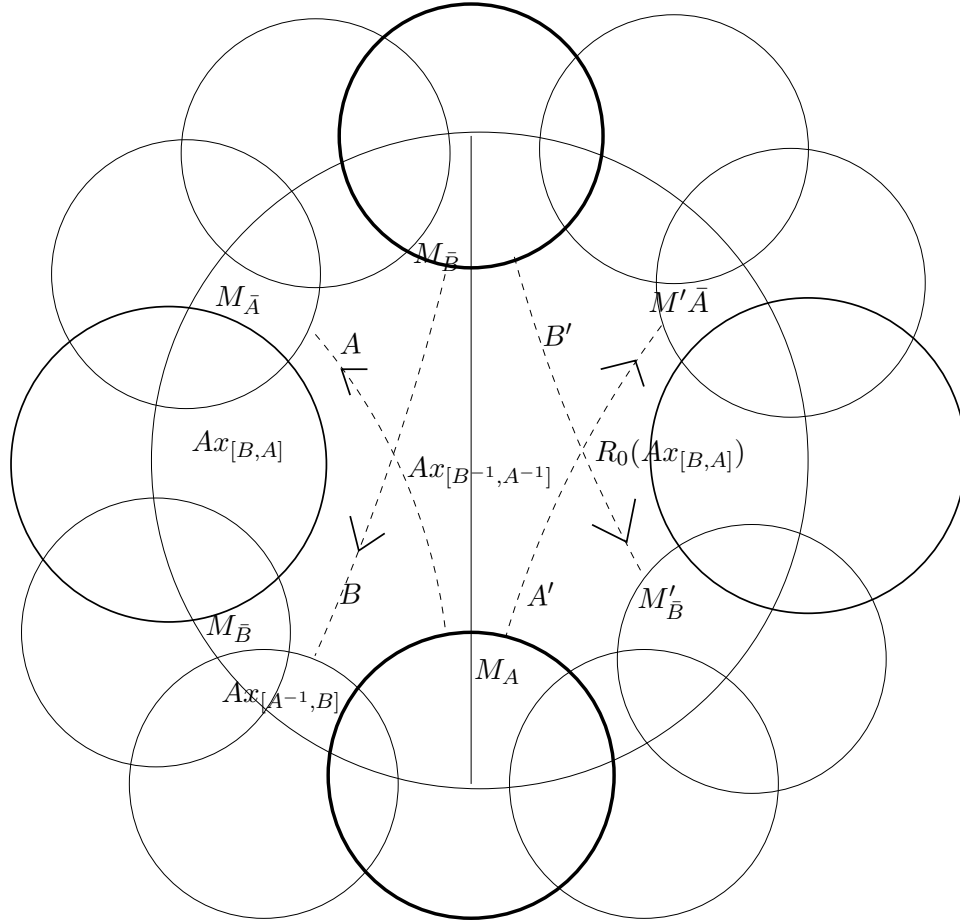


Figure 10. The domain  $F_K$  and its reflection in  $Ax_{[B^{-1}, A^{-1}]}$  forms a fundamental domain for  $\Gamma$

four vertices where there angles are all right and two cycles of three vertices, two where there are right angles and one where there is a straight angle. That  $\Gamma$  is Fuchsian and  $F_\Gamma$  is a fundamental domain now follows from Poincaré's theorem. Identifying the sides  $M_A$  and  $M_{\bar{A}}$  and the sides  $M_B$  and  $M_{\bar{B}}$  of  $F_K$  yields a torus with a hole where the remaining unidentified sides (arcs of commutator axes) fit together to form the boundary of the hole. Similarly, identifying corresponding sides of  $R_0(F_K)$  yields another torus with a hole. In  $F_\Gamma$ , the commutator axis sides are identified and the two tori with a hole join up to form the Nielsen double  $S_K$  as a compact surface of genus two.  $\square$



## 10.2. THE PLEATING MAP

We now construct the pleating map  $pl : \Delta/\Gamma \rightarrow \mathbb{H}^3/G$  just as we did in section 6.2. We use the definitions of  $F_\Gamma \subset \Delta$  as a union of two copies of  $F_K$ , and the domains  $F_{int} \in \mathbb{P}_{int}$  and  $F_{ext} \subset \mathbb{P}_{ext}$  from section 9.1. Again we begin by defining a pleating map  $PL : \Delta \rightarrow \mathbb{H}^3$ . We first define  $PL$  on  $F_\Gamma$  and then extend by the groups  $\Gamma$  and  $G$ .

Set  $PL : F_K \rightarrow F_{ext}$ . Next we set  $PL : R_0(F_K) \rightarrow F_{int}$ . Define the group homomorphism  $\phi : \Gamma \rightarrow G$  by first defining it on the generators:

$$A \mapsto A, B \mapsto B, A' \mapsto id, B' \mapsto id.$$

To show that this map on generators gives a group homomorphism from  $\Gamma$  to  $G$  we must show that  $\phi$  preserves the defining relation(s) of  $\Gamma$ . As in the previous case, we know that there are no reflection relations and the cycle relations form a complete set of relations for  $\Gamma$ . Again, it is easy to calculate that substituting  $\phi(A)$ ,  $\phi(A')$ ,  $\phi(B)$ , and  $\phi(B')$  into the cycle relation gives the identity.

We now extend  $PL$  to a map from  $\Delta$  into  $\mathbb{H}^3$ . That is, for  $x \in \Delta$  let  $g \in \Gamma$  be chosen so that  $g(x) \in F_\Gamma$  and set  $PL(x) = \phi(g^{-1}) \circ PL \circ g(x)$ . This is well defined when  $g(x)$  lies interior to  $F_\Gamma$  and when  $g(x)$  lies on a boundary curve of  $F_\Gamma$  or on  $Ax_{[B,A]}$  it is defined by continuity. Thus,  $(\Delta, PL)$  is a pleated surface with image in  $\mathbb{H}^3$  whose pleating locus is the image of the axis sides of  $F_\Gamma$  and all their images under  $\Gamma$ .

We now define the map  $pl : S_K \rightarrow \mathbb{H}^3/G$  by taking quotients. The map  $pl$  is clearly a hyperbolic isometry since  $PL$  is and its pleating locus consists of the images of the commutator axes.

By corollary 9.2, we can apply the group  $G$  to identify  $PL(F_\Gamma)$  with  $\partial\mathcal{C}$ , and taking quotients, identify  $pl(S_K)$  with the convex core boundary  $\partial N = \mathcal{C}_{int}/G \cup \mathcal{C}_{ext}/G$ .

10.3. THE WEIERSTRASS POINTS OF THE NIELSEN DOUBLE  $S_K$ 

We again now determine the Weierstrass points of  $S_K$  in a manner reminiscent of our discussion of the Schottky double. The map  $E_p$  is a conformal involution that maps  $F_K$  to itself and maps  $R_0(F_K)$  to its image under  $AB(B')^{-1}(A')^{-1}$ . Therefore  $E_p$  preserves the  $F_\Gamma$  tiling of  $\Delta$  which implies that  $E_p$  conjugates  $\Gamma$  to itself. It follows that  $E_p$  induces a conformal involution  $j$  of  $S_K$ . It is easy to check that the points  $p, p_A, p_B, p', p'_A, p'_B$  are mapped by  $E_p$  to points that are equivalent under the action of  $\Gamma$  and thus project to fixed points of  $j$ . This characterizes  $j$  as the hyperelliptic involution of  $S_K$ .

Note that  $PL(p) \in F_{int}$  and  $PL(p') \in F_{ext}$ , but as points in  $\mathbb{P}$ ,  $PL(p) = PL(p')$  and similarly for the other pairs of Weierstrass points.

For the other elliptics fixing these six points in  $F_\Gamma$  we have:

$$E_{p_A} = E_p A, E_{p_B} = E_p B, E_{p'} = E_p AB(B')^{-1}(A')^{-1},$$

$$E_{p'_A} = E_{p'} A', E_{p'_B} = E_{p'} B'$$

so that they also induce the involution  $j$ .

## 11. Weierstrass points and lines in the handlebody

In this section we give an explicit description of the handlebody  $H$  as  $\mathcal{S} \times I$  in the case where the axes of the generators intersect. We show that again,  $H$  admits a unique order two isometry  $j$  that we call the *hyperelliptic isometry of  $H$* . It fixes six unique geodesic line segments, which we call the *Weierstrass lines of  $H$* .

We show, again, that  $H$  is foliated by surfaces  $\mathcal{S}(s) = (\mathcal{S}, s)$  that are at distance  $s$  from the convex core, and that each  $\mathcal{S}(s)$  is fixed by  $j$ . The restriction of  $j$  to  $\mathcal{S}(s)$  fixes six *generalized Weierstrass points* on  $\mathcal{S}(s)$ . Moreover, on each  $\mathcal{S}(s)$  we find an orientation reversing involution  $J$  that interchanges pairs of generalized Weierstrass points.

### 11.1. CONSTRUCTION OF THE FOLIATION

We modify our construction in section 7 to construct the *hyperelliptic isometry for  $H$*  as follows. In section 9.1, we defined the half-turns about lines orthogonal to  $\mathbb{P}$  at the points  $p, p_A, p_B$  in  $\mathbb{P}$ . Set  $p_{int}$  and  $p_{ext}$  as the point  $p$  considered in  $\mathbb{P}_{int}$  or  $\mathbb{P}_{ext}$  respectively, and similarly for the other points.

Following our notation in section 7.1 we parameterize these orthogonal lines by arc length and denote them by  $V_p(t), V_{p_A}(t)$  and  $V_{p_B}(t)$  respectively, such that  $V_p(0) = p$ ,  $V_{p_A}(0) = p_A$  and  $V_{p_B}(0) = p_B$ . Direct them so that for  $t > 0$ ,  $V_p(t)$  is in the half-space with boundary  $\mathbb{P}_{ext}$  and similarly for the other lines. With this convention we write  $V_p(\pm s), V_{p_A}(\pm s)$  and  $V_{p_B}(\pm s)$  where  $s = |t|$ .

Since the fundamental domains  $F_{ext}$  and  $F_K$  are related by orthogonal projection we see that for  $p, q \in F_K$  we have  $p_{ext} = PL(q) \in F_{ext}$  is equal to  $\lim_{s \rightarrow 0} V_p(s) = V_p(0^+)$  and  $p_{int} = PL(p) \in F_{int}$  is equal to  $\lim_{s \rightarrow 0} V_p(-s) = V_p(0^-)$ . Also,  $p \in F_K \subset \Delta_{int}$  is equal to  $\lim_{s \rightarrow \infty} V_p(-s)$  and  $q \in R_{\partial\Delta}(F_K) \subset \Delta_{ext}$  is equal to  $\lim_{s \rightarrow \infty} V_p(s)$  and similarly for the other points.

We saw that  $A$  and  $B$  may be factored into products of pairs of these half turns and it follows that  $G$  is a normal subgroup of index two in the group generated by these half turns. Since the half-turns all

differ by elements of  $G$ , their actions by conjugation on  $G$  all induce the same order two automorphism  $j$  under the projection  $\pi : \mathbb{H}^3 \rightarrow \mathbb{H}^3/G = H$ . The fixed points of  $j$  in  $H$  are precisely the points on the images of the six lines  $V_p(\pm s), V_{p_A}(\pm s)$  and  $V_{p_B}(\pm s)$ : denote them by  $\pi(V_p)(\pm s), \pi(V_{p_A})(\pm s)$  and  $\pi(V_{p_B})(\pm s)$ .

We work with the lines  $V_p$  and  $\pi(V_p)$  but we have analogous statements for the lines  $V_{p_A}$  and  $V_{p_B}$  and their projections. As we saw in theorem 8.3, the endpoints  $p \in \Delta_{int}$  and  $q \in \Delta_{ext}$  of  $V_p$  project to Weierstrass points on the Schottky double  $S = \partial H = \mathcal{S}(\infty)$ .

In theorem 6.2 we found the Weierstrass points of the Nielsen double  $S_K$ . In the fundamental domain  $F_\Gamma \subset \Delta$  for  $\Gamma$  we found points  $p, q \in F_\Gamma \subset \Delta$  such that  $p_{int} = PL(p) = V_p(0^-)$  and  $p_{ext} = PL(q) = V_p(0^+)$ . For the moment, call the hyperelliptic involution on  $S_K$ ,  $j_{S_K}$  and the hyperelliptic involution on  $H$ ,  $j_H$ . By construction, the pleating map  $pl$  commutes with these involutions:  $pl \circ j_{S_K} = j_H \circ pl$ . It follows that the projections  $\pi(PL(p)) \in N_{int}$  and  $\pi(PL(q)) \in N_{ext}$  are fixed under  $j_H$ . By the same argument, we find two other pairs of points  $\pi(PL(p_A)), \pi(PL(q_A))$  on  $\pi(L_A)$  and  $\pi(PL(p_B)), \pi(PL(q_B))$  on  $\pi(L_B)$  fixed by  $j_H$ . They lie on the genus two surface  $\partial N = \mathcal{S}(0)$  and are its *generalized Weierstrass points*.

Each of the segments of the line  $\pi(V_p(\pm s))$  is point-wise fixed under  $j_H$ . We call each a *generalized Weierstrass line of  $H$* . Similarly, we have two other pairs of generalized Weierstrass lines  $\pi(V_{p_A}(\pm s))$ , and  $\pi(V_{p_B}(\pm s))$ .

We next construct the family of surfaces  $\mathcal{S}(s)$ . We will show that  $j_H$ , which we denote again simply by  $j$  since it will not cause confusion, is an order two isometry of  $\mathcal{S}(s)$  that has as its set of fixed points, the intersection points of these six lines with the surface.

We again consider the family of equidistant surfaces  $\Pi(\pm s)$  with boundary  $\partial\Delta$  at distance  $s$  from the plane  $\mathbb{P}$ . Using orthogonal projection, we can project  $\mathcal{C}_{ext}$  onto  $\Pi(s)$  to obtain  $\mathcal{C}_{ext}(s)$  and project  $\mathcal{C}_{int}$  onto  $\Pi(-s)$  to obtain  $\mathcal{C}_{int}(s)$ .

Each quotient  $N_{ext}(s) = \mathcal{C}_{ext}(s)/G$  and  $N_{int}(s) = \mathcal{C}_{int}(s)/G$  is topologically a torus with one boundary curve. In analogy with our construction of the Schottky double, we want to join these boundary curves, by a funnel to form the surface  $\mathcal{S}(s)$ .

Let  $Eq_{[B,A]}(s)$  be the equidistant cylinder in  $\mathbb{H}^3$  about  $Ax_{[B,A]}$ . Note that, for each  $s > 0$ ,  $[B, A]$  maps  $Eq_{[B,A]}(s)$  to itself. Because  $Ax_{[B,A]}$  lies in  $\mathbb{P}$ ,  $[B, A](\pm s) = Eq_{[B,A]}(s) \cap \Pi(\pm s)$  is the orthogonal projection of  $Ax_{[B,A]}$  to  $\Pi(\pm s)$ . The curves  $[B, A](\pm s)$  intersect  $\mathcal{C}_{ext}(s)$  and  $\mathcal{C}_{int}(s)$  in their boundary curves. The curves  $[B, A](\pm s)$  divide  $Eq_{[B,A]}(s)$  into two pieces, each an infinite topological strip. Both of these strips intersect

$\mathbb{P}$ , one on each side of the axis of  $[B, A]$ . Thus one strip intersects  $\mathcal{C}$  and one does not; we denote the strip disjoint from  $\mathcal{C}$  by  $Rec_{[B,A]}(s)$ .

Similarly, we form strips for each of the boundary curves of  $\mathcal{C}$ . The quotient of each of these strips under the action of  $G$  is a funnel; there is only one conjugacy class of boundary curves, so only one funnel, denoted  $F_{[B,A]}(s)$ .

The surface  $\mathcal{S}(s)$  is defined as

$$\mathcal{S}(s) = N_{ext}(s) \cup N_{int}(s) \cup F_{[B,A]}(s)$$

We have

**Proposition 11.1.** *The surface  $\mathcal{S}(s)$  is invariant under the involution  $j$ .*

*Proof.* By construction, the distance from each point of  $\mathcal{S}(s)$  to the convex core  $N$  is  $s$ . As the involution  $j : H \rightarrow H$  is an isometry and it preserves  $N$ , it leaves  $\mathcal{S}(s)$  invariant.  $\square$

We want to find the fixed points of  $j$  acting on  $\mathcal{S}(s)$ . To this end consider the intersection points of the line segments  $V_p(\pm s)$  with  $\Pi(\pm s)$ ; denote them respectively by  $q(s)$  and  $p(s)$ . Similarly, define the points  $q_A(s)$  and  $p_A(s)$  as the intersection points of the line segments  $V_{p_A}(\pm s)$  with  $\Pi(\pm s)$  and define the points  $q_B(s)$  and  $p_B(s)$  as the intersection points of the line segments  $V_{p_B}(\pm s)$  with  $\Pi(\pm s)$ .

**Proposition 11.2.** *The projections of the points  $p(s), q(s), p_A(s), q_A(s)$  and  $p_B(s), q_B(s)$  lie on the surface  $\mathcal{S}(s) \subset H$  and comprise the set of fixed points of  $j$  restricted to  $\mathcal{S}(s)$ .*

*Proof.* Each of these points lies on one of the six generalized Weierstrass lines in  $H$  at distance  $s$  from the convex core. These lines are distinct so the points are distinct. Since these lines are fixed point-wise by  $j$ , and since  $j$  leaves  $\mathcal{S}(s)$  invariant, these are fixed points of  $j$  on  $\mathcal{S}(s)$  as claimed.  $\square$

We call these projected points the *generalized Weierstrass points* of  $\mathcal{S}(s)$ .

We summarize these results as

**Theorem 11.3.** *The projections of the line segments  $V_p(\pm s), V_{p_A}(\pm s)$  and  $V_{p_B}(\pm s)$ , orthogonal to  $\mathbb{P} \subset H$  are point-wise fixed by the involution  $j$ . The endpoints of each segment, lying in the boundary of the convex core, are the generalized Weierstrass points of  $(\mathcal{S}, 0)$ . Each line is parameterized by hyperbolic arc length and is called a *generalized Weierstrass line*. The points at distance  $s$  on the generalized Weierstrass*

lines are the generalized Weierstrass points of the surface  $(\mathcal{S}, s)$ . The endpoints of the Weierstrass lines lie on the boundary the handlebody and are the Weierstrass points of the boundary surface  $S = (\mathcal{S}, \infty)$ .

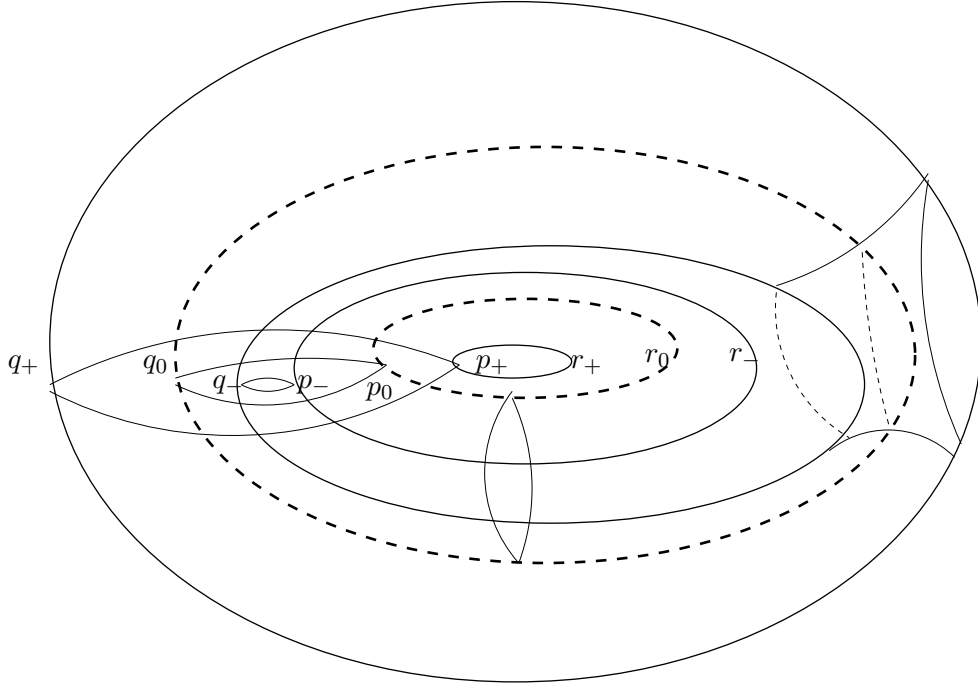


Figure 11. The convex core is a one holed torus inside the handlebody. Its boundary is indicated in dotted lines. The Weierstrass points are marked as follows:  $p_- = \pi(V_p(-\infty))$ ,  $p_0 = \pi(V_p(0))$ ,  $p_+ = \pi(V_p(+\infty))$ ,  $q_- = \pi(V_{p_A}(-\infty))$ ,  $q_0 = \pi(V_{p_A}(0))$ ,  $q_+ = \pi(V_{p_A}(+\infty))$ ,  $r_- = \pi(V_{p_B}(-\infty))$ ,  $r_0 = \pi(V_{p_B}(0))$ ,  $r_+ = \pi(V_{p_B}(+\infty))$ . The Weierstrass lines join  $q_+$  to  $q_0$ ,  $q_0$  to  $q_-$ ,  $p_-$  to  $p_0$ ,  $p_0$  to  $p_+$ ,  $r_-$  to  $r_0$ , and  $r_0$  to  $r_+$ .

Note that there are no generalized Weierstrass points on the central curve of the funnel. It is invariant under  $j$  and is mapped to its inverse.

## 11.2. ANTICONFORMAL INVOLUTIONS

We can also construct the anti-conformal involution  $J$  acting on  $H$ , by defining it to be the self-map of  $H$  induced by the anti-conformal reflection in the plane  $\mathbb{P}$ ,  $R_{\mathbb{P}}$ . Since the axes of the generators  $A$  and  $B$  of  $G$  lie in  $\mathbb{P}$ ,  $R_{\mathbb{P}}$  fixes these axes point-wise and since  $R_{\mathbb{P}}$  is an orientation reversing map, it induces an orientation reversing map on  $H$ .

We want to see how  $J$  acts on

$$\mathcal{S}(s) = N_{ext}(s) \cup N_{int}(s) \cup F_{[B,A]}(s)$$

Since  $R_{\mathbb{P}}$  is an isometry,  $R_{\mathbb{P}}(\mathcal{C}_{ext}(s)) = \mathcal{C}_{int}(s)$  and  $J(N_{ext}) = N_{int}$ . As  $J$  is orientation reversing, it maps the projection of the line  $V_p(s)$  to the projection of the line  $V_p(-s)$  and interchanges the projections of  $p(s)$  and  $q(s)$ . It acts on the other lines and generalized Weierstrass points similarly.

The involution maps  $Rec_{[B,A]}(s)$  to itself, fixing point-wise the curve that is the intersection of the plane  $\mathbb{P}$  with  $Rec_{[B,A]}(s)$ . We call the projection of this curve the central curve of the funnel. Since  $R_{\mathbb{P}}$  interchanges the part of  $Rec_{[B,A]}(s)$  on one side of  $\mathbb{P}$  with the part on the other, we deduce that  $J$  maps the funnel  $F_{[B,A]}(s)$  to itself, interchanging its boundary curves and sends the central curve to its inverse.

There is another anti-conformal self-map of  $H$  induced by  $R_{\mathbb{P}} \circ H_L$ ;  $\hat{J} = J \circ j$ . To see how  $\hat{J}$  acts on  $\mathcal{S}(s)$  we first look at the fundamental domains  $F_{ext}(s)$  and  $F_{int}(s)$ :  $J$  interchanges these domains but  $j$  leaves them invariant, so  $\hat{J}$  interchanges them.

It follows that  $\hat{J}$  maps  $N_{ext}(s)$  to  $N_{int}(s)$  maps the funnel to itself, interchanging the boundary curves and interchanges the generalized Weierstrass points.

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