Complex and Real Dynamics for the family $\lambda \tan(\mathbf{z})$

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1 Introduction

This article is based on my lecture at the Complex Dynamics Workshop of the Research Institute of Mathematics at Kyoto University in October 2001. It is an exposition of joint work with Janina Kotus.

The tangent family $f_\lambda(z) = \lambda \tan(z)$ is the meromorphic analogue of the quadratic family $z^2 + c$. The functions $f_\lambda(z)$ are characterized by their mapping properties: they are, up to scale, the only meromorphic functions fixing zero with no critical points and two symmetric asymptotic (omitted) values. The classification of stable behavior is essentially the same as for the quadratic family. Again like the quadratic family, the parameter plane has a combinatorial description based on the orbit of the singular value, appropriately interpreted.

The real axis plays a special role for the quadratic family. For real values of the parameter, the critical value is real and so is its forward orbit. Studying the orbit of the critical value, we can understand the observed period doubling and renormalization. For the tangent family, the imaginary axis plays a similar role. If the parameter lies on the imaginary axis, the asymptotic values are real as are even of iterates. Restricting our attention to the second iterate f_λ^2 for $\lambda = iy \in \Im$, we again observe period doubling.

In this paper we give an overview of the dynamical theory for the tangent family and describe the period doubling phenomena and some of its consequences. For details and proofs see [1, 3, 4, 5, 6] and the references cited therein.

2 Mapping Properties of the Tangent

Recall that the map $f(z) = \tan z$ maps the complex plane $\mathbb{C}$ onto the Riemann sphere $\hat{\mathbb{C}}$ minus the two points $\{\pm i\}$. It has period π. The strip $\{z = x + iy : -\pi/2 < x \leq \pi/2\}$ is mapped 1-1 onto $\hat{\mathbb{C}} \setminus \{\pm i\}$ as follows: the real axis in the strip maps to the full real axis; the imaginary axis maps to the interval $(-i, i)$ in the imaginary axis; the vertical line $z = \pi/2 + iy, y > 0$ is mapped to the imaginary axis $(i\infty, i)$ and $y < 0$ is mapped to $(-i\infty, -i)$; the regions $y > c$ and $y < -c$, $c > 0$ are mapped onto open topological disks punctured at $\pm i$. The regions are called *asymptotic tracts* and the image of any curve $\gamma(t) = x(t)+iy(t)$ such that $y(t) \to \infty$ as $t \to \infty$ is an asymptotic curve ending at the asymptotic value i. Thus, it is sometimes convenient to think of the asymptotic value as the "image" of infinity.

The derivative $f'(z) = \sec^2 z \neq 0$ for any $z \in \mathbb{C}$ so there are no critical points (and thus no critical values).

To get the mapping properties of $f_\lambda(z)$ we just multiply the image plane by λ; the asymptotic values are now $\pm\lambda i$.

There is symmetry in both the variable and parameter:

$$f_\lambda(-z) = -f_\lambda(z) \text{ and } f_{-\lambda}(z) = -f_\lambda(z)$$

It follows that if z_0 is a periodic point of period p, $f_\lambda^p(z_0) = z_0$ then $-z_0$ is also periodic of period p. Thus, for the periodic orbit, $z_0, z_1, \ldots, z_{p-1}$, either there is a symmetric orbit $-z_0, -z_1, \ldots, -z_{p-1}$ or p is even and the symmetric points are contained in the orbit, $z_{\frac{p}{2}+j} = -z_j$, $j = 0, \ldots \frac{p}{2} - 1$.

3 The Dynamic Plane

We define stability for meromorphic functions just as we do for rational maps.

A point z is *stable* if all the iterates $f_\lambda^n(z)$ are defined and if there is a neighborhood on which these iterates form a normal family. The set of stable points is also called the *Fatou set* and is denoted F_λ. It is clearly open and completely invariant. It may be empty.

The *chaotic or Julia set* is defined as $J_\lambda = \hat{\mathbb{C}} \setminus F_\lambda$. It is closed, backward invariant and forward invariant whenever the iterates are defined. It is never empty because it always contains the poles.

The ω-limit set is the accumulation set of $\overline{\cup_n f_\lambda^n(\pm i\lambda)}$. It is denoted ω_λ. It is forward invariant and controls the dynamics.

Points that eventually land on poles are called *prepoles.* In [**?**] we proved

Theorem 3.1. *The Julia set is the closure of the prepoles.*

A periodic cycle is *repelling, attracting or neutral* as the *multiplier* $m(\lambda, z_0) = |df_\lambda^p(z_0)/dz|$ is greater, less or equal to 1. Note that the multiplier doesn't depend on the point of the cycle at which it is evaluated. If, for a neutral cycle, $m(\lambda, z_0) = e^{2\frac{p}{q}\pi i}$, the cycle is *parabolic.*

We also proved, again in [**?**],

Theorem 3.2. *The Julia set is the closure of the repelling periodic cycles.*

Summarizing results in [**?**] and [4] we have have the following classification of stable behavior for the tangent family.

Theorem 3.3. *If $F_\lambda \neq \emptyset$ then either*

- $0 < |\lambda| < 1$ *and F_λ is the complement of a Cantor set in $\hat{\mathbb{C}}$ or*
- $|\lambda| \geq 1$ *and either*

  1. (a) *There are two symmetric periodic cycles of simply connected components $\pm\{D_i\}_0^{p-1}$ each with the same multiplier and there is an attractive cycle contained inside each cycle of components; ω_λ is contained inside $\pm D_i$.*
     (b) *There are two symmetric periodic cycles of simply connected components $\pm\{D_i\}_0^{p-1}$ each with the same multiplier and there is a parabolic cycle on the boundary of each cycle; again, ω_λ is contained inside $\pm D_i$.*
     (c) *Each cycle $\pm\{D_i\}_0^{p-1}$ is a cycle of* Siegel disks. *That is, $f_\lambda^p|_{\pm D_i}$ is holomorphically conjugate to an irrational rotation and the boundary of $\pm D_i$ is contained in ω_λ.*
  2. (a) *There is a single symmetric eventually periodic cycle of simply connected components $\{D_i\}_0^{2p-1}$ such that for $i = 0, \dots p-1$ $D_i = -D_{p+i}$. There is a single attractive cycle contained inside these components; ω_λ is contained inside the components.*
     (b) *There is a single symmetric eventually periodic cycle of simply connected components as above with a single parabolic cycle contained on their boundary; again, ω_λ is contained inside the components.*
     (c) *The components form a single cycle of Siegel disks of period $2p$ and the boundary of the components $\pm D_i$ is contained in ω_λ.*

Every component of F_λ eventually lands on these periodic components.

It follows from this theorem that if $F_\lambda \neq 0$ there is a uniquely defined multiplier $m(\lambda)$, the multiplier of one or both of the cycles corresponding to the periodic stable domains. We also conclude that either there is a single infinitely connected component of F_λ, ($|\lambda| < 1$), or all the components of F_λ are simply connected.

4 The Parameter Plane

If $F_\lambda \neq 0$ and if the corresponding multiplier $|m(\lambda)| < 1$ we say $f_\lambda(z)$ is *hyperbolic*. The *hyperbolic locus* of the parameter plane is then

$$\mathcal{H} = \{\lambda : F_\lambda \neq 0 \text{ and } 0 < |m(\lambda)| < 1\}$$

Denote a generic connected component of $\mathcal{H}$ by Ω. We have the following possibilities for which we set notation

1. $\Omega = \Delta^* = \{\lambda : 0 < |\lambda| < 1\}$
2. $\Omega = \Omega_p$: for $\lambda \in \Omega_p$, $f_\lambda(z)$ has two attractive cycles of period p
3. $\Omega = \Omega'_p$: for $\lambda \in \Omega'_p$, $f_\lambda(z)$ has a single attractive cycles of period $2p$

We have

Theorem 4.1. *[5] Either $\Omega = \Delta^*$ or the multiplier $m(\lambda)$ defines a holomorphic universal covering $m : \Omega \to \Delta^*$.*

Thus, for $\Omega \neq \Delta^*$, Ω is simply connected and holomorphically conjugate to the upper half plane. Figure 1 was made by W.H. Jiang [3]. We see a well defined structure to the components of $\mathcal{H}$. As we saw in theorem 3.3, the punctured unit disk is analogous to the exterior of the Mandelbrot set. The maps are hyperbolic and the Julia sets are Cantor sets.

The parameter values such that the asymptotic value lands on a pole play a special role analogous to the centers of the components of the Mandelbrot set.

For each integer, $p > 0$, set

$$\mathcal{C}_p = \{\lambda : f_\lambda^{p-1}(\lambda i) = \infty\}$$

and set $\mathcal{C} = \cup_p \mathcal{C}_p$. Then $\mathcal{C}_1 = \{\infty\}$ and $\mathcal{C}_2 = \{(k + 1/2)\pi, k \in \mathbb{Z}\}$.

For $\lambda \in \mathcal{C}_p$, the asymptotic values can be thought of as belonging to "virtual periodic cycles":$f^p(\lambda i) = \lambda i$. Computing

$$m(\lambda) = \lambda^p \Pi_{j=0}^p \sec^2 f_\lambda^j(\lambda i)$$

we see that for $\lambda \in \mathcal{C}_p$, $m(\lambda) = 0$. We therefore call these parameter values *virtual centers*.

Note that, unlike the centers of the Mandelbrot set, for any $\lambda \in \mathcal{C}$, by theorem 3.3, $J_\lambda = \hat{\mathbb{C}}$. Like the centers though, they can be used to enumerate the hyperbolic components.

Theorem 4.2. *[4] All components of $\mathcal{H}$, except Δ^* come in pairs (Ω_p, Ω'_p) with a common boundary point $\lambda \in \mathcal{C}_p$. If $p = 1$ there is a unique pair of components (Ω_1, Ω'_1) containing the positive and negative axes with $|x| > 1$ respectively. These are the only unbounded components. The pairs can be enumerated by the prepoles of the tangent map, $\tan z$ so that there is a one to one correspondence between the prepoles and the virtual centers.*

5 The Imaginary Parameters

For quadratic maps, the real parameters play a special role. The hyperbolic components exhibit period doubling and the period doubling cascades come with a Sharkovskii ordering. There is an analogous phenomenon for tangent maps with imaginary parameters.

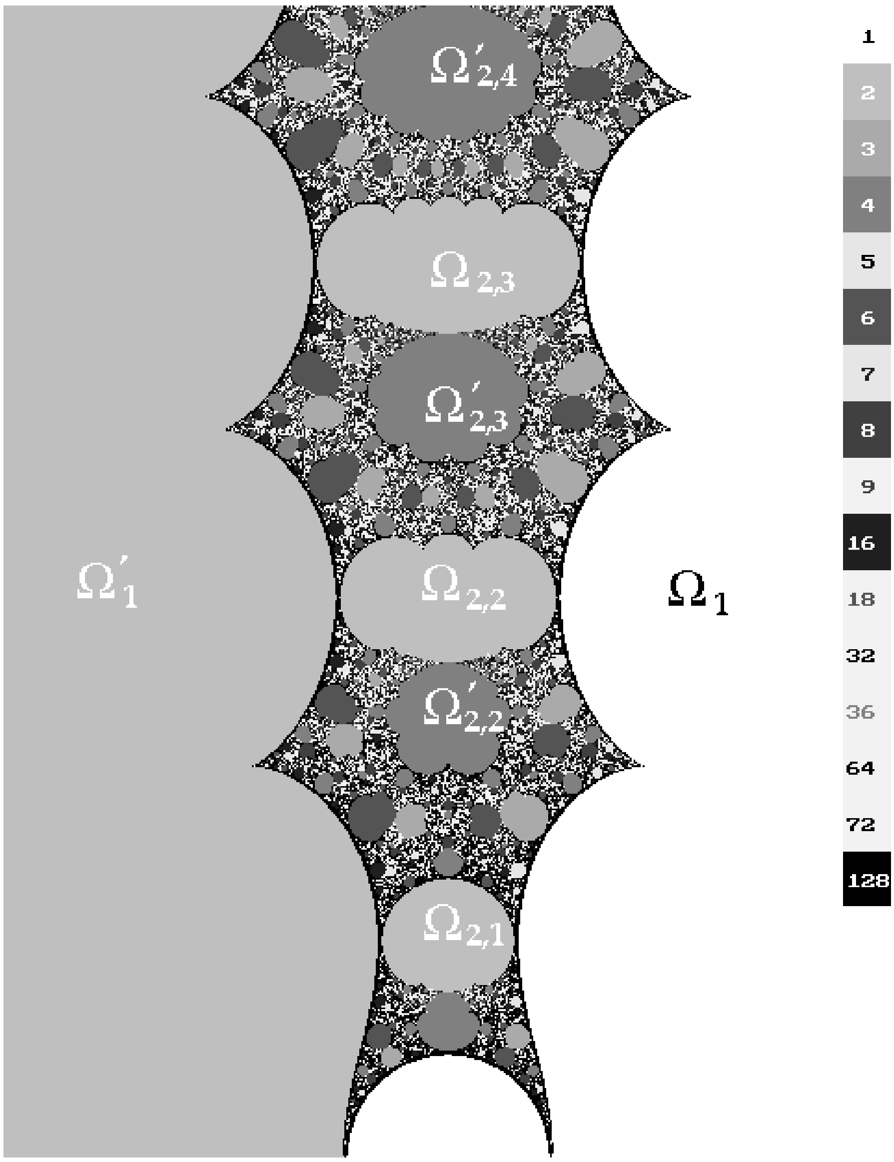

Figure 1: Hyperbolic components on the imaginary axis

5.1 The Period Doubling Cascade

Consider $f_\lambda^2(z) = \lambda \tan(\lambda \tan z)$. For $\lambda = iq$, $x, q \in \mathbb{R}$ we have

$$f_{iq}^2(x) = iq \tan(iq \tan x) = -q \tanh(q \tan x)$$

and we see that f_{iy}^2 maps the real axis to itself. Moreover, the asymptotic values $\mp q$ are both real. We know that if f_{iq} has an attractive cycle, the cycle attracts the asymptotic values. The orbits of the asymptotic values lie in the real and imaginary axes so the attractive cycle must also. The points alternate between the real and imaginary axes so that the period must be even. Similarly, if there is a parabolic cycle for f_{iq} it lies in the real and imaginary axes. Looking at the formula for the multiplier of these cycles, it is obviously real. Since $\omega_{iq} \subset \mathbb{R} \cup \Im$, we deduce that there can be no Siegel disks for purely imaginary parameters.

For hyperbolic components intersecting the imaginary axis we have the following propositions proved in [5]. We tacitly assume that the component is not Δ^*.

Proposition 5.1. *If $\Omega \cap \Im \neq \emptyset$ then $\Omega = -\overline{\Omega}$.*

Proposition 5.2. *If $\Omega \cap \Im \neq \emptyset$, the period of the attracting cycle(s) is even.*

Proposition 5.3. *If $\Omega \cap \Im = J \neq \emptyset$, then J is an interval with a virtual center at one end. At the other, the multiplier is ± 1.*

There is a natural ordering for components Ω intersecting the imaginary axis based on the order of the intervals $J = \Omega \cap \Im$. In the following theorems, we restrict ourselves to those components intersecting the positive imaginary axis. We reflect to obtain theorems for those on the negative imaginary axis. We write the elements of a pair (Ω, Ω') or (Ω, Ω') to reflect this order, and similarly for the pairs themselves.

We have

Theorem 5.4. *[5] Each parameter $\lambda_k = (k + 1/2)\pi$, $k = 0, 1, 2, \ldots$ is a virtual center for a pair (Ω_2', Ω_2). Set $\lambda = iq$. As q increases past $(k + 1/2)\pi$ we have a sequence of adjacent pairs of hyperbolic components with a common boundary point exhibiting a period doubling cascade*

$$(\Omega_2', \Omega_2)(\Omega_4, \Omega_4')(\Omega_8, \Omega_8')(\Omega_{16}, \Omega_{16}' \ldots$$

These components accumulate to a unique point iq_∞. The left endpoint of Ω_2' is a cusp like the cusp of the cardiod.

Note the difference between the first bifurcation pairs $(\Omega_2', \Omega_2)(\Omega_4, \Omega_4')$ and the second $(\Omega_4, \Omega_4')(\Omega_8, \Omega_8')$.

In Ω_2 there are two cycles of period 2 and in Ω_4 there are two cycles of period 4. This is a standard bifurcation; the multiplier of the cycle in Ω_2 is negative and becomes -1 at the parabolic point. The orbit of the asymptotic value approaches the periodic point from both sides. As q increases, that cycle of period 2 becomes repelling and a pair of points, one to the right, and one to the left become part of a new attracting cycle of period 4. The new cycle has positive multiplier. The orbit of the asymptotic value lies to only one side of the new periodic points.

As iq moves to the virtual center joining Ω_4 and Ω_4', the point p^* of the attracting cycle closest to the pole reaches the pole. The attracting cycle becomes a virtual cycle. As iq increases into Ω_4', the relative positions of p^* and the pole interchange. It follows that the sign of $f_{iq}^2(p)$ changes and the two cycles of period 4 interleave to become one of period 8. If we look at what has happened to the multiplier of the original cycle, it is a square root of the multiplier of the new cycle; in fact, it is the negative square root. Now the orbits of the asymptotic values change also. Each periodic point has the orbit of one asymptotic value approaching from the right and the orbit of the other asymptotic value approaching from the left. At the parabolic endpoint of Ω_4', the multiplier is $+1$ but the square root is -1. As q increases, the cycle of period 8 becomes repelling, and a pair of points, one to the right, and one to the left, of each point in the cycle become part of an attracting cycle. Each of these points attract a different asymptotic value so they belong to different cycles each of period 8. Thus we have the non-standard bifurcation wherein the cycle of period 8 bifurcates to two new attracting cycles of period 8.

Thus, in each interval λ_k, λ_{k+1} of the imaginary axis we have a period doubling cascade reminiscent of the period doubling on the real axis for quadratic maps.

5.2 Periods of All Orders

Away from the virtual centers the functions $g_n(\lambda) = f_\lambda^n(\lambda i)$ are holomorphic in λ. They have essential singularities at the poles of $f_\lambda^j(\lambda i)$, $j \leq n-1$, and poles at the solutions of $f_\lambda^{n-1}(\lambda i) = (k+1/2)\pi$.

For $\lambda = qi \in \Im$, the functions $g_{2n}(q)$ become real valued functions of a real variable, real analytic away from the virtual centers. We define a *full branch* of $g_{2n}(q)$ to be a branch (with no singularities) between consecutive solutions of $g_{2n-2}(q) = (j-1/2)\pi$ and $g_{2n-2}(q) = (j+1/2)\pi$. Then

Proposition 5.5. *[5] If $q \in ((k+1/2)\pi, (k+3/2)\pi)$ any full branch of $g_{2n}(q)$ has range containing the interval $-(k+1/2)\pi, (k+1/2)\pi)$ and such branches exist for $-k < j \leq k$.*

Using this we prove

Theorem 5.6. *[5] In each interval $(\lambda_k, \lambda_{k+1})$ there exist hyperbolic components of every even period. The ordering of the periods reflects the ordering of the endpoints of the full branches of the g_{2n}. There is a period doubling cascade to the right of each left endpoint of a full branch of g_{2n}.*

5.3 Cantor Sets with Bounded Geometry

The limit point of the period doubling cascade for real quadratic maps is the Feigenbaum point. For this map the Julia set contains a forward invariant Cantor set, the accumulation set of the critical value. Moreover this Cantor set has bounded geometry; that is, the ratio of the sizes of the gaps to the remaining intervals is bounded at all levels. We see a similar phenomenon for the limit of period doubling for tangent maps.

Choose $q \in ((k+1/2)\pi, (k+3/2)\pi)$ and set

$$h_q(x) = -q \tanh(q \tan x)$$

for real x. Periodic points of $h_q(x)$ are the real periodic points of $f_{-iq}(z)$; their period for h_q is half the period for f_{-iq}. Therefore, by theorem 5.4, there is an interval $I_k = [k+1/2)\pi, q_\infty)$ such that as q moves from left to right, $h_q(x)$ goes through a period doubling cascade.

Let $h(x) = h_{q_\infty}(x)$. We are interested in the set $\omega = \omega_{iq_\infty}$. This is the part of the set ω_{iq} that lies in $\mathbb{R}$. To this end, we first study the combinatorial structure of the periodic cycles generated in the period doubling cascade. These are precisely the real periodic points of f^2_{-iq}.

The only periodic points we are interested in come from the cascade so, for readability, we mean only these when we say periodic points. Once periodic points appear, their positions relative to those there already do not change. At each step there is one new point appearing to the left and one to the right of each of the existing points. At step n there are 2^n periodic points; half lie in the positive real axis and the other half lie symmetrically in the negative real axis. We want to set up a notation for them that reflects their ordering on the real axis.

Let $s_n = x_1 \dots x_n$ where $x_i \in \{0,1\}$. We denote the periodic points $\pm p_{s_n}$. The map $s_n \mapsto \sum_1^n \frac{(-1)^{1-x_j}}{2^j}$ orders the points on $\mathbb{R}^+$; for a given s_n, the points $s_n 0$ and $s_n 1$ at the next level are to the left and right of s_n respectively. We also find it convenient to set $m_n = 2^{n-1}$.

In [6] we give a full description of the map Θ induced by h on the sequences s_n. In this article we only consider the periodic point $p_{s_n^*} = p_{0101\dots010}$ for odd n and $p_{s_n^*} = p_{0101\dots0101}$ for even n and the points $\pm p_n^1 = \pm p_{111\dots11}$. The following propositions and theorem are all proved in [6]. By continuity and the map Θ we have

Proposition 5.7. *The point $p_{s_n^*}$ is closer to the pole $(k+1/2)\pi$ than any other periodic point of period less or equal to 2^n. It is to the right or left of the pole as n is even or odd. A similar statement holds for $-p_{s_n^*}$. The points $\pm p_n^1$ are respectively the right and leftmost periodic points. If n is odd $h(p_{s_n^*}) = -p_n^1$ and if n is even $h(p_{s_n^*}) = p_n^1$.*

Again by continuity, since $h(q_\infty)$ cannot be preperiodic, we have

Proposition 5.8. *For all n,*

$$k\pi < h(q_\infty) < p_{s_n}$$

Using the map Θ and proposition 5.7 we have

Proposition 5.9. *For all $n > 1$,*

$$p^1_{n-1} < h^{m_n}(-q_\infty) < p^1_n < h^{m_{n+1}}(-q_\infty) < p^1_{n+1} < \ldots < q_\infty$$

This proposition, together with those above, enables us to define the *structure intervals* $\pm\mathcal{I}_n(1) = [h^{m_n}(-q_\infty), q_\infty]$ and $\pm\mathcal{I}_n(j) = [h^{m_n+j}(-q_\infty), h^j(q_\infty)]$, $j = 1, \ldots, m_n - 1$. We prove that for each n they are mutually disjoint.

We see that, for all j,

$$\mathcal{I}_{n+1}(j) \subset \mathcal{I}_n(j) \text{ and } -\mathcal{I}_{n+1}(m_n + j) \subset \mathcal{I}_n(j).$$

We define the *gap intervals* by $\pm G_n(1) = (h^m_n(-q_\infty), h^{m_{n+1}+m_n}(q_\infty))$ so that

$$\mathcal{I}_n(1) = -\mathcal{I}_{n+1}(m_n + 1) \cup G_n(1) \cup \mathcal{I}_{n+1}(1)$$

We also have $\pm G_n(1 + j) = h^j(G_n(1))$, $j = 1, \ldots m_n - 1$.

The theorem here is

Theorem 5.10. *The set obtained by successively removing the intervals $\pm G_n(j)$, $n = 1, 2, \ldots, j = 1, \ldots, m_n-1$ from the intervals $[-q_\infty, h(-q_\infty)]$ and $[h(q_\infty), q_\infty]$ is a Cantor set with bounded geometry. That is, using absolute values for lengths of intervals, the ratios,*

$$|\mathcal{I}_{n+1}(j)|/|III_n(j)|, |-\mathcal{I}_{n+1}(j + m_n)|/|III_n(j)|, |G_n(j)|/|III_n(j)|$$

are bounded above and below by bounds independent of n and j.

The techniques in the proof are similar to those for quadratic maps. One uses the fact that h has negative Schwarzian derivative to show h is quasisymmetric. The important difference is that we need go only half way around the cycle to get the bounds for quasisymmetry. To get bounds on the gaps, we use the fact that the map $H(x) = |h(x)|$ defined on $[h(q_\infty), q_\infty]$ has the points $+p_{s_n}$ as its periodic points, is continuous and is unimodal.

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