

# LIMIT POINTS OF ITERATED FUNCTION SYSTEMS

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ABSTRACT. This is a written version of a lecture given at the *International Workshop on Teichmüller theory and moduli problems* held at the Harish Chandra Research Institute, Allahabad, U.P., India, in January 2006. It is based on work that originally appeared in [3],[5], and [6].

## 1. INTRODUCTION

Let  $\Delta$  denote the unit disk and  $X$  denote a subdomain of  $\Delta$ . Let  $f_1, f_2, f_3, \dots$  be a randomly chosen sequence of holomorphic maps  $f_i : \Delta \rightarrow X \subset \Delta$  and consider the compositions

$$F_n = f_1 \circ f_2 \circ \dots \circ f_{n-1} \circ f_n.$$

**Definition 1.** *The sequence  $\{F_n\}$  is called the iterated function system IFS coming from the sequence  $f_1, f_2, f_3, \dots$*

The sequence  $F_n$  forms a normal family and the limit functions  $F$  of the family are either open maps or constant maps. The constant limit points may be located either inside  $X$  or on  $\partial X$ .

Note that the compositions are taken in the *reverse* of the usual order; that is, backwards. There is a theory for forward iterated function systems that is somewhat simpler (see [4]).

Iterated function systems can also be defined for families of holomorphic functions from any plane domain  $\Omega$  onto a subdomain  $X \subseteq \Omega$ . We will show what kinds of limit functions occur and whether a given sequence admits more than one limit function depends only on the relative geometry of  $X$  in  $\Omega$ .

Clearly if  $\Omega$  is the whole complex plane and  $X$  has more than two boundary points, the only functions from  $\Omega$  to  $X$  are constants and the iterated function systems are not interesting. We therefore will assume, in this paper, that all the domains we consider have at least two boundary points. Since such a domain admits a hyperbolic metric, we call it *a hyperbolic domain*. The results here originally appeared in [3],[5] and [6]. The reader is also referred to [7] for background and other results.

## 2. HISTORY

The simplest situation occurs when all of the functions in the family are the same. Then we have the theorem of Denjoy.

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**Theorem** (Denjoy). *Given an IFS such that  $f_n = f$  for all  $n$  and  $f$  is not a Möbius transformation. Then*

$$\lim_{n \rightarrow \infty} F_n = \lim_{n \rightarrow \infty} f^n \equiv k$$

$k \in \bar{X}$  is unique.

The first generalization of the Denjoy theorem was given independently by Gill and Lorentzen.

**Theorem** (Gill(88)-Lorentzen(90)). *If  $X$  is relatively compact in  $\Delta$  then*

$$\lim_{n \rightarrow \infty} F_n \equiv k$$

$k \in X$ , ( $k \notin \partial X$ ) is unique.

This completely settles the question of what happens when the subdomain  $X$  is relatively compact in  $\Delta$ . The next natural case to consider is case when the subdomain  $X \subset \Delta$  is not relatively compact in  $\Delta$ . To this end, we recall the following concept for subdomains of the complex plane.

**Definition 2.** *A domain  $E \subset \mathbb{C}$  is a Bloch domain if*

$$r_a = \sup_r (|z - a| < r \subset X) \text{ is bounded for all } a \in X.$$

An example of a non-compact Bloch domain is an infinite strip whose boundary is a pair of parallel lines.

In [2], Beardon, Carne, Minda and Ng generalize this concept as follows. Let  $\rho_\Delta$  denote the hyperbolic metric on  $\Delta$ .

**Definition 3.** *A domain  $X \subset \Delta$  is a hyperbolic Bloch subdomain of  $\Delta$ , if*

$$r_a = \sup_r (\rho_\Delta(z, a) < r \subset X) \text{ is bounded for all } a \in X.$$

Since, in this paper we are only concerned with hyperbolic domains, we will abuse notation and use the term Bloch subdomain to mean a hyperbolic Bloch subdomain.

Clearly if  $X$  is relatively compact in  $\Delta$ , it is a Bloch subdomain. There are, however, lots of non-relatively compact subdomains that are also Bloch subdomains. Let us consider some examples:

- The interior of a polygon of finite hyperbolic area is a Bloch domain. Such a polygon may have cusps on the boundary of  $\Delta$  where the adjacent sides are tangent. Fundamental domains of finitely generated Fuchsian groups that represent Riemann surfaces with punctures are thus Bloch subdomains.
- The hyperbolic analog of the Euclidean infinite strip is a polygon with the property that those sides that touch the boundary of  $\Delta$  meet there at a finite angle. Note that such a polygon has infinite area but it is a Bloch subdomain.
- A more interesting example is the following. Let  $G$  be a finitely generated Fuchsian group  $G$  whose quotient  $S = \Delta/G$  is a compact Riemann surface. Let  $D$  be a fundamental polygon for  $G$  and let  $V = \{v_1, \dots, v_n\}$  be the set of vertices of  $D$ . For any  $g \in G$  let  $g(V) = \{g(v_1), \dots, g(v_n)\}$ . Then

$$X = \Delta \setminus \cup_{g \in G} g(V)$$

is an example of a non-simply connected Bloch subdomain.

We have the following theorem about Bloch domains.

**Theorem** ([2],[3]). *All limit functions of any IFS from  $\Delta$  to  $X$  are constant if and only if  $X$  is a Bloch subdomain.*

The “If” statement was proved by Beardon, Carne, Minda and Ng in [2] and the “Only if” statement was originally proved in [3]. We prove it here as theorem 2 in the next section.

### 3. UNIQUENESS OF LIMITS

If  $X$  is relatively compact in  $\Delta$  the Gill-Lorentzen theorem says that every IFS has a unique limit. It is natural then to ask whether the same is true for limit functions of iterated function systems coming from non-relatively compact subdomains  $X$ . Answers to this question are the content of the following theorems:

**Theorem 1** ([6]). *If  $X$  is non-relatively compact in  $\Delta$  and  $n$  is any integer, then given any  $n$  points  $c_1, c_2, \dots, c_n \in X$ , there is an IFS with at least  $n$  limit functions  $G_i$ ,  $i = 1, \dots, n$  such that  $G_i(0) = c_i$ . If  $X$  is Bloch  $G_i \equiv c_i$  and these are the only limit functions.*

When  $X$  is non-Bloch we have the following results:

**Theorem 2** ([3]). *If  $X$  is non-Bloch then there always exist non-constant limit functions – the Bloch condition is necessary for all limits to be constant.*

**Theorem 3** ([5]). *Suppose  $X$  is non-Bloch in  $\Delta$  and  $f : \Delta \rightarrow X$  is any holomorphic map. Then  $f$  is the limit of an IFS.*

### 4. PROOF OF THEOREM 1

With no loss of generality we may assume that  $0 \in X$ . The idea of the proof is to construct functions  $f_k$  such that the set  $S = \{c_0 = 0, c_1 = F_1(0) = f_1(0), c_2 = F_2(0) = f_1 \circ f_2(0), \dots, c_{n-1} = F_{n-1}(0) = f_1 \circ f_2 \circ \dots \circ f_{n-1}(0)\}$  consists of distinct points and such that the *cycle relation*

$$(1) \quad f_i \circ f_{i+1} \circ \dots \circ f_{i+n-1}(0) = 0$$

holds for all integers  $i$ . From this relation we see that  $F_n(0) = 0$  and if  $m = qn + r$ ,  $0 \leq r < n$ ,  $F_m(0) = c_r$ . The point here is that because the iteration is backwards, we have to find successive pre-images of the points  $c_i$ .

Once we have done this, for any subsequence  $F_{n_k} = f_1 \circ f_2 \circ \dots \circ f_{n_k}$ , we see that  $F_{n_k}(0) \in S$  for all  $k$ . It follows that any limit function must map 0 to a point in  $S$ . Choosing subsequences appropriately, we can find  $n$  distinct limit functions  $G_i$  such that  $G_i(0) = c_i$ ,  $i = 0 \dots n - 1$ .

If  $X$  is Bloch, all limit functions must be constant so these are all the limit functions.

We divide the proof into three cases,  $n = 1$ ,  $n = 2$  and  $n = 4$ . It will be clear from the  $n = 4$  case how the proof for any  $n > 4$  should go.

$n = 1$ : This is the most trivial case. Take  $f_i : \Delta \rightarrow X$  as a covering map with fixed point  $c$ ,  $f_i(c) = c$ . Then clearly  $F_n(c) = c$  for all  $n$  and any limit function satisfies  $F(c) = c$ .

$n = 2$ : We can use covering maps again in this case. Assume  $0 \in X$  and  $c_0 = 0$ . Choose a covering map

$$f_1 : \Delta \rightarrow X, \text{ such that } f_1(0) = c_1.$$

The map  $f_1$  is defined up to a rotation. Choose it so that for some  $x_1$  with  $f_1(x_1) = 0$  we have  $x_1 \in X$ . Now choose  $f_2$  so that  $f_2(0) = x_1$  and for some  $x_2$

$$f_2(x_2) = 0, \text{ and } x_2 \in X.$$

Continue to find  $f_n$  and  $x_n$  such that

$$f_n(0) = x_{n-1}, x_n \in X, f_n(x_n) = 0.$$

We can always do this by the non-compactness of  $X$ . Then for all  $n$  we have

$$f_1 \circ f_2 \circ \dots \circ f_{2n}(0) = 0 \text{ and}$$

$$f_1 \circ f_2 \circ \dots \circ f_{2n+1}(0) = c_1.$$

For  $n \geq 2$  we need more than covering maps. In order to find the functions  $f_n$  and their successive pre-images we need a preparatory lemma.

**Lemma 1.** *Given distinct points  $a_1, \dots, a_n \in \Delta \setminus \{0\}$ , there  $\exists f : \Delta \rightarrow \Delta$  and  $x_i \in X$ , such that*

$$f(x_i) = \frac{a_i}{x_i}.$$

*The function  $f$  is rational of degree  $n - 1$ .*

*Proof.* Assume  $n \geq 2$  and set

$$A(a, z) = \frac{z - a}{1 - \bar{a}z}, \quad A(a, A(-a, z)) = z.$$

Let  $g_1 : \Delta \rightarrow \Delta$  be a holomorphic function to be determined later. Choose  $x_1 \in X, |x_1| > |a_1|$ . Set

$$f(z) = \frac{A(x_1, z)g_1(A(x_1, z)) + \frac{a_1}{x_1}}{1 + \frac{\bar{a}_1}{x_1}A(x_1, z)g_1(A(x_1, z))}$$

Then  $f(x_1) = a_1/x_1$ .

Rewrite this as

$$A(x_1, z)g_1(A(x_1, z)) = A\left(\frac{a_1}{x_1}, f(z)\right).$$

Want to choose  $x_2, |x_2| > |a_2|$  such that

$$A(x_1, x_2)g_1(A(x_1, x_2)) = A\left(\frac{a_1}{x_1}, \frac{a_2}{x_2}\right)$$

and so on. For  $1 \leq j \leq k \leq n$  set

$$a_{jk} = A(x_j, x_k).$$

Next, for  $k = 2, \dots, n$  set

$$(2) \quad b_{1k} = A\left(\frac{a_1}{x_1}, \frac{a_k}{x_k}\right)$$

For  $j = 2, \dots, n - 1$  and  $k = j, j + 1, \dots, n$  set

$$(3) \quad b_{jk} = A\left(\frac{b_{(j-1)j}}{a_{(j-1)j}}, \frac{b_{(j-1)k}}{a_{(j-1)k}}\right).$$

In order that our construction work we need to choose the  $x_i$  so that the following inequalities hold:

$$(4) \quad \left| \frac{a_i}{x_i} \right| < 1, \quad i = 1, \dots, n.$$

In step 1 we chose  $x_1$  so this holds for  $i = 1$ .

For all  $j, k$  such that  $j < k$  we also need to have

$$(5) \quad \left| \frac{b_{jk}}{a_{jk}} \right| < 1$$

We use the non-compactness of  $X$  to prove we can choose  $x_i$  satisfying these conditions. Note first that for fixed  $j$ , and all  $k > j$ ,  $|x_k| \rightarrow 1$  implies  $|a_{jk}| \rightarrow 1$ .

Next, as  $|x_j| \rightarrow 1$ ,

$$\limsup |b_{1j}| \leq |A\left(\frac{a_1}{x_1}, a_j e^{\theta_j}\right)| = B_{1j} < 1$$

where  $\theta_j$  is chosen so that  $\arg a_j e^{\theta_j} = \arg \frac{a_1}{x_1} + \pi$  and  $B_{1j}$  is maximal.

Since  $X$  is not relatively compact we get conditions on  $x_i$ ,  $i = 2, \dots, n$  so that all the inequalities (4) and all the inequalities (5) hold with  $j = 1$ .

Now fix  $x_2$  so that (4) and (5) with  $j = 1$  hold, assuming the remaining  $|x_i|$  are close enough to 1.

We now find bounds

$$\limsup_{|x_j| \rightarrow 1} |b_{2j}| \leq |A\left(\frac{b_{12}}{a_{12}}, B_{1j} e^{\theta_j}\right)| = B_{2j} < 1$$

where again  $\theta_j$  is chosen to maximize  $B_{2j}$ .

We repeat this process, choosing  $x_3, \dots, x_{n-1}, x_n$ , in turn so that all the inequalities above hold.

Next, define the functions  $g_k(z) : \Delta \rightarrow \Delta$ ,  $k = 2, \dots, n$  recursively by

$$(6) \quad \begin{aligned} & A(x_k, z) g_k(A(x_k, z)) = \\ & A\left(\frac{b_{(k-1)k}}{a_{(k-1)k}}, g_{(k-1)}(A(x_{(k-1)}, z))\right) \end{aligned}$$

Now take the function  $g_n(z) \equiv 0$ . Then work back through equations (6) to obtain the functions  $g_1$  and  $f$ . Note that  $f$  is rational of degree  $n - 1$ .

We check that  $f(x_i) = \frac{a_i}{x_i}$  for  $i = 1, \dots, n$ , so that we have the required points  $x_i$  and the function  $f$ .  $\square$

We show how the theorem is proved for  $n = 4$ .

Assume  $0 \in X$  and  $c_0 = 0, c_1, c_2, c_3 \in X$ . We want the cycle relation

$$f_i \circ f_{i+1} \circ f_{i+2} \circ f_{i+3}(0) = 0$$

to hold for all  $i$ .

Let  $\pi_1$  be a covering map such that  $\pi_1(0) = c_1$ . Choose  $y_1, b'_2, b'_3 \in \Delta$  such that

$$\pi_1(y_1) = 0, \quad \pi_1(b'_2) = c_2, \quad \pi_1(b'_3) = c_3.$$

That is,

$$\begin{array}{cccc} & y_1 & 0 & b'_2 & b'_3 \\ \pi_1 & \downarrow & \downarrow & \downarrow & \downarrow \\ & \cdot 0 & \cdot c_1 & \cdot c_2 & \cdot c_3 \end{array}$$

Use the lemma to find  $g$  and  $x_1, b_2, b_3 \in X$  so that

$$g(x_1) = \frac{y_1}{x_1}, \quad g(b_2) = \frac{b'_2}{b_2}, \quad g(b_3) = \frac{b'_3}{b_3}.$$

Let  $g_1(x) = xg(x)$ . Then

$$\begin{array}{cccc} & x_1 & 0 & b_2 & b_3 \\ g_1 & \downarrow & \downarrow & \downarrow & \downarrow \\ & y_1 & 0 & b'_2 & b'_3 \\ \pi_1 & \downarrow & \downarrow & \downarrow & \downarrow \\ & \cdot 0 & \cdot c_1 & \cdot c_2 & \cdot c_3 \end{array}$$

Set  $f_1 = \pi_1 g_1$ . Choose a covering map  $\pi_2$  such that  $\pi_2(0) = b_2$  and points  $y_{21}, y_2, b'_{32}$  so that

$$\begin{array}{cccc} & y_{21} & y_2 & 0 & b'_{32} \\ \pi_2 & \downarrow & \downarrow & \downarrow & \downarrow \\ & x_1 & 0 & b_2 & b_3 \\ g_1 & \downarrow & \downarrow & \downarrow & \downarrow \\ & y_1 & 0 & b'_2 & b'_3 \\ \pi_1 & \downarrow & \downarrow & \downarrow & \downarrow \\ & \cdot 0 & \cdot c_1 & \cdot c_2 & \cdot c_3 \end{array}$$

Now use the lemma to find  $g_2$  and points  $x_{21}, x_2, b_{32}$  so that

$$\begin{array}{cccc} & x_{21} & x_2 & 0 & b_{32} \\ g_2 & \downarrow & \downarrow & \downarrow & \downarrow \\ & y_{21} & y_2 & 0 & b'_{32} \\ \pi_2 & \downarrow & \downarrow & \downarrow & \downarrow \\ & x_1 & 0 & b_2 & b_3 \\ g_1 & \downarrow & \downarrow & \downarrow & \downarrow \\ & y_1 & 0 & b'_2 & b'_3 \\ \pi_1 & \downarrow & \downarrow & \downarrow & \downarrow \\ & \cdot 0 & \cdot c_1 & \cdot c_2 & \cdot c_3 \end{array}$$

Set  $f_2 = \pi_2 g_2$ . Now find a covering map  $\pi_3$  and points  $y_{321}, y_{32}, y_3$  so that

$$\begin{array}{cccc}
& y_{321} & y_{32} & y_3 & 0 \\
\pi_3 & \downarrow & \downarrow & \downarrow & \downarrow \\
& x_{21} & x_2 & 0 & b_{32} \\
g_2 & \downarrow & \downarrow & \downarrow & \downarrow \\
& y_{21} & y_2 & 0 & b'_{32} \\
\pi_2 & \downarrow & \downarrow & \downarrow & \downarrow \\
& x_1 & 0 & b_2 & b_3 \\
g_1 & \downarrow & \downarrow & \downarrow & \downarrow \\
& y_1 & 0 & b'_2 & b'_3 \\
\pi_1 & \downarrow & \downarrow & \downarrow & \downarrow \\
& \cdot 0 & \cdot c_1 & \cdot c_2 & \cdot c_3
\end{array}$$

Then use the lemma again to find  $g_3$  and points  $x_{321}, x_{32}, x_3$  so that

$$\begin{array}{cccc}
& x_{321} & x_{32} & x_3 & 0 \\
g_3 & \downarrow & \downarrow & \downarrow & \downarrow \\
& y_{321} & y_{32} & y_3 & 0 \\
\pi_3 & \downarrow & \downarrow & \downarrow & \downarrow \\
& x_{21} & x_2 & 0 & b_{32} \\
g_2 & \downarrow & \downarrow & \downarrow & \downarrow \\
& y_{21} & y_2 & 0 & b'_{32} \\
\pi_2 & \downarrow & \downarrow & \downarrow & \downarrow \\
& x_1 & 0 & b_2 & b_3 \\
g_1 & \downarrow & \downarrow & \downarrow & \downarrow \\
& y_1 & 0 & b'_2 & b'_3 \\
\pi_1 & \downarrow & \downarrow & \downarrow & \downarrow \\
& \cdot 0 & \cdot c_1 & \cdot c_2 & \cdot c_3
\end{array}$$

Set  $f_3 = \pi_3 g_3$ . Finally, find a covering map  $\pi_4$  and use the lemma to find  $g_4$  so that

$$\begin{array}{cccc}
& 0 & x_{432} & x_{43} & x_4 \\
g_4 & \downarrow & \downarrow & \downarrow & \downarrow \\
& y_{4321} & y_{432} & y_{43} & y_4 \\
\pi_4 & \downarrow & \downarrow & \downarrow & \downarrow \\
& x_{321} & x_{32} & x_3 & 0 \\
g_3 & \downarrow & \downarrow & \downarrow & \downarrow \\
& y_{321} & y_{32} & y_3 & 0 \\
\pi_3 & \downarrow & \downarrow & \downarrow & \downarrow \\
& x_{21} & x_2 & 0 & b_{32} \\
g_2 & \downarrow & \downarrow & \downarrow & \downarrow \\
& y_{21} & y_2 & 0 & b'_{32} \\
\pi_2 & \downarrow & \downarrow & \downarrow & \downarrow \\
& x_1 & 0 & b_2 & b_3 \\
g_1 & \downarrow & \downarrow & \downarrow & \downarrow \\
& y_1 & 0 & b'_2 & b'_3 \\
\pi_1 & \downarrow & \downarrow & \downarrow & \downarrow \\
& \cdot 0 & \cdot c_1 & \cdot c_2 & \cdot c_3
\end{array}$$

Set  $f_4 = \pi_4 g_4$ . Note  $f_1 \circ f_2 \circ f_3 \circ f_4(0) = 0$ . This is the first cycle. We continue in this way to add new points to the cycles: to find  $f_5, f_6, f_7$ , we replace  $c_1, c_2$  and

$c_3$  with  $x_{432}, x_{43}$  and  $x_4$ , respectively; for the rest of the  $f_n$ , we repeat the process, changing the indices in the obvious way.

Next we find universal covering maps  $\pi_n$  and apply lemma 1 to the preimages in  $\Delta$  under  $\pi_n$  of points in the incomplete cycles to get new points in  $X$  and new maps  $g_n$ . The map  $f_n = \pi_n g_n$  completes the next cycle and we have added points to the incomplete cycles. In the limit, for all  $i \geq 1$  we have the cycle relation

$$f_i \circ f_{i+1} \circ f_{i+2} \circ f_{i+3}(0) = 0$$

for all  $i$ .

Let

$$F_k = f_1 \circ \dots \circ f_k.$$

Then,  $F_k(0) = c_r$  where  $r = k \bmod 4$ . The accumulation points are limits of subsequences  $\{F_{n_k}\}$ . For any such limit  $F$ ,  $F(0) = c_r$  for some  $r = 0, \dots, 3$ . Because the  $c_r$  are distinct, we have at least 4 distinct accumulation points.

If  $X$  is Bloch, all the limit functions of this IFS are constant. Since  $F(0) = c_k$  for some  $k$  for every limit function there are exactly 4 possible constant functions.

## 5. PROOFS OF THEOREMS 2 AND 3

Theorem 2 follows from theorem 3, but we give a separate proof that makes it easier to understand how the Bloch condition controls the dynamics.

**Theorem 2** ([4]).  *$X$  is Bloch in  $\Delta$  only if all limit functions of any IFS are constant.*

The idea of the proof is: Given  $X$ , non-Bloch in  $\Delta$ , construct an IFS with a limit function that is not constant using covering maps and Blaschke products of degree 2. We will need two lemmas. The proofs are relatively straightforward and we omit them.

Denote the hyperbolic metric on the disk by  $\rho(a, z) = \rho_\Delta(a, z)$  and the hyperbolic metric on  $X$  by  $\rho_X(a, z)$ .

**Lemma 2** (Geometric Lemma). *Given  $a \in X$  and*

$$\rho_X(a, z) < 1 < C = \text{Bloch rad at } a,$$

*then there is a function  $\epsilon(C)$  such that*

$$\rho_X(a, z) \leq (1 + \epsilon(C))\rho(a, z)$$

*and  $\epsilon(C) \rightarrow 0$  as  $C \rightarrow \infty$ .*

This lemma says that in a relatively compact subset of  $X$ ,  $\rho$  and  $\rho_X$  are equivalent:

$$\rho(z, w) < \rho_X(z, w) < K\rho(z, w)$$

and “deep inside  $X$ ”,  $K$  is very close to 1.

**Lemma 3** (Blaschke product lemma). *Let  $c \neq 0$  be any point in  $\Delta$  such that  $\rho(0, c) < 1$ . If*

$$A_a(z) = \frac{z(z-a)}{1-\bar{a}z},$$

*then  $A_a^{-1}(c) = \{z_1, z_2\}$  (that is,  $A_a(z_1) = A_a(z_2) = c$ ) and*

$$\rho(0, z_1) = \rho(a, z_2) \rightarrow \rho(0, c) \text{ as } |a| \rightarrow 1.$$



This lemma says that Blaschke products contract, but the contraction constant is close to 1 near the boundary.

**Proof of Theorem 2:**

Pick any two distinct points  $a_0, w_0 \in X$  such that

$$\rho_X(a_0, w_0) < 1/2.$$

Now, using the lemmas we want to recursively find  $f_n$ , and compositions  $F_n \rightarrow F$  satisfying

$$F(0) = a_0 \text{ and } F(\tilde{w}) = w_0$$

for some  $\tilde{w}$ ,  $\rho(0, \tilde{w}) < 1$ . To do this we will need a sequence  $\epsilon_n \rightarrow 0$  such that  $\prod_1^\infty (1 + \epsilon_n)^2 \leq 2$ .

To get the first function, let  $\pi_1 : \Delta \rightarrow X$ ,  $\pi_1(0) = a_0$  be a universal covering map. Then  $\exists c_0 \in \Delta$ , such that

$$\pi_1(c_0) = w_0 \text{ and } \rho(0, c_0) = \rho_X(a_0, w_0)$$

For any choice of  $a_1 \in X$ , find  $w_1, \tilde{w}_1 \in \Delta$  with

$$A_{a_1}(\tilde{w}_1) = A_{a_1}(w_1) = c_0 \text{ and } \rho(0, \tilde{w}_1) = \rho(a_1, w_1)$$

Define  $f_1 = \pi_1 \circ A_{a_1}$  so that

$$f_1(0) = f_1(a_1) = a_0 \text{ and } f_1(w_1) = f_1(\tilde{w}_1) = w_0.$$

We need to make sure that  $w_1$  belongs to  $X$ . By the Blaschke product lemma, if  $|a_1|$  is close enough to 1

$$\begin{aligned} \rho(0, \tilde{w}_1) &= \rho(a_1, w_1) \\ &< (1 + \epsilon_1)\rho(0, c_0) \\ &= (1 + \epsilon_1)\rho_X(a_0, w_0) \end{aligned}$$

Moreover, we may assume that  $R(X, \Delta, a_1)$ , the radius in the  $\rho$  metric of the largest disk in  $X$  centered at  $a_1$ , satisfies  $R(X, \Delta, a_1) \gg 1$  so that  $w_1$  is inside a disk in  $X$  “deeply enough” that by the geometric lemma

$$\begin{aligned} \rho_X(a_1, w_1) &< (1 + \epsilon_1)\rho(a_1, w_1) \\ &< (1 + \epsilon_1)^2\rho_X(a_0, w_0) \\ &< 1 \end{aligned}$$

We find the rest of the functions inductively. By our choice of  $\epsilon_n$ , there exist points  $a_n, w_n \in X$  and  $\tilde{w}_n \in \Delta$  such that

$$(7) \quad f_n(0) = f_n(a_n) = a_{n-1} \text{ and } f_n(w_n) = f_n(\tilde{w}_n) = w_{n-1},$$

with

$$\rho(a_n, w_n) = \rho(0, \tilde{w}_n) < (1 + \epsilon_n)\rho_X(a_{n-1}, w_{n-1})$$

and

$$\begin{aligned} \rho_X(a_n, w_n) &< (1 + \epsilon_n)\rho(a_n, w_n) \\ &< (1 + \epsilon_n)^2\rho_X(a_{n-1}, w_{n-1}). \end{aligned}$$

Therefore

$$\begin{aligned} \rho(0, \tilde{w}_n) &< \prod_1^n (1 + \epsilon_i)\rho(0, c_0) < 1 \text{ and} \\ \rho_X(a_n, w_n) &< \prod_1^n (1 + \epsilon_i)^2\rho_X(a_0, w_0) < 1. \end{aligned}$$

Now if  $F_n = f_1 \circ f_2 \circ f_3 \circ \dots \circ f_n$ , formulas (7) yield

$$F_n(0) = a_0 \quad \text{and} \quad F_n(\tilde{w}_n) = w_0.$$

By Montel's theorem,  $F_n$  is a normal family so that any subsequence  $F_{n_j}$  converges locally uniformly to a holomorphic limit function  $F$ . Therefore,

$$F(0) = a_0 \quad \text{and} \quad F(\tilde{w}) = w_0,$$

where  $\tilde{w}$  is an accumulation point of the sequence  $\tilde{w}_{n_j}$ . Since, we have  $\rho(0, \tilde{w}_n) < 1$  for all  $n$ , the point  $\tilde{w}$  belongs to  $\Delta$ . This implies that  $F$  is a nonconstant function concluding the proof of theorem 2.

**Theorem 3** ([5]). *Suppose  $X$  is non-Bloch in  $\Delta$  and  $f : \Delta \rightarrow X$  is any holomorphic map. Then  $f$  is the limit of an IFS.*

### Sketch of proof of Theorem 3

The reader can find the full details in [6]. The main tool is the use of infinite Blaschke products:

Let  $a_1, a_2, a_3, \dots$  be a sequence of points in  $\Delta$  and let  $k$  be a positive integer. The finite Blaschke products

$$A_n(z) = z^k \frac{z - a_1}{1 - \bar{a}_1 z} \frac{z - a_2}{1 - \bar{a}_2 z} \cdots \frac{z - a_n}{1 - \bar{a}_n z},$$

form a normal family. Thus, some subsequence  $A_{n(l)}$  of  $A_n$  converges locally uniformly to an infinite Blaschke product

$$A(z) = z^k \frac{z - a_1}{1 - \bar{a}_1 z} \frac{z - a_2}{1 - \bar{a}_2 z} \cdots$$

If  $|a_n| \rightarrow a > 0$ , then  $A^{(k)}(0) \neq 0$  for any accumulation point  $A$  and  $A : \Delta \rightarrow \Delta$  is open.

A rough idea of the proof is:

Construct infinite Blaschke products  $B_i : X \rightarrow \Delta$ . Interleave the  $B_i$  with covering maps  $\pi_i : \Delta \rightarrow X$ . Set  $f_i = \pi_{i-1} B_i$  for  $i \geq 2$  and show that

$$B_1 f_2 f_3 \dots f_n \rightarrow e^{i\theta} z.$$

Next set  $f_1 = f(e^{-i\theta} B_1(z))$ .

Because  $X$  is non-Bloch we can find  $c_n \in X, |c_n| \rightarrow 1$  and  $D(c_n, n) \subset X$ . We need the following lemmas.

**Lemma 4.** *Given  $C > 1, \exists a_n = c_{i_n}$  such that a limit  $A$  of the Blaschke product for the  $a_n$  satisfies*

$$|A(z)| \geq e^{-k/2^{n-2}C} \tanh(\rho(a_n, z))$$

$\forall z \in D(a_n, (n+4)/2)$ .

**Lemma 5.** *For all  $(p, q) \in \Delta$  with  $\rho(p, q) < 1$ , and any  $\sigma > 0, \exists C > 1$  such that if  $n$  is sufficiently large,  $\exists p_n, q_n \in D(a_n, (n+4)/2)$  with*

$$A(p_n) = p, \quad A(q_n) = q, \quad \rho(p_n, q_n) \leq (1 + \sigma)\rho(p, q).$$

For the construction pick  $z_0 \in X$  with  $z_0 \in D(0, 1)$  and pick  $\epsilon_n$  with  $(1 + \epsilon_n)^{2n+1} \rightarrow 1$ . Find  $a_n$  such that  $D(a_n, \epsilon_n) \subset X$ .

To obtain  $B_1$ , take subsequence of  $a_n$  and form the infinite Blaschke product. Using lemmas 4 and 5 find

$$z_1 \in D(0, 1) \text{ with } B_1(z_1) = z_0$$

$$a_{n_1}, q_1 \in D(a_{n_1}, (n_1 + 4)/2)$$

with

$$B_1(a_{n_1}) = 0, B_1(q_1) = z_0$$

and

$$\rho(0, z_1) \leq (1 + \epsilon_1)\rho(0, z_0)$$

$$\rho_X(a_{n_1}, q_1) \leq (1 + \epsilon_2)\rho(0, z_0).$$

Observe that by lemma 4,  $B_1$  maps  $X$  onto  $\Delta$ .

Define  $\pi_1 : \Delta \rightarrow X$ ,  $\pi_1(0) = a_{n_1}$  and find  $w_1$  with  $\pi_1(w_1) = q_1$  and

$$\rho(0, w_1) = \rho_X(a_{n_1}, q_1) \leq (1 + \epsilon_2)\rho(0, z_0)$$

To obtain  $B_2$ , take a sub-subsequence and form its infinite Blaschke product. Using lemmas 4 and 5 find  $z_2 \in D(0, 1)$  with  $B_2(z_2) = w_1$ . Now pull back by  $B_1$  and  $\pi_1$  to get  $a_{n_2}, q_2 \in D(a_{n_2}, (n_2 + 4)/2)$  with

$$B_1\pi_1B_2(a_{n_2}) = 0, B_1\pi_1B_2(q_2) = z_0$$

$$\text{and } \rho(0, z_2) \leq (1 + \epsilon_2)^3\rho(0, z_0).$$

Note that you get an extra factor of  $(1 + \epsilon_2)$  for each pullback.

Set  $f_2 = \pi_1B_2$ .

Continuing in this way, we obtain sequences of maps  $\pi_{n-1}$  and  $B_n$  with

$$f_n = \pi_{n-1} \circ B_n$$

and a sequence of points  $z_n$  with  $\rho(0, z_n) \rightarrow \rho(0, z_0)$  such that for

$$G_n = B_1 \circ f_2 \circ f_3 \dots \circ f_n$$

we have

$$G_n(z_n) = z_0, G_n(0) = 0$$

and

$$\rho(0, z_n) \leq (1 + \epsilon_n)^{2n+1}\rho(0, z_0).$$

Each  $G_n$  is holomorphic, maps the unit disk to itself and satisfies  $G_n(0) = 0$  and  $G_n(z_n) = z_0$ . Therefore any accumulation point  $G$  of  $G_n$  is a holomorphic self map of the unit disk that maps an accumulation point of  $z_n$  to  $z_0$ . Thus,  $G(z) = e^{i\theta}z$ . Now to complete the proof of theorem 3, set  $f_1 = f(e^{-i\theta}z)$ . Then the IFS  $F_n$  with  $F_n = f_1 \circ \dots \circ f_n$  has  $f(z) = f_1 \circ G$  as a limit function.

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