Forward Iterated Function Systems

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Abstract

We consider the iterated function system $F_n = f_n \circ \ldots \circ f_2 \circ f_1$ formed from the holomorphic functions in the family $\mathcal{H}(\Delta, \Omega)$. The analog of stable behavior for such systems is that the limit functions be constant. We prove that a necessary and sufficient condition for stable behavior for all iterated function systems formed from $\mathcal{H}(\Delta, \Omega)$ is that Ω be a proper subset of Δ. We also prove that for a given iterated function system the constant limit functions are unique if and only if Ω is relatively compact in Δ.

1 Introduction

Suppose that we are given a random sequence of holomorphic self maps $f_1, f_2, f_3, \ldots$ of the unit disk Δ. We consider the iteration scheme

$$F_n = f_n \circ f_{n-1} \circ \ldots f_2 \circ f_1$$

for this system. This is called the *forward iteration function systems* made from the sequence $f_1, f_2, f_3, \ldots$. There is a theory for "backward iterated function systems" but we do not consider it here (see [1, 8, 9]) so we simply call F_n an iterated function system. By Montel's theorem (see for example [2]), the sequence F_n is a normal family, and every convergent subsequence converges uniformly

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on compact subsets of Δ. The limits of those uniformly convergent subsequences are called accumulation points. Every accumulation point F of F_n is a holomorphic map. Therefore, every such map F is either a nonconstant open self map of the unit disk or a constant map. The constant accumulation points may be located either inside the unit disk or on the boundary of unit disk.

We may look at the iterated function system as a dynamical system acting on Δ. If z is an arbitrary point of Δ, its orbit under the iterated function system, $F_n(z)$ converges to $F(z)$. Hence, if the only limit functions are constants, the orbits of all points tend to periodic cycles. As we will see, we can find conditions so that whether this happens depends only on the subdomain $\Omega \subset \Delta$ and not on the particular system chosen from $\mathcal{H}(\Delta, \Omega)$.

If all maps f_n are the same, the well known Denjoy-Wolff Theorem determines all possible accumulation points.

The Denjoy-Wolff Theorem. Let f be a holomorphic self map of the unit disk Δ that is not a conformal automorphism. Then the iterates f^n of f converge locally uniformly in Δ to a constant value t, where $|t| \leq 1$.

Therefore, the only accumulation points of the system f^n where f is not a biholomorphic isometry of Δ are constants.

Many articles have studied possible generalizations of the Denjoy-Wolff Theorem to general iterated function systems. One result of Lorentzen [10] and Gill [7] is

Theorem(Lorentzen-Gill.) If an iterated function system is formed from functions in $\mathcal{H}(\Delta, K)$ where K is relatively compact in Δ then the system F_n converges locally uniformly in Δ to a unique constant. This constant is, of course, located in the compact set K.

In this paper we will prove the following theorems:

Theorem 1 *Let Ω be a subdomain of the unit disk Δ. Then all accumulation points of any iterated function system of maps in $Hol(\Delta, \Omega)$ are constant functions if and only if $\Omega \neq \Delta$.*

and

Theorem 2 *The accumulation points of any iterated function system of maps in $Hol(\Delta, \Omega)$ are unique if and only if Ω is a relatively compact subdomain of Δ.*

This paper as well as the work in [8] were motivated by the work in [1] which concentrated on backward iterated function systems. The authors there exploit the natural hyperbolic geometry of the domains Δ and Ω and the fact that holomorphic functions are contractions with respect to the hyperbolic metric. In section 2 we will prove the first theorem and in section 3 we will prove the second theorem.

2 Constant Limit Functions

The proof of theorem 1 will use the reasoning developed in [1] which compares the Poincaré distances in Δ and in Ω. First we recall some basic facts.

The Poincaré density of the unit disk Δ is defined as

$$\rho(z) = \frac{1}{1 - |z|^2}$$

for each z in Δ and the Poincaré distance on Δ between two points z and w in Δ is defined by

$$\rho(z, w) = \inf \int_\gamma \rho(t)|dt|$$

where the infimum is over all rectifiable curves γ joining z and w. It is easy to check that for any Möbius transformation $A(z)$ preserving Δ, $\rho(A(z))|A'(z)| = \rho(z)$ so that for any two points z and w in Δ,

$$\rho(z, w) = \rho(A(z), A(w))$$

Every open set Ω in the extended complex plane with at least three boundary points admits Δ as a universal covering surface. The projection of the Poincaré density in Δ defines a Poincaré density ρ_Ω on Ω as follows:

$$\rho_\Omega(z) = \frac{\rho(a)}{|\pi'(a)|}$$

where $a \in \Delta$, $\pi : \Delta \to \Omega$ is the universal covering projection and $\pi(a) = z$. From this formula and the local injectivity of π we deduce that the density ρ_Ω is continuous. The Poincaré distance on Ω between any pair of points $z, w \in \Omega$ is defined by

$$\rho_\Omega(z, w) = \inf \int_\gamma \rho_\Omega(t)|dt|$$

where the infimum is over all rectifiable curves γ joining z and w.

The Schwarz lemma for holomorphic functions defined on Δ says that for any $z \in \Delta$, and any $f \in \mathcal{H}((\Delta, \Delta)$, $\rho(f(z))|f'(z)| \leq \rho(z)$ with equality if and only if f is a Möbius transformation preserving Δ. Using this we can also characterize ρ_Ω as

$$\rho_\Omega(z) = \inf \frac{1}{|f'(0)|}$$

where the infimum is taken over all $f \in \mathcal{H}(\Delta, \Omega)$ such that $f(0) = z$.

Using this characterization we can show that for domains X, Y and any $f \in \mathcal{H}(X, Y)$, then $\rho_Y(fz)|f'z| \leq \rho_X(z)$. Applying this to the identity map with $X \subset Y$ we have $\rho_Y(z) \leq \rho_X(z)$.

Suppose now that Ω is a subset of the unit disk Δ such that the complement of Ω in Δ is nonempty. Let f_n be an arbitrary sequence of holomorphic functions in $\mathcal{H}(\Delta, \Omega)$. The sequence f_n generates the iterated function system $F_n = f_n \circ f_{n-1} \circ \ldots \circ f_1$. We want to show that all accumulation points of F_n are constant functions. Since Ω is a proper subset of Δ, the universal covering map $\pi : \Delta \to \Omega$ is not surjective. Applying the Schwarz lemma to the map π we have $\rho(z) < \rho_\Omega(z)$ for each z in Ω. Thus if N is any relatively compact subset of Ω we can find a constant $k(N) < 1$ such that $\rho(z) \leq k(N)\rho_\Omega(z)$ for all $z \in N$. iTherefore, we have the following lemma.

Lemma A. Let $N = N(a, r)$ be the set of all points in Ω whose $\rho_\Omega-$distance from some point a in Ω is less than r. Let f be any holomorphic map from the unit disk Δ into Ω. If z and w are any two points in Δ such that $f(z)$ and $f(w)$ belong to N, then

$$\rho(f(z), f(w)) \leq C\rho(z, w),$$

where the constant $C = C(N) < 1$ depends only on the neighborhood N.

Lemma A was one of the key ingredients in [1]. For the reader's benefit, we illustrate the proof here. Let $\tilde{N} = N(a, 3r)$. If γ is a geodesic in Ω which joins $f(z)$ and $f(w)$, then it has to stay in $\tilde{N}$. Therefore, we have

$$c(\tilde{N})\rho_\Delta(z,w) \geq c(\tilde{N})\rho_\Omega(fz, fw) = \int_\gamma c(\tilde{N})\rho_\Omega(t)|dt| \geq$$

$$\int_\gamma \rho(t)|dt| \geq \rho(fz, fw).$$

The proof of Theorem 1

Suppose that the sequence F_{n_k} converges locally uniformly on Δ to some holomorphic map F. Since the image of each F_{n_k} is a subset of the closure of Ω, the same holds for the limit F. Suppose that F is non-constant. Then F is an open map, and so there exists a point z_0 in Δ, such that $F(z_0)$ is in Ω. If w_0 is any point in Δ, with $\rho(z_0, w_0) < 1$, then since each f_i is a weak contraction, so is F_{n_k} and we have $\rho_\Omega(F_{n_k}(w_0), F_{n_k}(z_0)) \leq \rho(z_0, w_0) < 1$. Therefore, $F_{n_k}(w_0)$ and $F_{n_k}(z_0)$ both belong to $N = N(F(z_0), 2)$ for all sufficiently large k. Lemma A yields

$$\rho(F_{n_k}(z_0), F_{n_k}(w_0)) \leq C(N)\rho(f_{n_k-1} \circ \ldots f_1(z_0), f_{n_k-1} \circ \ldots f_1(w_0)) \leq$$

$$C(N)\rho(f_{n_k-2} \circ \ldots f_1(z_0), f_{n_k-2} \circ \ldots f_1(w_0)) \leq \ldots \leq$$
$$C(N)\rho(f_{n_{k-1}} \circ \ldots f_1(z_0), f_{n_{k-1}} \circ \ldots f_1(w_0)) \leq$$
$$C(N)^2\rho(f_{n_{k-1}-1} \circ \ldots f_1(z_0), f_{n_{k-1}-1} \circ \ldots f_1(w_0)) \leq \ldots$$

Therefore $\rho(F_{n_k}(z_0), F_{n_k}(w_0)) \to 0$ as $k \to \infty$ and $F(z_0) = F(w_0)$. This shows that F is constant in a neighborhood of z_0, but since F is holomorphic, we conclude that F is a constant map. That is a contradiction and finishes the proof of the theorem.

3 Uniqueness of limits

In this section we study the uniqueness of the accumulation points of an iterated system of functions f_n which map the unit disk to a proper subdomain Ω of the unit disk. We show that the accumulation points are not necessarily unique. By the Lorentzen-Gill theorem, if the subdomain Ω is relatively compact the limits are unique so we assume that Ω is an arbitrary subdomain of Δ, whose boundary intersects the unit circle.

Theorem 3 *Suppose that Ω is any subdomain of the unit disk Δ such that the closure of Ω is not a subset of Δ. Then there exists a sequence f_n of holomorphic mappings from Δ to Ω such that the iterated system $F_n = f_n \circ f_{n-1} \circ \dots f_3 \circ f_2 \circ f_1$ has more than one accumulation point.*

PROOF. Let Ω be any subdomain of the unit disk Δ such that the closure of Ω is not a subset of Δ. We will construct the sequence of maps f_i from a sequence of covering maps π_i of Ω together with Möbius transformations A_i of Δ.

We start with an arbitrary point a in Ω. If we choose a point c in Ω, which is sufficiently close, but not equal to a, then there exists a universal covering map π_1 from Δ to Ω and a point $a_1 \neq a$ in Ω, such that $\pi_1(a) = c$ and $\pi_1(a_1) = a$.

Let h_2 be a covering map from Δ onto Ω such that $h_2(a) = a_1$. Then there exists a point c_2 in Δ such that $h_2(c_2) = a$ and $\rho(a, c_2) = \rho_\Omega(a, a_1)$. We cannot assume, however, that $c_2 \in \Omega$. Since Ω is not relatively compact in Δ, the boundary of the hyperbolic disk (with respect to ρ) with center at a and radius $\rho(a, c_2)$ intersects Ω. Let a_2 be a point that belongs to this intersection, and let A_2 be a Möbius self map of the unit disk that is a rotation around the point a and sends a_2 to c_2. The covering map $\pi_2 = h_2 \circ A_2$ of Δ onto Ω satisfies $\pi_2(a) = a_1$ and $\pi_2(a_2) = a$. Furthermore $\rho(a_2, a) = \rho(c_2, a) = \rho_\Omega(a, a_1)$.

Continuing this process we obtain a sequence of covering maps π_n and a sequence of points a_n in Ω such that

$$\rho(a, a_n) = \rho_\Omega(a_{n-1}, a) \tag{1}$$

$$\pi_n(a) = a_{n-1} \tag{2}$$

and

$$\pi_n(a_n) = a \tag{3}$$

for all n.

We slightly alter the sequence π_n to obtain the desired sequence f_n. We let

$$f_{2n-1} = \pi_{2n}$$

and

$$f_{2n} = \pi_{2n-1}$$

for all n. If $F_n = f_n \circ f_{n-1} \circ \ldots f_3 \circ f_2 \circ f_1$ is the iterated function system made of covering maps $f_1, f_2, f_3, \ldots$, then the equalities (2) and (3) imply $F_{2n}(a) = a$ and $F_{2n-1}(a) = \pi_{2n}(a) = a_{2n-1}$. The equalities (1), (2) and (3) together with the fact that inclusion maps are contractions, yield

$$\rho_\Omega(a, a_1) = \rho(a, a_2) \leq \rho_\Omega(a, a_2) = \rho(a, a_3) \leq \ldots \leq \rho_\Omega(a, a_{2n-2}) = \rho(a, a_{2n-1}).$$

Therefore $\rho(a, a_{2n-1}) \geq \rho(a, a_1) > 0$. This implies that F_{2n} and F_{2n+1} have different accumulation points. $\square$

We remark that the same construction gives us non-uniqueness of limit points for backwards iterated function systems as well. Set $G_n = \pi_1 \circ \ldots \circ \pi_n$ and study the even and the odd subsequences of G_n. The equalities (2) and (3) imply that the even subsequence G_{2n} satisfies $G_{2n}(a) = a$, and the odd subsequence G_{2n+1} satisfies $G_{2n+1}(a) = c \neq a$. Therefore these two subsequences have different accumulation points.

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