

Pleating invariants for punctured torus groups

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Abstract

In this paper we give a complete description of the space \mathcal{QF} of quasifuchsian punctured torus groups in terms of what we call *pleating invariants*. These are natural invariants of the boundary $\partial\mathcal{C}$ of the convex core of the associated hyperbolic 3-manifold M and give coordinates for the non-Fuchsian groups $\mathcal{QF} - \mathcal{F}$. The pleating invariants of a component of $\partial\mathcal{C}$ consist of the projective class of its bending measure, together with the lamination length of a fixed choice of transverse measure in this class. Our description complements that of Minsky in [35], in which he describes the space of all punctured torus groups in terms of *ending invariants* which characterize the asymptotic geometry of the ends of M .

Pleating invariants give a quasifuchsian analog of the Kerckhoff-Thurston description of Fuchsian space by critical lines and earthquake horocycles. The critical lines extend to *pleating planes* on which the pleating loci of $\partial\mathcal{C}$ are constant and the horocycles extend to *BM-slices* on which the pleating invariants of one component of $\partial\mathcal{C}$ are fixed.

We prove that the pleating planes corresponding to rational laminations are dense and that their boundaries can be found *explicitly*. This means, answering questions posed by Bers in the late 1960's, that it is possible to compute an arbitrarily accurate picture of the shape of any embedding of \mathcal{QF} into \mathbf{C}^2 .

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1 Introduction

In his recent paper [35], Minsky gave a full description of the space of punctured torus groups in terms of their *ending invariants*. These invariants are the conformal structures of the quotient surfaces of the regular set of the group acting on the Riemann sphere, or, if a component is degenerate, the corresponding *ending lamination* of Thurston.

In this paper we give an alternative description of quasifuchsian space \mathcal{QF} in terms of what we call *pleating invariants*. These replace conformal structures at infinity by natural invariants of the geometry of the boundary of the convex core of the associated three manifold. These invariants again extend naturally to ending laminations for groups on the boundary of \mathcal{QF} . Pleating invariants have considerable computational advantages: we show how they can be used to explicitly locate the group with given invariants, and to compute the shape and boundary of \mathcal{QF} , for any embedding into \mathbf{C}^2 .

A *punctured torus group* $\langle G; A, B \rangle$ is a free marked two generator discrete subgroup of $PSL(2, \mathbf{C})$ such that the commutator of the generators is parabolic. Such a group is the image of a faithful representation ρ of the fundamental group of a punctured torus \mathcal{T}_1 with presentation $\pi_1(\mathcal{T}_1) = \langle \alpha, \beta \rangle$; the commutator of the generators represents a loop around the puncture and the ordered pair $(A, B) = (\rho(\alpha), \rho(\beta))$ is the marking. The group G acts as a discrete group of isometries of hyperbolic space \mathbf{H}^3 and the quotient hyperbolic manifold $M = \mathbf{H}^3 / G$ is a product $\mathcal{T}_1 \times (-1, 1)$.

A punctured torus group also acts as a group of conformal automorphisms of the Riemann sphere $\hat{\mathbf{C}}$ and partitions it into two invariant subsets, the open (possibly empty) regular set Ω and the closed limit set Λ . The group G is *quasifuchsian* if Ω consists of two non-empty simply connected invariant components denoted Ω^\pm . The quotients Ω^\pm / G are punctured tori with conformal structures inherited from $\hat{\mathbf{C}}$.

Quasifuchsian space \mathcal{QF} is the space of quasifuchsian marked punctured torus groups modulo conjugation in $PSL(2, \mathbf{C})$; *Fuchsian space* \mathcal{F} is the subset such that the components Ω^\pm are round disks.

The convex hull \mathcal{C} of Λ in \mathbf{H}^3 is also invariant under G . The hyperbolic manifold \mathcal{C}/G is called the *convex core* of G . If G is quasifuchsian, but not Fuchsian, $\partial\mathcal{C}/G$ consists of two components, $\partial\mathcal{C}^\pm/G$. Each component is homeomorphic to \mathcal{T}_1 and admits an intrinsic hyperbolic structure making it a *pleated surface* in the sense of Thurston. Such a surface is a hyperbolic surface “bent” along a geodesic lamination called the *pleating locus* or *bending lamination*. The pleating locus carries a natural transverse measure, the bending measure

$pl^\pm(G)$.

For any measured geodesic lamination μ on a hyperbolic surface Σ , we denote the projective class of μ by $[\mu]$ and the underlying lamination by $|\mu|$. Writing l_μ for the lamination length of μ , we note that if μ, μ' are in the same projective class, so that $\mu = c\mu', c > 0$, then $l_\mu = cl_{\mu'}$. We define the *pleating invariants* for $G \in \mathcal{QF} - \mathcal{F}$ to be the projective class of the pair (μ^\pm, l_{μ^\pm}) for any choice of measured laminations μ^\pm in $[pl^\pm]$.

We prove

Theorem 1 *A non-Fuchsian quasifuchsian marked punctured torus group is determined by its pleating invariants, uniquely up to conjugacy in $PSL(2, \mathbf{C})$.*

The essential idea is to study the sets in \mathcal{QF} on which some or all of the pleating invariants are constant; in particular, we study the set $\mathcal{P}_{\mu, \nu} \subset \mathcal{QF}$ for which $[pl^+] = [\mu], [pl^-] = [\nu]$. Clearly $\mathcal{P}_{\mu, \nu}$ depends only on the projective classes $[\mu], [\nu]$ of μ, ν . We prove that these sets are connected real two dimensional submanifolds of \mathcal{QF} whose boundaries meet \mathcal{F} and $\partial\mathcal{QF}$ in specific analytic curves; as the projective classes vary, the sets $\mathcal{P}_{\mu, \nu}$, which for obvious reasons we call pleating planes, foliate $\mathcal{QF} - \mathcal{F}$. We are also able to describe exactly how the closure of $\mathcal{P}_{\mu, \nu}$ meets \mathcal{F} .

The space \mathcal{QF} has a natural \mathbf{C}^2 -holomorphic structure induced from $PSL(2, \mathbf{C})$. Let $U \subset \mathcal{QF}$. An \mathbf{R}^2 -locus in U is a set $f^{-1}(\mathbf{R}^2) \cap U$ where $f : U \rightarrow \mathbf{C}^2$ is a non-constant holomorphic function defined on U . A *singularity* is a point where $Det(Jac_f(z)) = 0$. For example, Fuchsian space is an \mathbf{R}^2 -locus in \mathcal{QF} , (see section 7.1).

The starting point for our analysis of $\mathcal{P}_{\mu, \nu}$ is to prove that for $\mu \in ML$, the length function l_μ on \mathcal{F} extends to a holomorphic function λ_μ , called the *complex length* of μ , on \mathcal{QF} , and that λ_μ is real valued at points where the projective class of pl^\pm is $[\mu]$. Thus $\mathcal{P}_{\mu, \nu}$ is contained in the \mathbf{R}^2 -locus of the holomorphic function $L_{\mu, \nu} = \lambda_\mu \times \lambda_\nu$ from \mathcal{QF} to \mathbf{C}^2 .

To describe $\mathcal{P}_{\mu, \nu}$ more precisely, we recall some facts about Fuchsian space \mathcal{F} . Let μ be a measured geodesic lamination on a hyperbolic surface Σ . The distance t earthquake $\mathcal{E}_\mu(t)$ along μ gives a one parameter family of deformations of \mathcal{F} which generalize Fenchel-Nielsen twists along simple closed geodesics. For a point $p \in \mathcal{F}$, we denote the earthquake path $\{\mathcal{E}_\mu(t)(p) : t \in \mathbf{R}\}$ through p by \mathcal{E}_μ^p . The earthquake path is contained in \mathcal{F} and meets $\partial\mathcal{F}$, the Thurston boundary of \mathcal{F} , in the point $[\mu]$. Kerckhoff proved that for each measured lamination ν whose intersection $i(\mu, \nu)$ with μ is non-zero, the length function l_ν has a unique minimum along \mathcal{E}_μ^p .

In the special case of the punctured torus, it is an easy consequence of Kerckhoff's results that for each $c > 0$, there is a unique earthquake path $\mathcal{E}_{\mu,c}$ on which $l_\mu \equiv c$. We denote the point at which l_ν is minimal on this path by $p_{\mu,\nu,c}$, and set $f_{\mu,\nu}(c) = l_\nu(p_{\mu,\nu,c})$. For fixed μ, ν and variable c , the points $p_{\mu,\nu,c}$ define an analytic path $\mathcal{F}_{\mu,\nu}$, which we call a *critical line*; it meets $\partial\mathcal{F}$ in the points $[\mu], [\nu]$. The length functions l_μ, l_ν are monotonic on $\mathcal{F}_{\mu,\nu}$ and $f_{\mu,\nu}(c)$ is continuous, decreasing monotonically from ∞ to 0 on its domain $(0, \infty)$.

The following result completely describes the pleating planes $\mathcal{P}_{\mu,\nu}$; in particular it shows that $\mathcal{P}_{\mu,\nu}$ can be viewed as an extension into \mathcal{QF} of the critical line $\mathcal{F}_{\mu,\nu}$.

Theorem 2 *Let (μ, ν) be measured laminations on \mathcal{T}_1 with $i(\mu, \nu) > 0$. Then $\mathcal{P}_{\mu,\nu}$ is a non-empty connected non-singular component of the \mathbf{R}^2 -locus in $\mathcal{QF} - \mathcal{F}$ of the function $L_{\mu,\nu}$. The restriction of $L_{\mu,\nu}$ to $\mathcal{P}_{\mu,\nu}$ is a diffeomorphism to the open region under the graph of the function $f_{\mu,\nu}$ in $\mathbf{R}^+ \times \mathbf{R}^+$.*

The boundary of the closure of $\mathcal{P}_{\mu,\nu}$ in \mathcal{QF} is the critical line $\mathcal{F}_{\mu,\nu} \subset \mathcal{F}$; it is mapped homeomorphically by $L_{\mu,\nu}$ to the graph of $f_{\mu,\nu}$. The planes $\mathcal{P}_{\nu,\mu}$ and $\mathcal{P}_{\mu,\nu}$ are disjoint with common boundary $\mathcal{F}_{\mu,\nu}$ in \mathcal{QF} . The set $\mathcal{P}_{\mu,\nu} \cup \mathcal{P}_{\nu,\mu} \cup \mathcal{F}$ is an \mathbf{R}^2 -locus in \mathcal{QF} and the union $\mathcal{P}_{\mu,\nu} \cup \mathcal{P}_{\nu,\mu} \cup \mathcal{F}_{\mu,\nu}$ may be regarded as the extension of the μ, ν critical line to \mathcal{QF} .

The three components of the boundary of the image of $\mathcal{P}_{\mu,\nu}$ in $\mathbf{R}^+ \times \mathbf{R}^+$ correspond to three distinct parts of its closure in the set of algebraic limits of groups in \mathcal{QF} . As above, the component corresponding to the graph of $f_{\mu,\nu}$ represents groups on the critical line $\mathcal{F}_{\mu,\nu} \subset \mathcal{F}$. For limit groups corresponding to the axis $\lambda_\mu = 0$ the component Ω^+ has degenerated and the support $|\mu|$ of μ is an ending lamination; the bending measure of $\partial\mathcal{C}^-$, however, is still in the projective class of ν . Likewise, for limit groups corresponding to the axis $\lambda_\nu = 0$, the component Ω^- has degenerated and the ending lamination is $|\nu|$. The boundary point $(0, 0)$ represents a doubly degenerate group, unique by the results of [35] (or [17] in the rational case), with the two ending laminations $|\mu|$ and $|\nu|$.

Theorems 1 and 2 together show that we have a nice coordinate system on $\mathcal{QF} - \mathcal{F}$: theorem 1 shows that the map to pleating invariants is injective and theorem 2 describes the image.

The measured lamination μ is called *rational* if its support is a simple closed geodesic. Such a geodesic can only belong to the pleating locus $|pl^\pm|$ if its representatives $V \in G$ are purely hyperbolic and hence have real trace. Given any embedding \mathcal{QF} into \mathbf{C}^2 , the generators of G are holomorphic functions of the embedding parameters and $\text{Tr } V$ is a polynomial in the entries of the generators. In particular, given any elements $V, W \in G$ representing distinct

simple closed curves $|\mu|, |\nu|$ on \mathcal{T}_1 , one can compute the position of the critical line $\mathcal{F}_{\mu,\nu}$. If both laminations μ, ν are rational, we call $\mathcal{P}_{\mu,\nu}$ a *rational pleating plane*. Theorem 2 implies

Theorem 3 *Let μ, ν be rational laminations represented by non-conjugate elements $V, W \in G$. Then $\mathcal{P}_{\mu,\nu}$ and $\mathcal{P}_{\nu,\mu}$ are the unique components of the \mathbf{R}^2 -locus of the function $\text{Tr } V \times \text{Tr } W$ in $\mathcal{QF} - \mathcal{F}$ whose closures meet \mathcal{F} in $\mathcal{F}_{\mu,\nu}$. On $\mathcal{P}_{\mu,\nu} \cup \mathcal{P}_{\nu,\mu}$ the function $\text{Tr } V \times \text{Tr } W$ is non-singular and the boundary of $\mathcal{P}_{\mu,\nu} \cup \mathcal{P}_{\nu,\mu}$ can be computed by solving $\text{Tr } V = \pm 2$ and $\text{Tr } W = \pm 2$ on this component.*

We also prove

Theorem 4 *The rational pleating planes are dense in \mathcal{QF} .*

In the late 1960's, Bers asked whether it was possible to find the shape of quasifuchsian space by explicit computation; one would expect the punctured torus to be the easiest case. Partial results were obtained by a number of people, some using computational methods, among them [16,41,47], others developing new tools and techniques [13,30]. For the punctured torus, the above results give an effective means of finding the boundary of the image of any chosen embedding of \mathcal{QF} into \mathbf{C}^2 , answering Bers' question in full.

We also study the way in which the pleating planes fit together transversally to the real locus of $L_{\mu,\nu}$. This is done by fixing the pleating invariants of one side of $\partial\mathcal{C}$; one can regard this as analogous to fixing the ending invariant on one side in \mathcal{QF} , to obtain the classical *Bers slice* [1]. Thus for a fixed measured lamination μ and $c > 0$, we define the *BM-slice* $BM_{\mu,c}^+$ as the subset of \mathcal{QF} on which $[pl^+] = [\mu]$ and $\lambda_\mu = c$. The BM-slices are subsets of the *quakebend planes* $\mathcal{Q}_{\mu,c}$ obtained by Thurston's quakebend construction along the measured lamination μ (see [9] and section 7 below). These are extensions of the earthquake path $\mathcal{E}_{\mu,c}$ into \mathcal{QF} . Unlike the path $\mathcal{E}_{\mu,c}$ which is completely contained in \mathcal{F} , the quakebend plane $\mathcal{Q}_{\mu,c}$ is not totally contained in \mathcal{QF} . We prove

Theorem 5 *Let μ be a measured lamination on \mathcal{T}_1 and let $c > 0$. Then the closures in \mathcal{QF} of exactly two of the connected components of $\mathcal{Q}_{\mu,c} \cap (\mathcal{QF} - \mathcal{F})$ meet \mathcal{F} . These components are the slices $BM_{\mu,c}^\pm$ and the closure of each slice meets \mathcal{F} precisely in the earthquake path $\mathcal{E}_{\mu,c}$. Furthermore, each slice is simply connected and retracts onto $\mathcal{E}_{\mu,c}$.*

Thus, just like the Bers slices, the *BM-slices* are complex planes in \mathcal{QF} and like them, they foliate $\mathcal{QF} - \mathcal{F}$. We note that while the boundary of the pleating planes consists of smooth curves, the boundary of a *BM-slice* is typically a fractal-like curve. Pictures of such curves may be found in [18,38,47].

The basis of the proofs of the above results are two important theorems which control the local behavior of pleating invariants. We call these the *limit pleating theorem* and *local pleating theorem* respectively. Roughly, the limit pleating theorem states that if the pleating invariants of a sequence of groups in \mathcal{QF} converge, then the groups converge to an algebraic limit; furthermore the limit group is in \mathcal{QF} provided the limit pleating lengths are non-zero. It is closely related to Thurston's double limit theorem [44], and also to the 'Lemme de fermeture' in [5].

The local pleating theorem makes essential use of the complex length function λ_μ . As mentioned above, if $q \in \mathcal{QF}$, then $\lambda_{pl^+}(q) \in \mathbf{R}$. In general, the converse of this result is false; however the local pleating theorem gives a partial result: if $q \in \mathcal{P}_\mu$ so that $\lambda_\mu(q) \in \mathbf{R}$, then for q' near q , the condition $\lambda_\mu(q') \in \mathbf{R}$ implies that $q' \in \mathcal{P}_\mu$. (As discussed in the introduction of [20] this result does not hold for higher genus.)

The theory of quakebends as developed in [9] allows us to extend the earthquake paths $\mathcal{E}_{\mu,c}$ into a family of holomorphic planes $\mathcal{Q}_{\mu,c}$ in \mathcal{QF} . We reduce the problem of studying the sets $\mathcal{P}_{\mu,\nu}$ by restricting to the subset $\mathcal{P}_{\mu,\nu,c}$ of $\mathcal{P}_{\mu,\nu}$ on which the value of λ_μ is fixed at $c \in \mathbf{R}^+$. For reasons that will be clear below, we call such a set a *pleating ray*. In $\mathcal{Q}_{\mu,c}$, the complex length λ_ν restricts to a holomorphic function of one variable and it follows from the limit and local pleating theorems that $\mathcal{P}_{\mu,\nu,c}$ is both open and closed in the \mathbf{R} -locus of λ_ν in $\mathcal{Q}_{\mu,c} \cap \mathcal{QF}$.

The fact that the pleating rays are non-empty and the discussion of how they meet Fuchsian space \mathcal{F} results from the detailed study of the situation near \mathcal{F} which was carried out in [20]. We also have detailed information from [39] about rays for which the laminations μ, ν are rational and correspond to a pair of generators of \mathcal{T}_1 . Combining this information allows us to prove

Theorem 6 *Let μ, ν be measured laminations on \mathcal{T}_1 with $i(\mu, \nu) > 0$ and let $c > 0$. Then the set $\mathcal{P}_{\mu,\nu,c} \subset \mathcal{QF}$ on which $[pl^+] = [\mu]$, $[pl^-] = [\nu]$ and $l_\mu = c$, is a non-empty connected non-singular component of the \mathbf{R} -locus of the restriction of λ_ν to $\mathcal{Q}_{\mu,c}$. This restriction is a diffeomorphism onto its image $(0, f_{\mu,\nu}(c)) \subset \mathbf{R}^+$.*

Theorem 2, and hence also theorem 1, are immediate consequences of this result. We also easily deduce theorem 5.

For groups on the boundary of \mathcal{QF} , at least one of the components Ω^\pm degenerates and it is clear that our pleating invariants extend naturally to the corresponding ending laminations for which the length (and also the complex length) is always 0. It is also clear that these invariants should also characterize boundary groups; careful analysis requires the study of generalized Maskit

slices in which the fixed ending lamination is irrational, see [31].

The reader is referred to [35] for a good outline of the history relating to the study of punctured torus groups.

Some of the ideas of this paper, in particular the relation of pleating planes to the Kerckhoff picture of \mathcal{F} and the idea of looking at the BM -slices, grew out of discussions with John Parker, and we should like to thank him for his input into this work. We should also like to thank our referees for their detailed reading of earlier versions of this paper, in particular, for having signalled, in view of the examples in [24], a gap in our proof of theorem 15, as well as having suggested a more direct proof of lemma 41 and a simplification of the proof of theorem 23. We would also like to thank Yair Minsky for conversations which helped us precisely locate the above mentioned gap, and Francis Bonahon and Cyril Lecuire for very useful discussions about how to rectify it.

The paper is organized as follows. Section 2 contains background on the punctured torus, geodesic laminations and surfaces. Section 3 explains the picture of earthquake paths and critical lines in \mathcal{F} and in section 4 we review results on pleated surfaces and the convex hull boundary. We prove the limit pleating theorem in section 5. In section 6 we show how to complexify the length functions and show that the complex length of the pleating locus is real. In section 7 we review results about quakebends and the convex hull boundary and then in section 8 prove the local pleating theorem. We also derive various important consequences of this result, including the proof of theorem 4. In section 9 we prove our main results, theorem 6 on pleating rays and theorem 2 on pleating planes. In section 10 we study BM -slices, proving theorem 5, and we conclude in section 11 with a discussion of rational pleating planes, computation, and some explicit examples. For readability, the proofs of three technical results are deferred to the appendix.

2 Background

2.1 Punctured torus groups and markings

Let \mathcal{T}_1 be a torus with one puncture and a fixed orientation. Any pair of simple closed loops on \mathcal{T}_1 that intersect exactly once are free generators of $\pi_1(\mathcal{T}_1)$. Let (α, β) be such an ordered pair of free generators, chosen so that their commutator $\alpha\beta\alpha^{-1}\beta^{-1}$ represents a loop around the puncture that is positively oriented around the component of \mathcal{T}_1 not containing the puncture. The ordered pair (α, β) is called a *marking*.

A *punctured torus group* is a discrete subgroup $G \subset PSL(2, \mathbf{C})$ that is the image of a faithful representation ρ of $\pi_1(\mathcal{T}_1)$ such that the image of the loop around the puncture is parabolic. If (α, β) is a marking of \mathcal{T}_1 , and if $A = \rho(\alpha), B = \rho(\beta)$, then the commutator $K = ABA^{-1}B^{-1}$ is parabolic and the ordered pair $(A, B) = (\rho(\alpha), \rho(\beta))$ is called a *marking* of G . If γ is any simple closed curve on \mathcal{T}_1 , then we can always choose a curve δ such that (γ, δ) is a marking of \mathcal{T}_1 . Setting $\rho(\gamma) = V, \rho(\delta) = W$, then all possible markings $(V, W'), W' \in G$ of G are of the form $(V, V^m W), m \in \mathbf{Z}$.

The group G is *quasifuchsian* if the regular set Ω consists of two non-empty simply connected invariant components Ω^\pm . The limit set $\Lambda(G)$ is topologically a circle. *Quasifuchsian space* \mathcal{QF} is the space of marked quasifuchsian punctured torus groups modulo conjugation in $PSL(2, \mathbf{C})$; it has a holomorphic structure induced from the natural holomorphic structure of $SL(2, \mathbf{C})$. *Fuchsian space* \mathcal{F} is the subset such that the components Ω^\pm are round disks. It is canonically isomorphic to the Teichmüller space of marked conformal structures on \mathcal{T}_1 .

The quotients Ω^\pm/G are punctured tori with conformal structures, and hence also orientations, inherited from $\hat{\mathbf{C}}$; the orientations of Ω^+/G and \mathcal{T}_1 agree whereas those of Ω^-/G and \mathcal{T}_1 are opposite. This means $\Omega^+(G)$ is the component such that A^-, B^+, A^+, B^- occur in counterclockwise order around its boundary $\Lambda(G)$, where, for a loxodromic $g \in SL(2, \mathbf{C})$, g^+ and g^- denote its attracting and repelling fixed points respectively. Thus an alternative way to choose a marking of G is to choose *any* pair of generators X, Y of G , and to specify the choice of Ω^+ by choosing it to be the component such that the fixed points X^-, Y^+, X^+, Y^- run counterclockwise around its boundary.

A point $q \in \mathcal{QF}$ represents an equivalence class of marked groups in $PSL(2, \mathbf{C})$. We choose once and for all a triple of distinct points in $\hat{\mathbf{C}}$ and let $G = G(q)$ denote the representative normalized by choosing A^-, A^+, K^∞ to be this fixed triple, where K^∞ is the fixed point of the parabolic K . We will refer to this as the *standard normalization*. If it is clear from the context, for readability, we suppress the dependence on q .

Note that throughout this paper, \mathcal{QF} and \mathcal{F} refer to the special case of the once punctured torus \mathcal{T}_1 only.

2.2 Laminations

Let Σ be a hyperbolic surface. We denote by \mathcal{S} the set of all simple closed geodesics on Σ . There is one such geodesic in each free homotopy class of simple closed non-boundary parallel loops, and the set \mathcal{S} is independent of the hyperbolic structure on Σ .

Geodesic laminations were introduced by Thurston [45] as a generalization of simple closed geodesics. A geodesic lamination on Σ is a closed set that is a union of pairwise disjoint simple geodesics called its *leaves*. We denote by $GL = GL(\Sigma)$ the set of all geodesic laminations on Σ ; $GL(\Sigma)$ is also independent of the hyperbolic structure, see e.g. [7] section 4.1.4 and [19] section 3.7.

The Hausdorff topology on the set of closed subsets of Σ induces a topology on GL . Two laminations are close in this topology if any long segment of a leaf of either one is closely approximated by a long segment of a leaf of the other. See [7,9,40] for a complete discussion.

A *measured lamination* μ on Σ is a geodesic lamination, called the *support* of μ and denoted $|\mu|$, together with a transverse measure, also denoted μ . We denote the set of all measured laminations on Σ by $ML(\Sigma)$. The space ML is topologized by defining laminations to be close in ML if the measures they assign to any finite set of transversals are close, for details see [7] or [19]. Notice that the support of any measured lamination always avoids a definite neighborhood of each cusp. The relationship between the topologies on ML and GL is discussed in section 2.3 below.

Any element $\gamma \in \mathcal{S}$ carries a natural transverse measure δ_γ which assigns unit mass to each intersection with γ . We call a measured geodesic lamination on Σ *rational* if its support is a union of curves in \mathcal{S} . The maximum number of disjoint loops in \mathcal{S} on the punctured torus \mathcal{T}_1 is one, so that rational measured laminations are of the form $\mu = k\delta_\gamma, k > 0$. We denote the set of all rational measured laminations on Σ by $ML_Q(\Sigma)$; the set ML_Q is dense in ML .

Two measured laminations $\mu, \mu' \in ML$ are *projectively equivalent* if $|\mu| = |\mu'|$ and if there exists $k > 0$ such that for any arc σ transverse to the leaves of $|\mu|$, $\mu'(\sigma) = k\mu(\sigma)$. We write $[\mu]$ for the projective class of $\mu \in ML(\Sigma)$. We denote the set of projective equivalence classes on Σ by $PML(\Sigma)$. It is well known that $PML(\mathcal{T}_1)$ is homeomorphic to $S^1 \simeq \mathbf{R} \cup \{\infty\}$ (see for example [45]).

The length l_γ of a geodesic $\gamma \in \mathcal{S}$ generalizes to arbitrary laminations. Let ϕ represent a hyperbolic structure on Σ . For $\mu \in ML$, the length $l_\mu(\phi)$ is the total mass, on the surface with structure ϕ , of the measure that is the product of hyperbolic distance along the leaves of μ with the transverse measure μ . In particular, if $\mu \in ML_Q(\Sigma)$ with $\mu = \delta_\gamma$, then $l_\gamma = \int_\Sigma d\delta_\gamma ds$ is just the hyperbolic length of γ .

Clearly, if $\mu' = k\mu$ then $l_{k\mu} = kl_\mu$. We define

$$[\mu, l_\mu] \stackrel{\text{def}}{=} \{k\mu, kl_\mu \in ML \times \mathbf{R}^+ : k > 0\}$$

and call it the *projective class* of the pair (μ, l_μ) .

The geometric intersection number $i(\gamma, \gamma')$ of two geodesics $\gamma, \gamma' \in \mathcal{S}$ extends to a continuous function $i(\mu, \nu)$ on $ML(\Sigma)$ (see for example [21]). For $\Sigma = \mathcal{T}_1$, $i(\mu, \nu) > 0$ is equivalent to $[\mu] \neq [\nu]$. We also recall the well known fact that on \mathcal{T}_1 , measured laminations are uniquely ergodic; that is, if $\mu, \mu' \in ML(\mathcal{T}_1)$ with $|\mu| = |\mu'|$, then $[\mu] = [\mu']$.

2.3 The convergence lemma

In general, laminations which are close in ML may not be close in the Hausdorff topology on GL . For example, one can put a transverse measure ν' on a long closed geodesic γ' spiralling in to a closed geodesic γ with transverse measure ν , such that ν, ν' are close in ML but γ' has arcs far from γ . A sequence of laminations may converge in ML to a measured lamination ν_0 with support in one part of Σ , while simultaneously limiting on a closed curve with support disjoint from $|\nu_0|$.

The following lemma gives conditions under which Hausdorff convergence is a consequence of convergence in ML . We note that the lemma depends crucially on the fact that on \mathcal{T}_1 , any irrational measured lamination is maximal. As stated, it is false for more general surfaces, and it is false if $\nu_0 \in ML_Q(\mathcal{T}_1)$.

Lemma 1 *Suppose that $\nu_0 \in ML(\mathcal{T}_1) - ML_Q(\mathcal{T}_1)$, and that ν and ν_0 are close in $ML(\mathcal{T}_1)$. Then $|\nu|$ and $|\nu_0|$ are close in the Hausdorff topology on $GL(\mathcal{T}_1)$.*

This lemma is proved in appendix 12.1.

From now on, unless specifically stated, GL, ML, PML will always refer to \mathcal{T}_1 .

3 Fuchsian space

Kerckhoff and Thurston used earthquake deformations to study the set of hyperbolic structures on a surface Σ . For \mathcal{T}_1 the description is especially simple. For an unpunctured torus, the Teichmüller space is a disk. Thinking of this disk as the hyperbolic plane \mathbf{D} with boundary circle S^1 , for each boundary point ξ there is a foliation of \mathbf{D} by horocycles tangent to $\partial\mathbf{D}$ at ξ . Joining each pair of distinct boundary points ξ, η is a unique geodesic $\gamma_{\xi, \eta}$ which, for fixed ξ and varying η , give another foliation of \mathbf{D} . It follows from Kerckhoff's results [21,23] and Thurston's compactification of Teichmüller space [10] (see also [12]), that there is an analogous picture for \mathcal{F} , the Teichmüller space of

\mathcal{T}_1 . This picture is certainly well known and described for Teichmüller spaces of compact surfaces in [23]. As it is of central importance for us we explain it in detail here.

Since the torus is homogeneous, \mathcal{F} is holomorphically the same as the Teichmüller space of the unpunctured torus, namely \mathbf{D} . The Thurston boundary of \mathcal{F} is naturally identified with the circle S^1 . The classical Fenchel-Nielsen coordinates for \mathcal{F} are the length l_α of a generating curve α and a corresponding twist parameter t_α . In [21], [22], the Fenchel-Nielsen deformation defined by varying the twist parameter t_α is generalized to a map $\mathcal{E}_\mu(t) : \mathcal{F} \rightarrow \mathcal{F}$ defined relative to a measured lamination $\mu \in ML$. The map $\mathcal{E}_\mu(t)$ is called the time t earthquake along μ ; when needed for clarity we write the parameter t as t_μ . The family $\mathcal{E}_\mu(t), t \in \mathbf{R}$ is a one parameter family of deformations of \mathcal{F} ; in particular $\mathcal{E}_\mu(0) = \text{id}$.

For $p \in \mathcal{F}$, we define the *earthquake path* along μ through p by

$$\mathcal{E}_\mu^p = \{\mathcal{E}_\mu(t)(p) \in \mathcal{F} : t \in \mathbf{R}\}.$$

Clearly, \mathcal{E}_μ^p is invariant under the earthquakes $\mathcal{E}_\mu(t)$. In [22], Kerckhoff showed that \mathcal{E}_μ^p is a real analytic path in \mathcal{F} . Along \mathcal{E}_μ^p , the length l_μ is constant. Thus for every $p \in \mathcal{F}$, $\mathcal{E}_\mu(t)(p)$ tends to the same point $[\mu] \in \partial\mathcal{F}$ as $t \rightarrow \pm\infty$.

In [21], Kerckhoff showed that if $\nu \in ML$ with $i(\mu, \nu) > 0$, then along an earthquake path \mathcal{E}_μ^p , the length l_ν is a strictly convex real analytic function of t and $l_\nu(t) \rightarrow \infty$ as $t \rightarrow \pm\infty$. Thus l_ν has a unique minimum on \mathcal{E}_μ^p ; at this point we say that l_ν is minimal with respect to \mathcal{E}_μ^p . Wolpert showed in addition, that at the minimum, $d^2l_\nu/dt_\mu^2 > 0$. (Actually Wolpert proved this only when μ, ν are rational; this case follows by inspection of his second derivative formula [46] Theorem 3.4, in which all terms are clearly positive. The case μ general but ν rational follows by inspection of [9] Theorem 3.10.1. For general ν , see Corollary 21 below.)

It follows from the anti-symmetry of the derivative formula

$$dl_\nu/dt_\mu = \int_{\mathcal{T}_1} \cos \theta d\mu d\nu,$$

(where θ is the angle, measured counterclockwise, from a leaf of $|\mu|$ to a leaf of $|\nu|$ at each intersection point of the laminations $|\mu|, |\nu|$), that the minimum points for l_ν along \mathcal{E}_μ and l_μ along \mathcal{E}_ν coincide, and that at this minimum point p we have $D\mathcal{E}_\nu(t_\nu)(p) = -D\mathcal{E}_\mu(t_\mu)(p)$.

The results which follow are simple consequences of Kerckhoff's results applied to \mathcal{T}_1 .

Proposition 2 *For any $c \in \mathbf{R}^+$ and $\mu \in ML$, there is at most one earthquake*

path \mathcal{E}_μ^p along which $l_\mu = c$.

Proof: Suppose that there are two such paths, $\mathcal{E}_1, \mathcal{E}_2$. They are clearly disjoint, moreover since $\mathcal{F} \cup \partial\mathcal{F}$ is a closed disk and both \mathcal{E}_1 and \mathcal{E}_2 meet $\partial\mathcal{F}$ at the same point $[\mu]$, one path, \mathcal{E}_1 say, separates $\mathcal{F} \cup \partial\mathcal{F}$ so that one component of the complement contains both \mathcal{E}_2 and $\partial\mathcal{F} - \{[\mu]\}$. Choose $\nu \in ML$ with $i(\mu, \nu) > 0$ and let p be the minimum point for l_ν on \mathcal{E}_1 . Then \mathcal{E}_2 separates \mathcal{E}_1 from $\partial\mathcal{F}$ and hence p from $[\nu]$, so that \mathcal{E}_ν^p must also cut \mathcal{E}_2 at a point p' . Since p is the unique minimum point for l_μ on \mathcal{E}_ν^p , and since $l_\mu(p) = l_\mu(p')$ we have a contradiction. \square

We denote the unique earthquake path on which $l_\mu = c$ by $\mathcal{E}_{\mu,c}$. It follows easily from proposition 3 below that $\mathcal{E}_{\mu,c} \neq \emptyset$. Since for $s > 0$, $\mathcal{E}_{s\mu}(t) = \mathcal{E}_\mu(st)$ and $l_{s\mu} = sl_\mu$, we have $\mathcal{E}_{s\mu,sc} = \mathcal{E}_{\mu,c}$. For $\nu \notin [\mu]$, we denote the minimum point for l_ν on $\mathcal{E}_{\mu,c}$ by $p(\mu, \nu, c)$. We define a function $f_{\mu,\nu} : \mathbf{R}^+ \rightarrow \mathbf{R}^+$ by $f_{\mu,\nu}(c) = l_\nu(p(\mu, \nu, c))$. Notice that from the definition, $f_{\mu,\nu} = f_{\nu,\mu}^{-1}$.

For each pair $\mu, \nu \in ML \times ML$, $\mu \notin [\nu]$, set

$$\mathcal{F}_{\mu,\nu} = \{p \in \mathcal{F} \mid dl_\nu/dt_\mu(p) = 0\}$$

Note that $\mathcal{F}_{\mu,\nu}$ depends only on $[\mu], [\nu]$, and that by the antisymmetry of the derivative, $\mathcal{F}_{\mu,\nu} = \mathcal{F}_{\nu,\mu}$. We call $\mathcal{F}_{\mu,\nu}$ the μ, ν -critical line. This is justified by the following proposition.

Proposition 3 *For each pair μ, ν , $i(\mu, \nu) > 0$, the locus $\mathcal{F}_{\mu,\nu}$ is a real analytic path in \mathcal{F} with endpoints at $[\mu]$ and $[\nu]$ in $\partial\mathcal{F}$. Both l_μ and l_ν are strictly monotonic on $\mathcal{F}_{\mu,\nu}$ and vary from 0 to ∞ in opposite directions.*

Proof: By Wolpert's result, $d^2l_\nu/dt_\mu^2 > 0$ at every point of $\mathcal{F}_{\mu,\nu}$. Therefore $\mathcal{F}_{\mu,\nu}$ is a disjoint union of real analytic arcs.

We claim the function l_μ is strictly monotonic on each component of $\mathcal{F}_{\mu,\nu}$. If not, there is an earthquake path $\mathcal{E}_{\mu,c}$ that meets $\mathcal{F}_{\mu,\nu}$ in two distinct points. Both these points are critical for l_ν on $\mathcal{E}_{\mu,c}$ which is impossible.

Since l_μ is real analytic, its restriction to $\mathcal{F}_{\mu,\nu}$ is open and proper and hence its range must be $(0, \infty)$. Clearly, as $l_\mu(p) \rightarrow 0$ along $\mathcal{F}_{\mu,\nu}$, we have $p \rightarrow [\mu] \in \partial\mathcal{F}$. Thus each component of $\mathcal{F}_{\mu,\nu}$ is an embedded arc with endpoints $[\mu]$ and $[\nu]$ in $\partial\mathcal{F}$.

If $\mathcal{F}_{\mu,\nu}$ had two components, then, for some $c > 0$, we could find a path $\mathcal{E}_{\mu,c}$ intersecting both components of $\mathcal{F}_{\mu,\nu}$. Thus l_ν would be minimal at two points on $\mathcal{E}_{\mu,c}$ which is impossible.

By the anti-symmetry in the formulas, we see that l_ν also varies monotonically from 0 to ∞ along $\mathcal{F}_{\mu,\nu}$ but in the opposite direction. \square

Corollary 4 *For any $c \in \mathbf{R}^+$ and $\mu \in ML$ there is a unique earthquake path \mathcal{E}_μ^p along which $l_\mu = c$.*

Remark 5 In [23], Kerckhoff proves that given $(\mu, \nu) \in ML$ with $i(\mu, \nu) > 0$ and such that μ, ν fill up the surface (that is, the complement of their union consists of pieces which are either simply connected or a neighborhood of the puncture), then for each $t \in (0, 1)$ there is a unique $p \in \mathcal{F}$ at which the function $tl_\mu(p) + (1-t)l_\nu(p)$ attains minimum. As t varies keeping μ, ν fixed, the set of these minima is a line. For the punctured torus, any pair $(\mu, \nu) \in ML$ with $i(\mu, \nu) > 0$ fills up the surface. While not strictly needed for our development, the following lemma confirms that for the punctured torus, Kerckhoff's *line of minima* is identical with our critical line, see also [23] theorem 3.4.

Lemma 6 *Suppose that $i(\mu, \nu) > 0$. Then $p \in \mathcal{F}_{\mu,\nu}$ if and only if p is the global minimum for some function $tl_\mu(p) + (1-t)l_\nu(p)$ for some $t \in (0, 1)$.*

Proof: At a minimum of $tl_\mu(p) + (1-t)l_\nu(p)$, since l_μ is constant along the earthquake path $\mathcal{E}_\mu(p)$, we find $dl_\nu/dt_\mu(p) = dl_\mu/dt_\nu(p) = 0$ so that $p \in \mathcal{F}_{\mu,\nu}$. Conversely, if $dl_\nu/dt_\mu(p) = 0$, the earthquake paths $\mathcal{E}_\mu(p)$ and $\mathcal{E}_\nu(p)$ must be tangent at p because p is the unique minimum of l_ν on $\mathcal{E}_\mu(p)$. Thus $\mathcal{E}'_\mu(p) = -k\mathcal{E}'_\nu(p)$ for some $k \neq 0$, where $'$ denotes the tangent vector to the corresponding earthquake path. From the derivative formula $dl_\nu/dt_\mu = -dl_\mu/dt_\nu$ it follows that $k > 0$. We get $dl_\eta/dt_\mu(p) = -kdl_\eta/dt_\nu(p)$ for any $\eta \in ML$, which, using the derivative formula again, gives $dl_\mu/dt_\eta(p) = -kdl_\nu/dt_\eta(p)$. Since the tangent vectors $\mathcal{E}'_\eta(p), \eta \in ML$ certainly span the tangent space to \mathcal{F} at p , we must be at a critical point of $l_\mu + kl_\nu$. \square

Using the identification of the critical line $\mathcal{F}_{\mu,\nu}$ with the Kerckhoff line of minima, the following proposition follows immediately from [23] theorem 2.1. Here is another proof.

Proposition 7 *Fix $[\mu] \in PML$. Then the arcs $\mathcal{F}_{\mu,\nu}, [\nu] \in PML - \{[\mu]\}$ are pairwise disjoint and foliate \mathcal{F} .*

Proof: Given $p \in \mathcal{F}$, following Kerckhoff we define $\beta = \beta_p : ML \rightarrow T_p\mathcal{F}$ to be the map which takes $\mu \in ML$ to $D\mathcal{E}_\mu(t_\mu)(p)|_{t_\mu=0}$, the derivative with respect to t_μ of the earthquake path $\mathcal{E}_\mu(t_\mu)(p)$ through p evaluated at p . By [23], Theorem 3.5 the map β is a homeomorphism. Clearly, β induces a homeomorphism between PML and the set of rays through the origin in $T_p\mathcal{F}$.

Suppose $[\mu], [\nu], [\nu'] \in PML$ are distinct, and suppose that $p \in \mathcal{F}_{\mu,\nu} \cap \mathcal{F}_{\mu,\nu'}$.

Pick representatives μ, ν, ν' of $[\mu], [\nu], [\nu']$ and let $c = l_\mu(p), d = f_{\mu, \nu}(c), d' = f_{\mu, \nu'}(c)$. The earthquake paths $\mathcal{E}_{\nu, d}$ and $\mathcal{E}_{\nu', d'}$ both go through p and, because l_μ is minimal at p with respect to both \mathcal{E}_ν and $\mathcal{E}_{\nu'}$, from the derivative formula we see that $D\mathcal{E}_\nu(t_\nu)(p)|_{t_\nu=0} = D\mathcal{E}_{\nu'}(t_{\nu'})(p)|_{t_{\nu'}=0}$. By the injectivity of β_p on PML , $[\nu] = [\nu']$.

Now let $p \in \mathcal{F}$. By the surjectivity of β , there is some $\nu \in ML$ such that $D\mathcal{E}_\nu(t_\nu)(p)|_{t_\nu=0} = -D\mathcal{E}_\mu(t_\mu)(p)|_{t_\mu=0}$. Therefore the earthquake paths $\mathcal{E}_{\nu, l_\nu}(p)$ and $\mathcal{E}_{\mu, l_\mu}(p)$ are tangent at p . Since earthquake paths can intersect in at most two points it follows that l_ν is minimal at p with respect to \mathcal{E}_μ , so that $p \in \mathcal{F}_{\mu, \nu}$.

These two facts show that the sets $\mathcal{F}_{\mu, \nu}$ foliate \mathcal{F} . \square

We shall also need

Corollary 8 *For fixed $\mu \in ML$, $c \in \mathbf{R}^+$, the map $\psi : PML - \{[\mu]\} \rightarrow \mathcal{E}_{\mu, c}$, $\psi([\nu]) = p(\mu, \nu, c)$, is a homeomorphism.*

Proof: Proposition 7 shows that ψ is well defined and a bijection. It is also clear, thinking of $PML - \{[\mu]\}$ and $\mathcal{E}_{\mu, c}$ as intervals, that ψ is monotonic. The result follows. \square

Corollary 4 implies that for $\mu \in ML$, the paths $\mathcal{E}_{\mu, c}$, $c \in \mathbf{R}^+$ are pairwise disjoint and foliate \mathcal{F} . This is the analogue of the foliation of the hyperbolic disk \mathbf{D} by horocycles tangent at to a point on the boundary. Likewise, the critical lines $\mathcal{F}_{\mu, \nu}$ are the analogue of the geodesics in \mathbf{D} joining a pair of distinct points in S^1 . For fixed $[\mu]$ the foliation by leaves $\mathcal{F}_{\mu, \nu}$, $[\nu] \neq [\mu]$ is clearly transverse to that by the earthquake paths $\mathcal{E}_{\mu, c}$.

This is the picture that we shall extend to \mathcal{QF} below.

4 Hyperbolic 3-manifolds

4.1 The pleating locus

Let $q \in \mathcal{QF}$ and let $G = G(q)$ be a group representing q with the standard normalization of section 2.1. The group G acts as a discrete group of isometries of hyperbolic space \mathbf{H}^3 and the quotient hyperbolic manifold $M = \mathbf{H}^3/G$ is a product $\mathcal{T}_1 \times (-1, 1)$. If G is quasifuchsian, but not Fuchsian, the boundary $\partial\mathcal{C}$ of the hyperbolic convex hull \mathcal{C} of Λ in \mathbf{H}^3 has two components $\partial\mathcal{C}^\pm$ each of which is also G -invariant. Each quotient $\partial\mathcal{C}^\pm/G$ is homeomorphic to \mathcal{T}_1 , see

for example [19] proposition 3.1. The metric induced on the components $\partial\mathcal{C}^\pm$ from \mathbf{H}^3 makes them *pleated surfaces*. This means, see for example [9], that there are surjective isometric maps $\psi^\pm : \mathbf{D} \rightarrow \partial\mathcal{C}^\pm$ such that for each point z in \mathbf{D} there is at least one geodesic segment through z that is mapped to a geodesic segment in $\partial\mathcal{C}^\pm$. The group G acts as a discrete group of isometries on each component $\partial\mathcal{C}^\pm$. Since $\partial\mathcal{C}^\pm/G$ are both homeomorphic to \mathcal{T}_1 , these two groups of isometries are both isomorphic to $\pi_1(\mathcal{T}_1)$ and inherit a marking in the obvious way. (The marking on $\partial\mathcal{C}^-/G$ has its orientation reversed.) The isometries ψ^\pm induce isomorphisms to marked Fuchsian punctured torus groups $F^\pm = F^\pm(q)$ acting on \mathbf{D} , which we may again take to have the standard normalization. We refer to both the marked groups $F^\pm(q)$ and the quotients $\mathbf{D}/F^\pm(q)$ as the *flat structures* of either the surfaces $\partial\mathcal{C}^\pm/G(q)$ or of their universal covers $\partial\mathcal{C}^\pm(q)$.

The bending laminations of $\partial\mathcal{C}^\pm/G$ carry natural transverse measures, the *bending measures* $pl(q)^\pm$, see [9,19]. The underlying laminations $|pl(q)^\pm|$ are the *pleating loci* of G . If $G \in \mathcal{QF}$ is a Fuchsian group acting on the hyperbolic disk $\mathbf{D} \subset \mathbf{H}^3$, then $\mathcal{C} = \mathbf{D}$ is degenerate and we regard $\partial\mathcal{C}$ and $\partial\mathcal{C}/G$ as 2-sided surfaces, each side of which is a pleated surface with empty pleating locus (and zero measure).

The following proposition follows immediately from [20] proposition 3.3 and corollary 3.4.

Proposition 9 *Suppose that $q \in \mathcal{QF} - \mathcal{F}$. Then the projective class of the bending measure cannot be the same on both sides of the convex core; that is, $[pl^+(q)] \neq [pl^-(q)]$.*

Remark 10 The work in [20] depends heavily on the λ -lemma and the theory of holomorphic motions which is usually stated in the context of one complex variable. In the present case we shall be studying families of groups parameterized by a two dimensional complex manifold; in fact the theory of holomorphic motions extends to motions over any complex manifold, see [33].

In [19] we prove:

Theorem 11 *The map $\mathcal{QF} \rightarrow \mathcal{F}$ which sends $q \mapsto F^\pm(q)$, and the map $\mathcal{QF} - \mathcal{F} \rightarrow ML$ which sends $q \mapsto pl^\pm(q)$, are continuous.*

4.2 Pleating Varieties

Given $\mu \in ML$ we set

$$\mathcal{P}_\mu^\pm = \{q \in \mathcal{QF} : [pl^\pm(q)] = [\mu]\} \text{ and } \mathcal{P}_\mu = \mathcal{P}_\mu^+ \cup \mathcal{P}_\mu^-.$$

We call these sets the μ -pleating varieties.

Given the ordered pair $(\mu, \nu) \in ML \times ML$, we set

$$\mathcal{P}_{\mu, \nu} = \{q \in \mathcal{QF} : [pl^+(q)] = [\mu], [pl^-(q)] = [\nu]\}.$$

We call this set the μ, ν -pleating plane. Note that two these definitions depend only on the projective classes $[\mu], [\nu]$.

Finally, given the ordered pair $\mu, \nu \in ML \times ML$, and $c > 0$ we set

$$\mathcal{P}_{\mu, \nu, c} = \{q \in \mathcal{P}_{\mu, \nu} : l_\mu(q) = c\}.$$

We call this set a *pleating ray*. Note that for $s \in \mathbf{R}^+$, $\mathcal{P}_{\mu, \nu, c} = \mathcal{P}_{s\mu, \nu, sc}$. Thus $\mathcal{P}_{\mu, \nu, c}$ depends on the projective class of the pair (μ, c) , (recall section 2.2), and on the projective class $[\nu]$.

Theorems 2 and 6, to be proven in section 9 below, will justify the terminology *rays* and *planes*.

Proposition 9 implies $\mathcal{P}_{\mu, \mu} = \emptyset$. It is also clear that $\mathcal{P}_{\mu, \nu} \cap \mathcal{P}_{\mu', \nu'} = \emptyset$ unless $[\mu] = [\mu'], [\nu] = [\nu']$. In particular $\mathcal{P}_{\mu, \nu} \neq \mathcal{P}_{\nu, \mu}$ whenever $i(\mu, \nu) > 0$.

Remark 12 Whether a group is in $\mathcal{P}_{\mu, \nu}$ or in $\mathcal{P}_{\nu, \mu}$ depends on our conventions in labelling the sides $\partial\mathcal{C}^\pm$ of $\partial\mathcal{C}$. This is based on the labelling of the components of the regular set Ω^\pm . The point here is that two groups which differ only in the labelling of their + side and their - side are *not the same* as marked groups in \mathcal{QF} .

The main result of [20] is that the pleating varieties are non-empty. Precisely, we prove

Theorem 13 *Let $\mu, \nu \in ML$, $[\mu] \neq [\nu]$. Then $\mathcal{P}_{\mu, \nu} \neq \emptyset$.*

We shall need to study the ideas in the proof of this result in some detail; see 7.2 below.

4.3 Lamination length in $M = \mathbf{H}^3 / G$.

For the proof of theorem 15 below, we need also to discuss briefly the length $l_\mu(M)$ of a measured lamination $\mu \in ML$ in the hyperbolic 3-manifold $M = \mathbf{H}^3 / G$. First, suppose that $\mu = \delta_\gamma$ where $\gamma \in \mathcal{S}$ is represented by an element $V \in G$. The multiplier λ_V is related to its trace by the formula $\text{Tr } V = 2 \cosh \lambda_V / 2$. The translation length of V , $\Re\lambda_V$, is the minimum distance that V moves a point in \mathbf{H}^3 . Equivalently it is the length of the geodesic representative of γ in M , so that $l_{\delta_\gamma}(M) = \Re\lambda_V$.

In [45], p.9.21 and [3], p.117, it is shown that this definition can be extended by linearity and continuity to define the lamination length $l_\mu(M)$ for an arbitrary $\mu \in ML$. In the proof of theorem 15 below, we shall need to make crucial use of the fact that one can extend this definition continuously to the algebraic closure of \mathcal{QF} .

Suppose G is a (discrete) punctured torus group associated to the faithful representation $\rho : \pi_1(\mathcal{T}_1) \rightarrow G \subset PSL(2, \mathbf{C})$. This representation *marks* the associated hyperbolic 3-manifold $M = \mathbf{H}^3/G$. One says that a lamination $|\mu|$ on \mathcal{T}_1 is *realized* in M relative to the marking ρ , if there is a Fuchsian group Γ , a homeomorphism $h : \mathcal{T}_1 \rightarrow S = \mathbf{H}^2/\Gamma$, and a pleated surface $f : S \rightarrow M$ with pleating locus containing $|\mu|$, such that fh induces ρ .

Let $AH(\mathcal{T}_1)$ denote the set of Kleinian once punctured torus groups as defined in section 2.1, modulo conjugation in $PSL(2, \mathbf{C})$. By abuse of notation, we also denote by $AH(\mathcal{T}_1)$ the set of hyperbolic 3-manifolds $\{M = \mathbf{H}^3/H : [H] \in AH(\mathcal{T}_1)\}$, where $[H]$ is the conjugacy class of H in $PSL(2, \mathbf{C})$.

Clearly, whether or not a lamination is realized is a conjugacy invariant. Simple closed curves are always realized in any hyperbolic 3-manifold $M \in AH(\mathcal{T}_1)$ unless they are represented by parabolics. The closed geodesics are dense in the set of realizable laminations, [7] theorem 5.3.11. Since length is a conjugacy invariant, the above definition of lamination length $l_\mu(M)$ extends by continuity to any $M \in AH(\mathcal{T}_1)$ containing a realization of $|\mu|$. If $|\mu|$ is connected and not realized in M , set $l_\mu(M) = 0$. (If the closure $\overline{|\mu|}$ of $|\mu|$ is not connected one has to be more careful with this definition since some components of μ may be realized and others not; for example on a general surface, $|\mu|$ might consist of disjoint loops some but not all of whose components are accidentally parabolic. In this case only the accidental parabolics are not realized and l_μ must be defined by summing over the connected components of $\overline{|\mu|}$. In the case of a punctured torus $\overline{|\mu|}$ is always connected (since μ is measured) and this difficulty does not occur.)

In the next section, we shall make important use of the following result.

Proposition 14 *The function $L : AH(\mathcal{T}_1) \times ML \rightarrow \mathbf{R}$, $L(H, \mu) = l_\mu(\mathbf{H}^3/H)$ is continuous.*

Proof: This result was asserted by Thurston in [44]; detailed proofs appear in [36] Lemma 4.2 and [2] Theorem 5.1. We remark that the proof in [36] seems to have overlooked the above mentioned difficulties when $|\mu|$ is not connected. See [2] section 7 for a discussion of the general case. \square

Note that if a lamination $\mu \in ML$ is realized in $M \in AH(\mathcal{T}_1)$, then the length

of μ in M is equal to the hyperbolic length of μ on the surface Σ , where $\psi : \Sigma \rightarrow M$ is the pleated surface map realizing $|\mu|$, and so is strictly positive.

In general, the lamination lengths $l_\mu(\partial\mathcal{C})$ on $\partial\mathcal{C}$ and $l_\mu(M)$ in M are not the same, and we shall take care to indicate which length we mean. In the special case in which $q \in \mathcal{P}_\mu^+$, however, the lengths $l_\mu(\partial\mathcal{C}^+)$ and $l_\mu(M)$ coincide, and may be safely denoted by $l_\mu = l_\mu(q)$. This is the situation we are discussing in theorem 15 below.

In section 6, we shall show how to extend the holomorphic multiplier λ_V to a holomorphic function called the *complex length* λ_μ of μ on \mathcal{QF} . Again by linearity and continuity, we have $l_\mu(M) = \Re\lambda_\mu$. We also prove in section 6 that $q \in \mathcal{P}_\mu^+$ implies $\lambda_\mu \in \mathbf{R}$. Combining these observations gives that $q \in \mathcal{P}_\mu^+$ implies $\lambda_\mu = l_\mu(\partial\mathcal{C}^+) = l_\mu(M)$.

5 The limit pleating theorem

Classically, the ending invariants of a quasifuchsian group are the marked conformal structures $\omega^\pm(q)$ of the tori $\Omega^\pm(q)/G(q)$ and so are points in the Teichmüller space Teich . Suppose we have a sequence $q_n \in \mathcal{QF}$ with $\omega^\pm(q_n) \rightarrow \omega^\pm \in \text{Teich}$. It then follows from Bers' simultaneous uniformization theorem that the groups $G(q_n)$ have an algebraic limit in \mathcal{QF} . If both of the sequences $\omega^\pm(q_n)$ converge to distinct points in the Thurston boundary of Teich , then Thurston's double limit theorem [44] again asserts the existence of an algebraic limit G_∞ ; the intermediate situation works in a similar way and is discussed in [35].

We need an analogous result which asserts the existence of a limit group when our pleating invariants converge. We also need to understand the behavior of the pleating invariants when an algebraic limit exists. The results we need are collected in the following *limit pleating theorem*, which will be a key factor in the proof of our main results in section 9.

Theorem 15 Limit Pleating Theorem. *Let $\mu, \nu \in ML$, $[\mu] \neq [\nu]$ and suppose that $\{q_n\} \in \mathcal{P}_{\mu,\nu}$. Then*

- (1) *if $l_\mu(q_n) \rightarrow c \geq 0$ and $l_\nu(q_n) \rightarrow d \geq 0$, then there is a subsequence of the groups $\{G(q_n)\}$ with an algebraic limit G_∞ ;*
- (2) *if the sequence $\{G(q_n)\}$ has algebraic limit G_∞ , then the sequences $\{l_\mu(q_n)\}$ and $\{l_\nu(q_n)\}$ have finite limits $c \geq 0, d \geq 0$ respectively. The group G_∞ represents a point in \mathcal{QF} if and only if $c > 0$ and $d > 0$.*

We remark that in the case of a more general surface, the second statement as

it stands is false, as is seen by taking $|\mu|$ to be a multiple loop such that one, but not all of its components, becomes accidentally parabolic. It works in our case because any measured lamination on \mathcal{T}_1 is automatically connected. The result is closely related to, but not the same as, the ‘Lemme de fermeture’ in [5], which concerns the existence of the limit groups under hypotheses on the limits of bending measures as opposed to lengths.

The first statement, the existence of the algebraic limit, follows from a deep estimate of Thurston’s about lengths of geodesics in hyperbolic 3-manifolds, [44], theorem 3.3 (Efficiency of pleated surfaces). The same estimate is fundamental in Thurston’s proof of the double limit theorem in [44]. A detailed discussion and proof of Thurston’s estimate is to be found in [6], where a limit theorem similar to our first statement in the context of Schottky groups is proved.

To prove the second statement we use continuity of lamination length described in section 4.3 above. This allows us to deduce that the laminations μ, ν must be realized in the *algebraic* limit. We complete the proof by showing that the pleated surfaces which realize μ and ν are in fact components of the convex hull boundary of the algebraic limit. This idea is in essence the same as that used in [5], and we would like to thank F. Bonahon for suggesting this approach.

The statement, and the theorem on continuity of lamination length, conceals much subtlety. The hypothesis that $G_n \in \mathcal{P}_{\mu, \nu}$ is crucial; examples like the one described in [24] show that it is not enough just to require that some fixed curve on $\partial\mathcal{C}^+$ have bounded length. Again, if one takes a varying sequence $\mu_n \rightarrow \mu$ as in [5], then it is essential to add the hypothesis that the laminations converge in the Hausdorff topology as well as in measure, otherwise examples similar to the one in [24] again show that the convergence may not be strong.

Proof: First we suppose that $l_\mu(q_n) \rightarrow c \geq 0$ and $l_\nu(q_n) \rightarrow d \geq 0$, and show that there is some subsequence of $\{q_n\}$, along which an algebraic limit exists. Choose and fix an ideal triangulation λ on \mathcal{T}_1 ; specifically, take λ as the lines from the cusp to itself in the homotopy classes of the curves α, β and $\alpha\beta$, where $\langle \pi_1(\mathcal{T}_1); \alpha, \beta \rangle$ corresponds to $\langle G; A, B \rangle$.

Let $M_n = \mathbf{H}^3/G_n$ and realize λ as the pleating locus of a pleated surface S_n in M_n . The lamination λ has no closed leaves and its complement is a pair of ideal triangles. Pick $\xi \in ML$. When an oriented arc on a leaf $|\xi|$ cuts two consecutive sides of one of these complementary triangles T , the two sides meet in an ideal vertex which is either to its left or its right. The arc of leaf containing an intersection point P of $|\xi|$ and λ goes from one triangle T_1 to another T_2 . Following Thurston, [44], we call P a *boundary intersection* if the right-left location of the ideal vertex switches as we cross from T_1 to T_2 , and

we define the *alternation number* $a(\xi, \lambda)$ as the total ξ -measure of the set of boundary intersection points. Recall from section 4.3 that $l_\xi(S_n)$ denotes the length of the lamination ξ measured in the flat structure of S_n and $l_\xi(M_n)$ denotes the length of the lamination ξ in M_n . Then by [44] theorem 3.3, there exists a constant $C > 0$, depending only on a fixed choice of structure for \mathcal{T}_1 , such that

$$l_\xi(S_n) \leq l_\xi(M_n) + Ca(\xi, \lambda).$$

(We remark that since $a(\xi, \lambda) \leq i(\xi, \lambda)$ the usual intersection number would be just as good a bound in the present case.) Applying this inequality in our case to the pleating laminations $|\mu|$ and $|\nu|$ we find,

$$l_\mu(S_n) \leq l_\mu(q_n) + Ca(\mu, \lambda), \quad l_\nu(S_n) \leq l_\nu(q_n) + Ca(\nu, \lambda).$$

It follows that the sequences $\{l_\mu(S_n)\}$ and $\{l_\nu(S_n)\}$ are bounded.

Since $[\mu] \neq [\nu]$, the laminations $|\mu|, |\nu|$ fill up \mathcal{T}_1 and we conclude from [44] proposition 2.4 that the hyperbolic structures of the surfaces S_n lie in a bounded subset of \mathcal{F} and thus that the lengths $l_\alpha(S_n)$ and $l_\beta(S_n)$ of the geodesic representatives of the marking curves α and β on S_n are bounded. From the discussion in section 4.3, we conclude that, since $l_\alpha(M_n) \leq l_\alpha(S_n)$ and $l_\beta(M_n) \leq l_\beta(S_n)$, the sequences $\{|\text{Tr } A_n|\}$, $\{|\text{Tr } B_n|\}$ are also bounded. Therefore we can find a convergent subsequence along which $\text{Tr } A_n$ and $\text{Tr } B_n$ converge and thus, (because from the Markov identity $\text{Tr } A$ and $\text{Tr } B$ determine at most two normalized punctured torus groups up to conjugation) we conclude that a subsequence of $\{G_n\}$ has an algebraic limit G_∞ . This proves statement 1.

Now suppose that G_∞ is the algebraic limit of a sequence $G_n = G(q_n) \in \mathcal{P}_{\mu, \nu}$. By the continuity of lamination length on $AH(\mathcal{T}_1)$, the sequences $\{l_\mu(q_n)\}$, $\{l_\nu(q_n)\}$ converge to $\{l_\mu(G_\infty)\}$, $\{l_\nu(G_\infty)\}$, in particular the limits exist. We have to prove that $G_\infty \in \mathcal{QF}$ if and only if both limits are non-zero. We note immediately that if $G_\infty \in \mathcal{QF}$, then, using our assumption that $q_n \in \mathcal{P}_{\mu, \nu}$, we have $\{l_\mu(q_n)\} \rightarrow c > 0$ and $\{l_\nu(q_n)\} \rightarrow d > 0$ by the continuity theorem 11. This can also be seen from the fact that all laminations, in particular μ and ν , are realized in G_∞ , see [45], [7] theorem 5.3.11.

Suppose that one of the laminations μ or ν , for definiteness say μ , is not realized in G_∞ . Since $|\mu|$ is connected, $l_\mu(G_\infty) = 0$ and by the continuity of lamination length on $AH(\mathcal{T}_1)$ we deduce that $c = 0$. Thus we need only prove that if μ, ν are both realized in G_∞ , and if $c > 0, d > 0$, then $G_\infty \in \mathcal{QF}$.

Our strategy is to show that the lifts of the pleated surfaces which realize $|\mu|$ and $|\nu|$ are in fact invariant components of $\partial\mathcal{C}(G_\infty)$ which face simply connected invariant components of the regular set $\Omega(G_\infty)$. The key point is to show that if $|\mu|$ is realized in the algebraic limit $M_\infty = \mathbf{H}^3/G_\infty$, then the

lift of any leaf of $|\mu|$ to \mathbf{H}^3 is the limit of corresponding lifts of leaves of $|\mu|$ in their realizations in M_n .

To prove this involves comparing geodesics in the universal covers of M_∞ and M_n . We need to use the fact that geometric structures vary continuously with the holonomy. The clearest statement of what we need is in [15], Lemma 14.28. Our normalizations have been fixed in such a way that $\rho_n(g) \rightarrow \rho_\infty(g)$ for each $g \in \pi_1(\mathcal{T}_1)$. Let M_∞^1 be the manifold with boundary obtained by removing from M_∞ a horoball neighborhood of the cusp corresponding to $\rho_\infty(K)$, where K corresponds to a loop around the cusp of \mathcal{T}_1 . A relative compact core M_c for M_∞^1 is a compact submanifold $M_c \subset M_\infty^1$ such that the induced map on fundamental groups is an isomorphism, and such that $M_c \cap \partial M_\infty^1$ is a compact subsurface of ∂M_∞^1 . The assertion of Kapovich's lemma is that under the hypothesis that $\rho_n(g) \rightarrow \rho_\infty(g)$, if M_c is any relative compact core for M_∞^1 , then there exists a sequence of smooth maps $\tilde{j}_n : \tilde{M}_c \rightarrow \mathbf{H}^3$ which intertwine the actions of ρ_∞ and ρ_n and which converge C^1 to the identity, uniformly on compact subsets of \tilde{M}_c . Here $\tilde{M}_c \subset \mathbf{H}^3$ is the universal cover of M_c . For the proof see [15] theorem 7.2 or [7] theorem 1.7.1. Following [7], one can actually make \tilde{j}_n be C^r close to the identity for any r . It follows that the map $j_n : M_c \rightarrow M_n$ induced by \tilde{j}_n is close to a local isometry, see the similar assertion in [32] section 3.1.

Now we have to be careful about our set-up of pleated surfaces. Let Γ_0 be a fixed Fuchsian group acting on \mathbf{D} . Identify \mathcal{T}_1 with \mathbf{D}/Γ_0 , choosing a fixed isomorphism of $\pi_1(\mathcal{T}_1)$ with Γ_0 . The action of $G_n = \rho_n(\Gamma_0)$ on $\partial\mathcal{C}_n^+$ pulls back to the action of a Fuchsian group Γ_n on \mathbf{D} . This induces a pleated surface map $f_n : \mathbf{D} \rightarrow \mathbf{H}^3$ with image $\partial\mathcal{C}_n^+$, intertwining the action of Γ_n on \mathbf{D} and G_n on $\partial\mathcal{C}_n^+$. Let $h_n : \mathbf{D} \rightarrow \mathbf{D}$ denote a homeomorphism which intertwines the actions of Γ_0 and Γ_n , so that $f_n h_n : \mathbf{D} \rightarrow \mathbf{H}^3$ induces the representation $\rho_n : \Gamma_0 \rightarrow G_n$. Since $|\mu|$ is realized in M_∞ , there is also a Fuchsian group Γ_∞ , and a pleated surface $f : \mathbf{D} \rightarrow \mathbf{H}^3$ intertwining the actions of Γ_∞ and G_∞ with pleating locus containing $|\mu|$, together with a homeomorphism $h : \mathbf{D} \rightarrow \mathbf{D}$ intertwining the actions of Γ_0 and Γ_∞ such that fh induces $\rho_\infty : \Gamma_0 \rightarrow G_\infty$. Thus all the maps $f_n h_n$ and fh have the same domain \mathbf{D} and range \mathbf{H}^3 and intertwine the groups Γ_0 and G_n , and Γ_0 and G_∞ , respectively.

Now let l be a lift to \mathbf{D} of some leaf of μ in $\mathcal{T}_1 = \mathbf{D}/\Gamma_0$, that is, in the structure induced by Γ_0 . The corresponding leaves for the structures induced by Γ_n, Γ_∞ are the geodesics l_n, l_∞ which have the same endpoints on $\partial\mathbf{D}$ as $h_n(l), h(l)$ respectively. From the definition of pleated surfaces, under f_n and f these leaves are mapped to geodesics in \mathbf{H}^3 . To make precise the statement that leaves of $|\mu|$ in $M_\infty = \mathbf{H}^3/G_\infty$ are close to leaves of the corresponding realizations in M_n , we shall prove that $f_n(l_n) \rightarrow f(l_\infty)$.

Let S_0^1 be the surface with boundary obtained by removing from \mathbf{D}/Γ_0 a

horocycle neighborhood of the cusp. The pleating locus of the convex hull boundary cannot contain any leaf going out to the cusp (because otherwise the developing image of the boundary in a horoball neighborhood of the cusp would not be embedded), so we may suppose that the support of μ on \mathbf{D}/Γ_0 is contained in S_0^1 . Now M_∞ is geometrically tame and homeomorphic to $S \times \mathbf{R}$, see [3], or [15] theorem 14.17. It follows that M_∞ may be exhausted by relative compact cores; in particular we may assume that M_c contains the image under the map induced by fh of S_0^1 . Choose a map \tilde{j}_n as in Kapovich's lemma above. As already remarked, since \tilde{j}_n is C^∞ close to the identity on compact subsets of \tilde{M}_c , the induced map $j_n : M_c \rightarrow M_n$ is close to a local isometry. The same is therefore true of \tilde{j}_n on the whole of \tilde{M}_c . Now the image of a geodesic under a map which is close to a local isometry is clearly a quasi-geodesic with small constants, and hence close to its geodesic representative. Hence $\tilde{j}_n f(l_\infty)$ has definite endpoints on $\partial\mathbf{H}^3$ and is arbitrarily close (depending on n) to the geodesic A_n with the same endpoints.

Since $\tilde{j}_n \rightarrow \text{id}$ uniformly on compact sets in \tilde{M}_c and since $f(l_\infty) \subset \tilde{M}_c$, we see that $\tilde{j}_n f(l_\infty)$ converges to $f(l_\infty)$. (Here we use the fact that both curves are quasigeodesic, so it suffices to prove convergence on compact subsets of \mathbf{H}^3 .) Thus to complete the proof, it will suffice to show that $f_n(l_n) = A_n$. We shall do this by showing that the curves $f_n(l_n)$ and $\tilde{j}_n f(l_\infty)$ are a bounded distance apart.

First let us show that $f_n h_n(l)$ and $\tilde{j}_n f h(l)$ are a bounded distance apart. Pick a fundamental domain for the action of Γ_0 on \mathbf{D} , and let R be the closure of the intersection of this region with the lift \tilde{S}_0^1 of S_0^1 to \mathbf{D} . Since R is compact, there exists $K > 0$ such that $d_{\mathbf{H}^3}(\tilde{j}_n f h(x), f_n h_n(x)) \leq K$ for all $x \in R$. Now by construction, both maps $\tilde{j}_n f h, f_n h_n$ are equivariant, meaning that $\tilde{j}_n f h(\gamma x) = \rho_n(\gamma) \tilde{j}_n f h(x)$ for all $x \in \mathbf{D}$ and $\gamma \in \Gamma_0$, and similarly for $f_n h_n$. Thus $d_{\mathbf{H}^3}(\tilde{j}_n f h(x), f_n h_n(x)) \leq K$ for all $x \in \tilde{S}_0^1$. Parameterize the leaf l as $t \mapsto l(t)$ for $t \in \mathbf{R}$. Since l projects to S_0^1 , it follows that $d_{\mathbf{H}^3}(\tilde{j}_n f h(l(t)), f_n h_n(l(t))) \leq K$ for all $t \in \mathbf{R}$, in other words the two curves are a bounded distance apart over the whole of their lengths.

Since S_0^1 is compact, by equivariance the restriction of h_n to \tilde{S}_0^1 is Lipschitz with constant depending on n . Hence $h_n(l)$ is a quasigeodesic, and thus lies at a bounded distance from its geodesic representative l_n . Now the restriction of f_n to the compact set $h_n(\tilde{S}_0^1)$ is also Lipschitz, so that $f_n(l_n)$ and $f_n(h_n(l))$ are also a bounded distance apart (again with constant depending on n). Similarly, so are $\tilde{j}_n f(l_\infty)$ and $\tilde{j}_n f h(l)$. We conclude that $f_n(l_n)$ is a bounded distance from A_n (with bound depending on n). However $f_n(l_n)$ is a geodesic, and geodesics which are a bounded distance apart over the whole of their lengths coincide. Thus $f_n(l_n) = A_n$ as claimed.

We now use this fact to prove that the image of the pleated surface $f : \mathbf{D} \rightarrow \mathbf{H}^3$

is a component of the convex hull boundary of G_∞ . The projection of the pleating locus of f to \mathbf{D}/Γ_∞ is a geodesic lamination which contains $|\mu|$. If this pleating locus is not maximal, then by area considerations we can make it maximal by adding at most three extra leaves. (If μ is irrational, there may be extra leaves running from the cusp and spiralling into the two boundary leaves; if μ is rational there may be a further leaf running from the cusp to itself, or a leaf spiralling around μ at both ends.) Call this maximal lamination $\hat{\mu}$ and for simplicity, use the same symbols $\mu, \hat{\mu}$ to denote the lifts to \mathbf{D} . Notice that since the pleating locus of f_n actually equals $|\mu|$ (since the pleating locus of the convex hull boundary cannot contain any leaf going out to the cusp or spiralling onto a closed geodesic) the additional leaves of $\hat{\mu} - |\mu|$ are necessarily mapped to geodesics by f_n . Moreover any additional endpoints of these leaves are cusps and hence their lifts move continuously as $n \rightarrow \infty$.

We call any ideal triangle in \mathbf{H}^3 formed by the lifts of the images of the boundary leaves of a complementary region of $\hat{\mu}$ under a pleated map a *plaque*. The vertices of such a triangle are either the endpoints of leaves of the lamination or parabolic fixed points. For clarity, denote the images in \mathbf{H}^3 of $\hat{\mu}$ under the pleated surface maps f_n, f by $\hat{\mu}_n, \hat{\mu}_\infty$ respectively. We have just shown that any plaque of $\hat{\mu}_\infty$ is arbitrarily closely approximated in \mathbf{H}^3 by a plaque of $\hat{\mu}_n$ for all sufficiently large n . Notice also that any plaque of $\hat{\mu}_n$ is contained in a support plane for $\partial\mathcal{C}_n^+$.

Denote the image of f by Π^+ . We want to show that Π^+ is a component of $\partial\mathcal{C}(G_\infty)$. Let X be a plane containing a plaque of $|\mu_\infty|$ and let X_n be a sequence of planes containing approximating plaques for $\partial\mathcal{C}^+(G_n)$. We claim that all of Π^+ lies on the same side of X so that X is a support plane for Π^+ . If not, we can find points $y, y' \in \Pi^+$ on opposite sides of X so that the geodesic joining y to y' crosses X transversally. By choosing n sufficiently large, we can find y_n, y'_n near to y, y' in $\partial\mathcal{C}_n^+$, and a support plane X_n to $\partial\mathcal{C}_n^+$ close to X , such that the geodesic from y_n to y'_n crosses X_n , which is impossible.

Denote by H_X the closed half space bounded by X containing Π^+ and set $K = \bigcap_X H_X$ where X runs through all planes containing plaques of Π^+ . By the above, $\Pi^+ \subset K$ so $K \neq \emptyset$. By its construction, K is convex and closed. Moreover K is G_∞ invariant since the same is true of Π^+ . Since $\mathcal{C}(G_\infty)$ is the smallest closed convex G_∞ invariant set in \mathbf{H}^3 , we conclude $\mathcal{C}(G_\infty) \subset K$.

We claim $\Pi^+ \subset \partial\mathcal{C}(G_\infty)$. Let P be a plaque of Π^+ . Clearly $P \subset \mathcal{C}(G_\infty)$ and so $P \subset K$. Since P is by definition contained in a support plane for K , we conclude $P \subset \partial\mathcal{C}(G_\infty)$. Since Π^+ is the closure in \mathbf{H}^3 of the union of its plaques, the claim follows.

We prove in lemma 16 below that Π^+ is embedded in \mathbf{H}^3 . (This rules out the possibility that, for example, $|\mu|$ is rational and the bending angle is π .)

Thus Π^+ is isometric to a complete hyperbolic surface and hence is both open and closed in $\partial\mathcal{C}(G_\infty)$. Since Π^+ is connected, it must be a component of $\partial\mathcal{C}(G_\infty)$. As such, it faces a component Ω^+ of $\Omega(G_\infty)$. Moreover since Π^+ is simply connected, so is Ω^+ . (This also follows from the fact that the limit representation $\rho : \pi(\mathcal{T}_1) \rightarrow G_\infty$ is faithful.) Also G_∞ invariance of Ω^+ follows from that of Π^+ .

Now there is a similar image Π^- for the pleated surface map which realizes $|\nu|$, from which we deduce the existence of another simply connected invariant component Ω^- of $\Omega(G_\infty)$. We conclude, see for example [28] lemma 3.2, that $G_\infty \in \mathcal{QF}$. \square

Lemma 16 *With the notation and conditions above, the image Π^+ of the pleated surface map f is embedded in \mathbf{H}^3 .*

Proof: If Π^+ is not embedded then $f(x) = f(y)$ for some distinct points $x, y \in \mathbf{D}$; these cannot be in the same plaque since f is an isometry on plaques. We begin by reducing to the case in which x and y are both contained in leaves of $\hat{\mu}$. If not, suppose that x is in a complementary region of $\hat{\mu}$, and let P_x be the image plaque containing $f(x)$. Now y is either in a distinct complementary region with image plaque P_y , or on a leaf with image a geodesic L . If P_y or L cuts P_x transversally, then the same is true for all nearby pleated surfaces f_n , since the endpoints which determine plaques and leaves move continuously. This is impossible since $f_n(\mathbf{D}) = \partial\mathcal{C}_n^+$ is embedded. Thus P_y (or L) and P_x are in a common plane. In the first case there is some point on boundary leaves of both P_y and P_x , and in the second L meets some boundary point of P_x .

Now let l_x, l_y be leaves of the lift of $\hat{\mu}$ to \mathbf{D} through x, y respectively. We claim that the image leaves $f(l_x)$ and $f(l_y)$ meet at a non-zero angle in \mathbf{H}^3 . This follows from [43] Theorem 5.6. A pleated surface map k from a surface S into a 3-manifold M induces an obvious map K from its pleating locus to the projective unit tangent bundle $\mathbb{P}M$ of M . Thurston's result states that if k is weakly doubly incompressible, then K is an embedding. In our situation, the pleated surface map f induced on the quotient \mathbf{D}/Γ_∞ is weakly doubly incompressible since the induced map $\Gamma_\infty \rightarrow G_\infty$ is an isomorphism. Thus the conclusion of the theorem implies immediately that the image geodesics $f(l_x)$ and $f(l_y)$ are distinct. (Actually Thurston's proof simplifies slightly in our situation, see remark 17 below.)

Consider the plane P containing these two leaves. It meets $\hat{\mathbf{C}}$ in a circle C . Notice that any circle through the endpoints of $f(l_x)$ other than C separates the endpoints of $f(l_y)$. Now for any nearby group G_n , there are leaves $f_n(l_x), f_n(l_y)$ near $f(l_x), f(l_y)$. Any support plane to $\partial\mathcal{C}_n^+$ through either of these leaves meets $\hat{\mathbf{C}}$ in a circle which cannot separate the other pair of endpoints. One deduces easily that any pair of support planes for $\partial\mathcal{C}_n^+$ must meet

\hat{C} in circles both of which are close to C , and that $\Lambda(G_n)$ is contained in the thin ring or crescent between them. Now every support plane of $\partial\mathcal{C}_n^-$ is disjoint from every support plane of $\partial\mathcal{C}_n^+$; moreover one of the two disks defined by the circle it bounds on \hat{C} contains no limit points.

It follows that every support plane of $\partial\mathcal{C}_n^-$ must have very small diameter, and hence that the distance of any such support plane to $f(x)$ tends to ∞ with n . On the other hand, any support plane for $\partial\mathcal{C}_n^-$ contains points close to some plaque of the pleated surface which realizes $|\nu|$ in M_∞ . Pick a point $z \in \mathbf{H}^3$ on a lift of a leaf of $|\nu|$, at distance D say from $f(x)$. Since z is on a plaque of $|\nu|$ it is close to a support plane of $\partial\mathcal{C}_n^-$. This shows there are points in $\partial\mathcal{C}_n^-$ which stay at bounded distance, with bound close to D , from $f(x)$. This contradiction completes the proof. \square

Remark 17 We explain the simplification of Thurston's result alluded to above. Notice that μ (the lamination on the quotient \mathbf{D}/Γ_∞) is recurrent. Thus following the first part of the proof of [43] Theorem 5.5, one can use a shadowing argument to show that the canonical lift \bar{F} of \bar{f} induces a local embedding of $|\mu|$ into the projective unit tangent bundle $\mathbb{P}M_\infty$. The observation that the injectivity radius of M_∞ is bounded below in a neighborhood of $\bar{f}(|\mu|)$ follows since $|\mu|$ is compact. Since Margulis tubes in M_∞ are separated by a definite distance, only a finite number can intersect the image $\bar{f}(|\mu|)$.

Still following Thurston, one then argues that \bar{F} extends to a covering map on a neighborhood of $|\mu|$. (Notice that the inequality in [43] concerning the degree of the covering should be reversed.) We now need to see that the covering has degree 1. If μ is a closed geodesic, this follows immediately since f induces an isomorphism of fundamental groups. Otherwise, the complement of $|\mu|$ is a punctured bigon B with two boundary leaves. Thus any non-trivial covering must have degree two and identify the two boundary leaves. Choose a loop β in B which is very close to the boundary leaves but homotopic to a loop round the puncture. Since the covering is degree 2, $f(\beta)$ is the square of the generator of the maximal parabolic subgroup in G_∞ . This is impossible since f induces an isomorphism.

6 Complex length

In this section we introduce the *complex length* of a measured lamination. Just as lamination length as defined in section 2.2 is a real analytic function on \mathcal{F} , the complex lamination length is a holomorphic function on \mathcal{QF} . The relationship of this holomorphic function to pleating varieties, in particular theorem 23, is a central tool in everything which follows. Complex lamination length has also been introduced using somewhat different techniques by

Bonahon [4].

6.1 Complex length of a loxodromic

Let $M \in PSL(2, \mathbf{C})$. Its *complex translation length* $\lambda_M \in \mathbf{C}/2\pi i\mathbf{Z}$ is given by the equation

$$\pm \operatorname{Tr} M = 2 \cosh \lambda_M/2 \tag{1}$$

where $\operatorname{Tr} M$ is the trace of M and we choose the sign so that $\Re \lambda_M \geq 0$.

Complex length is invariant under conjugation by Möbius transformations and has the following geometric interpretation, provided M is not parabolic. Let $x \in AxM$ and let \bar{v} be a vector normal to AxM at x . Then $\Re \lambda_M$ is the hyperbolic distance between x and $M(x)$ and $\Im \lambda_M$ is the angle mod 2π between $M(\bar{v})$ and the parallel transport of \bar{v} to $M(x)$, measured facing the attracting fixed point M^+ of M . In particular, if M is loxodromic then $\Re \lambda_M > 0$ and if M is purely hyperbolic then in addition $\Im \lambda_M \in 2\pi\mathbf{Z}$; equivalently $\operatorname{Tr} M \in \mathbf{R}$, $|\operatorname{Tr} M| > 2$. (We refer to [39] for a detailed discussion of the sign ambiguity in equation 1; note that in our notation here λ_M is *twice* the multiplier denoted by λ_M in [39].)

Let $q \in \mathcal{QF}$, let $\gamma \in \mathcal{S}$ and denote the element representing γ in the group $G(q)$ by $W(q)$. Because the trace is a conjugation invariant, the complex translation length $\lambda_W(q)$ depends only on q and is independent of the normalization of $G(q)$. We want to define the complex length $\lambda_\gamma(q) = \lambda_W(q)$ as a holomorphic function on \mathcal{QF} with values in \mathbf{C} , not $\mathbf{C}/2\pi i\mathbf{Z}$. To do this, we choose the branch that is real valued on \mathcal{F} . Since $\lambda_\gamma \neq 0$ on \mathcal{QF} this choice uniquely determines a holomorphic function $\lambda_\gamma : \mathcal{QF} \rightarrow \mathbf{C}$. From now on, the term “complex length” will always refer to this branch.

We define the *complex length* of the rational lamination $\mu = c\delta_\gamma \in ML_Q$, $c > 0$, as $\lambda_\mu(q) = c\lambda_\gamma(q)$.

To define the complex length $\lambda_\mu(q)$ for arbitrary $\mu \in ML$ and $q \in \mathcal{QF}$, we would like to choose $\mu_n \in ML_Q$, $\mu_n \rightarrow \mu$ and set

$$\lambda_\mu(q) = \lim_{n \rightarrow \infty} \lambda_{\mu_n}(q).$$

To justify this, we need to show these limits exist and are independent of the sequence $\{\mu_n\}$.

We do this using the following theorem which summarizes the results of [22], lemma 2.4 and [21], theorem 1. In the statement, l_μ denotes lamination length

defined in section 2.2.

Theorem 18 *The function $(c\delta_\gamma, p) \mapsto cl_{\delta_\gamma}(p)$ from $ML_Q \times \mathcal{F}$ to \mathbf{R}^+ extends to a continuous function $(\mu, p) \mapsto l_\mu(p)$ from $ML \times \mathcal{F}$ to \mathbf{R}^+ . If $\mu_n \in ML_Q$, $\mu_n \rightarrow \mu$ then $l_{\mu_n}(p) \rightarrow l_\mu(p)$ uniformly on compact subsets of \mathcal{F} , and the limit function $l_\mu(p)$ is non-constant.*

We also need an elementary lemma about holomorphic functions.

Lemma 19 *If $f : \mathcal{QF} \rightarrow \mathbf{C}$ is holomorphic and if $f \equiv c$ on \mathcal{F} for some constant c , then $f \equiv c$ on \mathcal{QF} .*

Proof: Because \mathcal{F} is the \mathbf{R}^2 -locus of the complex Fenchel-Nielsen coordinates (λ, τ) in \mathcal{QF} , see [27], and section 7 below, the conclusion follows directly from the Cauchy-Riemann equations applied to each variable separately. \square

Theorem 20 *The function $(\mu, q) \mapsto \lambda_\mu(q)$ from $ML_Q \times \mathcal{QF}$ to \mathbf{C} extends to a continuous function from $ML \times \mathcal{QF}$ to \mathbf{C} , also denoted $\lambda_\mu(q)$. The function $q \mapsto \lambda_\mu(q)$ is holomorphic and non-constant for all μ and the family $\{\lambda_\mu\}$ is bounded and equicontinuous on compact subsets of \mathcal{QF} .*

Proof: By construction, the functions $\{\lambda_\mu\}$, $\mu \in ML_Q$, omit the half plane $\Re z < 0$ and thus form a normal family on compact subsets of \mathcal{QF} . It follows that if $\mu_n \rightarrow \mu$, $\mu_n \in ML_Q$, then suitable subsequences of $\{\lambda_{\mu_n}\}$ converge to limit functions that are holomorphic.

We note that on \mathcal{F} , if $\mu \in ML_Q$, then λ_μ is real and coincides with l_μ . By theorem 18, if $\mu_n \rightarrow \mu$, $\mu_n \in ML_Q$, then $\{l_{\mu_n}\}$ is uniformly convergent on compact subsets of \mathcal{F} ; further, the limit function l_μ is finite, non-constant and independent of the choice of the sequence $\{\mu_n\}$. The result now follows from lemma 19. \square

Corollary 21 *Let $\mu, \nu \in ML$. Then the zero of dl_ν/dl_μ at the minimum of l_ν along the earthquake path \mathcal{E}_μ^p is simple.*

Proof: From the discussion preceding Proposition 2, we have only to consider the case in which ν is not rational. By Theorem 20, l_ν extends to a holomorphic function λ_ν on \mathcal{QF} which is locally uniformly approximated by the complex lengths λ_{ν_n} of a sequence of rational laminations ν_n which converge to ν . Now apply Hurwitz' theorem. \square

For $\mu \in ML$, we call λ_μ the *complex length* of μ . Throughout this paper, the complex length functions are a fundamental tool. We remark that

- (1) Suppose $q \in \mathcal{QF}$ and let $F^\pm(q) \in \mathcal{F}$ denote the flat structures (see section 4.1) on the convex core boundary $\partial\mathcal{C}^\pm(q)/G(q)$. If $\mu \in [pl^\pm(q)]$, then $l_\mu(F^\pm(q)) = \Re\lambda_\mu(q)$, see Proposition 22 below.
- (2) For $q \in \mathcal{QF}$, $\mu \in ML$, $\Re\lambda_\mu$ coincides with the lamination length $l_\mu(M(q))$ in the 3-manifold $M(q) = \mathbf{H}^3/G(q)$ as discussed in section 4.3 above.

For $\mu \in ML_Q$, $\Re\lambda_\mu(q) = l_\mu(M(q))$, so by continuity, both statements hold for all $\mu \in ML$.

6.2 Complex length and Pleating varieties

The first step in proving our main theorems is to show that for any $\mu \in ML$, the complex length λ_μ is real valued on \mathcal{P}_μ .

First consider the case $\mu \in ML_Q$. We have

Proposition 22 *Suppose $\mu \in ML_Q$. Let $q \in \mathcal{QF}$ and suppose $pl^+(q) = \mu$. Then $\lambda_\mu(q) \in \mathbf{R}$.*

Proof: This is just a reformulation of the easy observation, proved in [18], lemma 4.6, that if a geodesic γ is contained in $|pl^\pm(G)|$, then any representative in G is purely hyperbolic. \square

We now extend proposition 22 to arbitrary laminations.

Theorem 23 *Suppose $\mu \in ML$. Let $q \in \mathcal{QF}$ and suppose $pl^+(q) = \mu$. Then $\lambda_\mu(q) \in \mathbf{R}$.*

Proof: For $[\mu] \in ML_Q$ this is proposition 22, so suppose $[\mu] \notin ML_Q$.

The map $\mathcal{QF} - \mathcal{F} \rightarrow ML \times \mathcal{F}$ that takes $q \in \mathcal{QF} - \mathcal{F}$ to $(pl^+(q), F^+(q))$ where $F^+(q)$ is the flat structure of $\partial\mathcal{C}^+/G(q)$ is continuous by theorem 11. The map is also injective because the hyperbolic structure F^+ together with the bending data pl^+ determine the group $G = G(q)$, see [9] Chapter 3 especially Lemma 3.7.1. Let $U \subset \mathcal{QF} - \mathcal{F}$ be an open ball containing q ; if $[pl^+(q')]$ were constant on U , a four dimensional neighborhood would have a three dimensional image, violating the invariance of domain for a continuous injective map.

By the continuity theorem 11, since PML is one dimensional, we may find a sequence $q_n \rightarrow q$ in U such that $pl^+(q_n) = \mu_n$ with $\mu_n \in ML_Q$. By proposition 22, $\lambda_{\mu_n}(q_n) \in \mathbf{R}$. By the continuity theorem again, $\mu_n \rightarrow \mu$ and hence $\lambda_{\mu_n} \rightarrow \lambda_\mu$ uniformly on compact subsets of \mathcal{QF} . Thus taking a diagonal limit we have $\lambda_{\mu_n}(q_n) \rightarrow \lambda_\mu(q)$ and $\lambda_\mu(q) \in \mathbf{R}$. \square

7 Twists and Quakebends

In this section we briefly discuss complex Fenchel-Nielsen coordinates and quakebends, and the connection with the convex hull boundary $\partial\mathcal{C}$. This circle of ideas is at the heart of the proof of the local pleating theorem 26 in section 8; some of the ideas are also needed in section 9, where we work in quakebend planes as defined in section 7.3 below.

7.1 Complex Fenchel Nielsen coordinates

Complex Fenchel Nielsen parameters were introduced in [27,42] (see also [20]) as a generalization to \mathcal{QF} of the classical Fenchel Nielsen coordinates for Fuchsian groups. Here we briefly summarize the main points as applied to \mathcal{T}_1 .

Let $\langle G; A, B \rangle$ be a marked quasifuchsian punctured torus group constructed from a pair of marked generators α, β of $\pi_1(\mathcal{T}_1)$ as described in 2.1. Complex Fenchel Nielsen coordinates $(\lambda_A, \tau_{A,B})$ for $\langle G; A, B \rangle$ are obtained as follows. The parameter $\lambda_A \in \mathbf{C}/2\pi i\mathbf{Z}$ is the complex translation length of the generator $A = \rho(\alpha)$, or equivalently the complex length λ_α . The twist parameter $\tau_{A,B} \in \mathbf{C}/2\pi i\mathbf{Z}$ measures the complex shear when the axis $Ax B^{-1}AB$ is identified with the axis $Ax A$ by B . More precisely, if the common perpendicular δ to $Ax B^{-1}AB$ and $Ax A$ meets these axes in points Y, X respectively, then $\Re\tau_{A,B}$ is the *signed* distance from X to $B(Y)$ and $\Im\tau_{A,B}$ is the angle between δ and the parallel translate of $B(\delta)$ along $Ax A$ to X , measured facing towards the attracting fixed point of A . On the critical line $\mathcal{F}_{\alpha,\beta}$, $\tau_{A,B} \equiv 0 \pmod{2\pi i}$ and $Ax A, Ax B$ intersect orthogonally. Thus a point on this line corresponds to a rectangular torus with generators (A, B) . The conventions for measuring the signed distance and the angle are explained in more detail in [20] but are not important here.

As shown in [11,20,27], given the parameters $\lambda_A, \tau_{A,B}$, and a fixed a normalization, one can explicitly write down the matrix generators for a marked two generator group $G(\lambda_A, \tau_{A,B}) \subset PSL(2, \mathbf{C})$ in which the commutator $[A, B]$ is parabolic. This group may or may not be discrete. The matrix coefficients of G depend holomorphically on the parameters. The construction thus defines a holomorphic embedding of \mathcal{QF} into a subset of $\mathbf{C}/2\pi i\mathbf{Z} \times \mathbf{C}/2\pi i\mathbf{Z}$, in which Fuchsian space \mathcal{F} is identified with the image of \mathbf{R}^2 .

We want to lift this to an embedding into \mathbf{C}^2 . In section 6 we discussed how to lift the length function λ_A on \mathcal{QF} to a holomorphic function on \mathbf{C} . We can similarly lift the twist parameter $\tau_{A,B}$ by specifying that it be real valued on \mathcal{F} .

On \mathcal{F} , the real valued parameters $\lambda_A, \tau_{A,B}$ reduce to the classical Fenchel Nielsen parameters $l_A, t_{A,B}$ defined by the above construction with λ_A the hyperbolic translation length l_A of A and $\tau_{A,B}$ the twist parameter $t_{A,B}$.

Clearly, the complex Fenchel Nielsen construction can be made relative to any marking V, W of G . As described in detail in section 5 of [20], for fixed $\lambda \in \mathbf{R}^+$ and $\tau \in \mathbf{C}$, the complex Fenchel Nielsen construction relative to V, W determines a map $\mathbf{D} \rightarrow \mathbf{H}^3$. This map is the composition of the earthquake $\mathcal{E}_\gamma(\Re\tau)$ along the geodesic γ represented by V with a map $\psi : \mathbf{D} \rightarrow \mathbf{H}^3$ which is an isometry restricted to each conjugate of $Ax A$ and also to each component of the complement of these axes in \mathbf{D} . The earthquake $\mathcal{E}_\gamma(\Re\tau) : \mathbf{D} \rightarrow \mathbf{D}$ intertwines the action of the rectangular torus group $G(\lambda, 0)$ with the group $G(\lambda, \Re\tau)$. The map ψ is a pleated surface map with with pleating locus γ and angle $\Im\tau$ between the outward normals to adjacent flat planes. It conjugates the actions of $G(\lambda, \Re\tau)$ on \mathbf{D} and $G(\lambda, \tau)$ on its image in \mathbf{H}^3 . We set $\mathbf{D}_\gamma(\lambda, \tau) = \psi(\mathbf{D})$. We note for future use that the bending measure of a transversal σ is $i(\gamma, \sigma)\Im\tau$.

7.2 Quakebends

Quakebends are a complex version of earthquakes. The construction was introduced by Thurston and is explained in detail in [9] and also summarized in [20]. An alternative discussion can be found in [31].

Let $p \in \mathcal{F}$ and let $G_0 = G(p)$ act on the disk $\mathbf{D} \subset \mathbf{H}^3$. For $\mu \in ML$ and $\tau \in \mathbf{C}$, the quakebend construction defines an isomorphism $\mathcal{Q}_\mu(\tau)$ from G_0 to its image $\mathcal{Q}_\mu(\tau)(G_0) = G_\mu^p(\tau)$, together with a pleated surface $\psi_\mu^p(\tau) : \mathbf{D} \rightarrow \mathbf{H}^3$ conjugating the actions of $\mathcal{E}_\mu(\Re\tau)(G_0) = G_\mu^p(\Re\tau)$ on \mathbf{D} and $G_\mu^p(\tau)$ on the image $\mathbf{D}_\mu^p(\tau) = \psi_\mu^p(\tau)(\mathbf{D})$. If $\Im\tau \neq 0$, then $\mathbf{D}_\mu^p(\tau)$ has pleating locus $|\mu|$. When $\tau = 0$, $\psi_\mu^p(\tau) = id$ and $G_\mu^p(\tau) = G_0$. When $\Im\tau = 0$ and $\Re\tau = t$, $\mathcal{Q}_\mu(\tau)$ coincides with the earthquake $\mathcal{E}_\mu(t)$, $\mathbf{D}_\mu^p(\tau) = \mathbf{D}$ and $G_\mu^p(t)$ is discrete and Fuchsian for all $t \in \mathbf{R}$. If $\Re\tau = 0$, we call the quakebend a *pure bend*.

If the lamination μ is rational, $\mu = k\delta_\gamma$, an earthquake along μ reduces to a Fenchel Nielsen twist. In terms of Fenchel Nielsen coordinates $(l_V, t_{V,W})$ relative to a marking (V, W) , where $V \in G$ represents the geodesic γ , this is given by the formula $(\lambda_V, \tau_{V,W}) \mapsto G(\lambda_V, \tau_{V,W} + kt)$. Likewise a quakebend along $k\delta_\gamma$ is the complex Fenchel Nielsen twist given by the formula $\mathcal{Q}_\mu(\tau) : G(\lambda_V, \tau_{V,W}) \mapsto G(\lambda_V, \tau_{V,W} + k\tau)$. In particular, if the base point $p \in \mathcal{F}$ is the rectangular group $G(\lambda, 0)$ relative to its marking (V, W) , the image pleated surface $\mathbf{D}_\mu^p(\tau)$ is exactly $\mathbf{D}_\gamma(\lambda, \tau)$ as described in 7.1 above. We shall make frequent use of this observation below. Note that the bending measure of a transversal σ to $\mathbf{D}_\mu^p(\tau)$ is always $i(\sigma, \mu)\Im\tau$.

So far, we have only discussed quakebends when the basepoint p is in \mathcal{F} . Examining [9], however, it is clear that one can make the same construction starting from a basepoint $q \in \mathcal{P}_\mu^+$. More precisely, let $pl^+(q)$ be the bending measure on $\partial\mathcal{C}^+(q)$, so that (by the unique ergodicity of measured laminations on a punctured torus) $pl^+(q) = k\mu$ for some $k > 0$. Let the flat structure of $\partial\mathcal{C}^+(q)$ be represented by the Fuchsian group $F^+(q)$ acting in \mathbf{D} . One can define the quakebend $\mathcal{Q}_\mu^q(\tau)$ as the group obtained by the quakebend $\mathcal{Q}_\mu^q(\tau + ik)$ acting on $F^+(q)$; in other words compose an earthquake along μ by $\Re\tau$ with a pure bend by $\Im\tau + k$. In this case, we should consider the time zero pleated surface $\mathbf{D}_\mu^q(0)$ to be the surface $\partial\mathcal{C}^+$. (See also [26], [31] for other versions of this construction.)

We shall not need to discuss here the problems associated with defining a quakebend from an arbitrary basepoint in \mathcal{QF} .

7.3 Quakebend planes

In what follows, we shall often want to regard the quakebend parameter τ as a holomorphic function on the space of representations $\rho : \pi_1(\mathcal{T}_1) \rightarrow PSL(2, \mathbf{C})$, modulo conjugation in $PSL(2, \mathbf{C})$. When the basepoint is Fuchsian, this is justified by the following proposition, which is [9], Lemma 3.8.1.

Proposition 24 *Let $p \in \mathcal{F}$, $\tau \in \mathbf{C}$, $\mu \in ML$, and let $G_\mu^p(\tau) = \mathcal{Q}_\mu(\tau)(G(p))$. Then the matrix coefficients of the elements of $G_\mu^p(\tau)$ are holomorphic functions of τ .*

It is clear that the Epstein-Marden proof still works when the basepoint q is in \mathcal{P}_μ^+ .

This result enables us to introduce *quakebend planes*, which are the device used in section 9 to reduce the investigation of pleating varieties to a tractable problem in one complex dimension.

For $q \in \mathcal{P}_\mu^+ \cup \mathcal{F}$, we set $\mathcal{Q}_\mu^q = \{G_\mu^q(\tau) : \tau \in \mathbf{C}\}$; we call \mathcal{Q}_μ^q the μ -quakebend plane based at q and sometimes write $\mathcal{Q}_\mu^q(\tau)$ for $G_\mu^q(\tau)$. By proposition 25 below, a neighborhood of q in \mathcal{Q}_μ^q is contained in \mathcal{P}_μ — but we emphasize once again that that in general the whole of \mathcal{Q}_μ^q is not contained in \mathcal{QF} (see proposition 35 below and [31]).

In the rational case $\mu \in ML_Q$, \mathcal{Q}_μ^q has a very easy description in terms of complex Fenchel Nielsen coordinates. Suppose that $\mu = \delta_\gamma$, $\gamma \in \mathcal{S}$ and that (γ, γ') are a pair of marked generators for $\pi_1(\mathcal{T}_1)$. Let $(\lambda_V, \tau_{V,W}) \subset \mathbf{C}^2$ be complex Fenchel Nielsen coordinates relative to corresponding marked generators (V, W) of G . Let $c = \lambda_\mu(q)$. Then it is clear from the discussion above that

\mathcal{Q}_μ^q is just the slice $\{(c, \tau)\} \subset \mathbf{C}^2$. We denote this slice by $\mathcal{Q}_{\gamma,c}$. Clearly, $\mathcal{Q}_{\gamma,c}$ meets \mathcal{F} along the earthquake path $\mathcal{E}_{\gamma,c}$.

More generally, if $\mu \in ML$ and $p, p' \in \mathcal{E}_{\mu,c}$, it is clear that $\mathcal{Q}_\mu^p = \mathcal{Q}_\mu^{p'}$; we denote this plane by $\mathcal{Q}_{\mu,c}$. Clearly, $\mathcal{Q}_{\mu,c}$ meets \mathcal{F} along the earthquake path $\mathcal{E}_{\mu,c}$. In general, however, if $q, q' \in \mathcal{P}_\mu$ and $\lambda_\mu(q) = \lambda_\mu(q')$, then it is not immediately clear whether or not $\mathcal{Q}_\mu^q = \mathcal{Q}_\mu^{q'}$. It is a consequence of our main results that $\lambda_\mu(q) = \lambda_\mu(q')$ always implies $\mathcal{Q}_\mu^q = \mathcal{Q}_\mu^{q'}$; this is proved in corollary 45 below.

As explained above, for a basepoint $q \in \mathcal{P}_\mu \cup \mathcal{F}$, the quakebend plane \mathcal{Q}_μ^q is not, in general, contained in $\mathcal{Q}\mathcal{F}$. We note that in the special case $p \in \mathcal{F}$, since $\mathcal{Q}\mathcal{F}$ is an open neighborhood of \mathcal{F} (in the space of representations into $PSL(2, \mathbf{C})$ modulo conjugation), it follows that for small τ , $G_\mu^p(\tau)$ is quasifuchsian. The following stronger result shows that, as one would naively expect, as one quakebends along μ away from a basepoint $q \in \mathcal{P}_\mu^+ \cup \mathcal{F}$ (for which $\partial\mathcal{C}^+ = \mathbf{D}_\mu^q(0)$), the pleated surface $\mathbf{D}_\mu^q(\tau)$ remains equal to $\partial\mathcal{C}^+$ for all small τ .

Proposition 25 *Given $\mu \in ML$ and $q \in \mathcal{P}_\mu^+ \cup \mathcal{F}$, there exists $\epsilon > 0$, depending on μ and q , such that if $|\tau| < \epsilon$, then $G_\mu^q(\tau) \in \mathcal{Q}\mathcal{F}$ and $\mathbf{D}_\mu^q(\tau)$ is a component of $\partial\mathcal{C}(G_\mu^q(\tau))$.*

Proof: This is proved in [20], prop 8.10 for the case in which the basepoint q is in \mathcal{F} . It is clear that the same proof works in our more general case. \square

We note that if $G_\mu^q(\tau) \in \mathcal{Q}\mathcal{F}$ and $\mathbf{D}_\mu^q(\tau) = \partial\mathcal{C}^+(G_\mu^q(\tau))$, then the flat structure of $\partial\mathcal{C}^+(G_\mu^q(\tau))$ is represented by the Fuchsian group $\mathcal{E}_{(\Re\tau)\mu}(F^+(q))$ obtained by earthquaking a distance $\Re\tau$ along the pull-back of μ to \mathbf{D} . This observation will be important in section 8 below.

8 The local pleating theorem

In this section we prove the local pleating theorem 26. We derive various consequences including the density theorem 4 of the introduction and a detailed description of how pleating varieties meet \mathcal{F} . The statement of the theorem is as follows.

Theorem 26 Local Pleating Theorem *Suppose that $\nu \in ML$ and $q_0 \in \mathcal{P}_\nu \cup \mathcal{F}$. Then there exists a neighborhood U of q_0 in $\mathcal{Q}\mathcal{F}$ such that if $q \in U$ and $\lambda_\nu(q) \in \mathbf{R}^+$, then $q \in \mathcal{P}_\nu \cup \mathcal{F}$.*

Our starting point for proving this theorem is proposition 7.6 of [20], part of whose content can be stated in the following way. We write $G_\gamma^{q_0}(\tau)$ for $G_{\delta_\gamma}^{q_0}(\tau) = \mathcal{Q}_{\delta_\gamma}(\tau)(G(q_0))$.

Proposition 27 *Suppose that $\gamma \in \mathcal{S}$ and $q_0 \in \mathcal{F}$. Then there exists $\eta > 0$ such that if $|\tau| < \eta$, then $G_\gamma^{q_0}(\tau) \in \mathcal{P}_\gamma \cup \mathcal{F}$.*

This proposition can be regarded as the special case of 26 in which $\nu = k\delta_\gamma \in ML_Q$, the basepoint q_0 is Fuchsian and we restrict the discussion to the quakebend plane $\mathcal{Q}_\gamma^{q_0}$ through q_0 . Notice that in this plane, $\lambda_\gamma(G_\gamma^{q_0}(\tau))$ is fixed and hence automatically real.

We begin by reviewing the argument in [20]. Suppose $\gamma \in \mathcal{S}$, let $V \in G$ represent γ and choose $W \in G$ such that (V, W) is a marking. Let $(\lambda_V, \tau_{V,W})$ be complex Fenchel Nielsen coordinates for \mathcal{QF} relative to (V, W) ; thus we regard $(\lambda_V, \tau_{V,W})$ as holomorphic functions on \mathcal{QF} . As described in section 7.1, whenever $\lambda_V = \lambda_V(q) \in \mathbf{R}^+$, the complex Fenchel Nielsen construction determines a pleated surface map $\mathbf{D} \rightarrow \mathbf{H}^3$ with pleating locus γ . To indicate more clearly the relevant variables, we shall write $\mathbf{P}_\gamma(q)$ for the image $\mathbf{D}_\gamma(\lambda_V, \tau_{V,W}) \subset \mathbf{H}^3$.

If $q_0 \in \mathcal{F}$, then $\Im\tau_{V,W}(q_0) = 0$, hence for q near q_0 , $\Im\tau_{V,W}(q)$ is small. In [20], we argued that for $\Im\tau$ sufficiently small, $\mathbf{P}_\gamma(q) = \mathbf{D}_\gamma(\lambda_V, \tau_{V,W})$ is embedded and bounds a convex half space in \mathbf{H}^3 . It follows by proposition 7.2 of [20], that $\mathbf{P}_\gamma(q)$ is a component of $\partial\mathcal{C}(q)$.

There are two problems in applying this argument in the present circumstances. First, we wish to include the case $q_0 \notin \mathcal{F}$, and thus can no longer assume that $\Im\tau_{V,W}$ is small. Second, we want to prove theorem 26 for an irrational lamination ν by taking a limit of rational laminations. Since the constant ϵ of proposition 25 depends on γ and is not uniform, (in fact $\epsilon \sim 2 \exp(-l_\gamma/2)$), the limiting process fails, indicating that we need to scale the approximating laminations properly. To resolve these problems, we digress to study the geometry of the pleated surfaces $\mathbf{P}_\gamma(q)$ more carefully.

Fix $q_0 \in \mathcal{QF}$, $\gamma \in \mathcal{S}$ and a marking (V, W) as above. Suppose that $q \in \mathcal{QF}$ and that $\lambda_V(q) \in \mathbf{R}^+$. Let $\phi_\gamma(q)$ be the normalized Fuchsian group with (real) Fenchel Nielsen coordinates $(\lambda_V(q), \Re\tau_{V,W}(q))$. The surface $\mathbf{P}_\gamma(q)$ is the image of the pleated surface map $\mathbf{D} \rightarrow \mathbf{H}^3$ defined by a pure bend along δ_γ by $i\Im\tau_{V,W}(q)$. We refer to $\phi_\gamma(q)$ as the *flat structure* of $\mathbf{P}_\gamma(q)$.

We can associate a transverse measure $b_\nu(q)$ to $\mathbf{P}_\gamma(q)$ in an obvious way: for any arc σ on $\mathbf{P}_\gamma(q)$ transverse to its pleating locus γ , set $b_\gamma(q)(\sigma) = i(\sigma, \gamma)\Im\tau_{V,W}(q)$. Thus we can also write $\mathbf{P}_\gamma(q) = \mathcal{Q}_{b_\gamma(q)}^p(i)$, where p is the image of $\phi_\gamma(q)$ in \mathcal{F} .

We remark that we are *not* making the assumptions that $\mathbf{P}_\gamma(q)$ is a component of $\partial\mathcal{C}(q)$, or that $\phi_\gamma(q)$ is one of the flat structures $F^\pm(q)$ of $\partial\mathcal{C}(q)$ (see section 4.1); in fact, this is exactly what we must prove. In particular, we cannot assume that $b_\gamma(q)$ is the bending measure $pl^\pm(q)$. The following result, however, gives information about $\phi_\gamma(q)$ and $b_\gamma(q)$ for q near $q_0 \in \mathcal{P}_{\nu_0}$ for irrational ν_0 .

Proposition 28 *Given $\nu_0 \in ML - ML_Q$, and $q_0 \in \mathcal{P}_{\nu_0}^+ \cup \mathcal{F}$, let $F^+(q_0) \in \mathcal{F}$ and $pl^+(q_0)$ be the flat structure and bending measure of $\partial\mathcal{C}^+(q_0)$ respectively. (If $q_0 \in \mathcal{F}$, then $pl^+(q_0) = 0$ and $F^+(q_0)$ is the Fuchsian group representing q_0 .) Then, given neighborhoods V of $F^+(q_0)$ in \mathcal{F} and W of $pl^+(q_0)$ in ML , there exist neighborhoods U of q_0 in \mathcal{QF} and X of $[\nu_0]$ in PML such that if $q \in U$, $[\delta_\gamma] \in X \cap PML_Q$ and $\lambda_\gamma(q) \in \mathbf{R}^+$, then the flat structure $\phi_\gamma(q)$ of $\mathbf{P}_\gamma(q)$ is in V and the transverse measure $b_\gamma(q)$ is in W .*

The idea of the proof of this proposition is that by the convergence lemma 1, for $\nu_0 \in ML - ML_Q$, nearby rational laminations are close in the Hausdorff topology, so that the bending loci and hence the structures of the associated pleated surfaces are also close. The details are a technical modification of the arguments in [19] and are given in appendix 12.2. (We remark that the result is still true for $\nu_0 \in ML_Q$, however the details of the proof differ since the convergence lemma does not apply. We omit this case since it is not needed here.)

The plan of the proof of theorem 26 is the following. The hard case to handle is $\nu \notin ML_Q$. We shall show in theorem 31 below, that if $q_0 \in \mathcal{P}_\nu$, then for q in a neighborhood of q_0 , if $[\delta_\gamma]$ is sufficiently close to $[\nu]$ in PML , the condition $\lambda_\gamma(q) \in \mathbf{R}^+$ implies that $\mathbf{P}_\gamma(q)$ is also a component of $\partial\mathcal{C}$. Theorem 26 then follows by an easy limiting argument using the continuity theorem 11.

We prove theorem 31 using an extension of proposition 25, which we state as proposition 29. Stated roughly it says that if $p \in \mathcal{F}$ and the pleated surface $\mathbf{D}_\mu^p(\tau)$ associated to the quakebend $\mathcal{Q}_\mu^p(\tau)$ is a component of $\partial\mathcal{C}$, then the same is true of any surface $\mathbf{D}_{\mu'}^{p'}(\tau')$ obtained by quakebending a nearby amount τ' from a nearby point $p' \in \mathcal{F}$ along a nearby lamination μ' . Now, a component of $\partial\mathcal{C}$ can be obtained from the Fuchsian group representing its flat structure by a pure bend along the pleating lamination $|\mu|$. Proposition 28 allows us to apply proposition 29 to $\mathbf{P}_\gamma(q)$ for $[\delta_\gamma]$ close to $[\nu]$ and q close to q_0 , thus proving theorem 31.

Proposition 29 *Let $p_0 \in \mathcal{F}$ be represented by $G_0 = G(p_0)$ and suppose that $\tau_0 \in \mathbf{C}$ is such that $q_0 = \mathcal{Q}_{\mu_0}^{p_0}(\tau_0) \in \mathcal{P}_{\mu_0}^+$. Then there exist neighborhoods X, Y and Z of μ_0, p_0 and τ_0 in ML, \mathcal{F} and \mathbf{C} respectively, such that if $\mu \in X$, $p \in Y$ and $\tau \in Z$, then $q = \mathcal{Q}_\mu^p(\tau) \in \mathcal{QF}$ and $\mathbf{D}_\mu^p(\tau)$ is a component of $\partial\mathcal{C}(q)$.*

The proof of this result is identical with the version in [20] once we note that the constants involved depend continuously on G and μ . This follows from the following variant of lemma 8.2 of [20].

Lemma 30 *Let X and Y be compact sets in ML and \mathcal{F} respectively. Then there exist constants $d > 0$ and $K > 0$ such that if $\mu \in X$ and $G \in Y$, and if σ is any geodesic segment on \mathbf{D}/G of length less than d , then $\mu(\sigma) < K$.*

We can now prove theorem 31, which is important in its own right.

Theorem 31 *Suppose $\nu_0 \in ML - ML_Q$ and $q_0 \in \mathcal{P}_{\nu_0}^+ \cup \mathcal{F}$. Then there are neighborhoods U of q_0 in \mathcal{QF} and X of $[\nu_0]$ in PML such that if $[\delta_\gamma] \in PML_Q \cap X$, $q \in U$ and if $\lambda_\gamma(q) \in \mathbf{R}^+$, then $\mathbf{P}_\gamma(q)$ is a component of $\partial\mathcal{C}(q)$.*

Proof: By proposition 28, there are neighborhoods X of $[\nu_0]$ in PML and U of q_0 in \mathcal{QF} such that for $q \in U$ and $[\delta_\gamma] \in X$, the flat structures $F^+(q_0)$ of $\partial\mathcal{C}^+(q_0)$ and $\phi(q)$ of $\mathbf{P}_\gamma(q)$ are close in \mathcal{F} , and the transverse measures $pl^+(q_0)$ and $b_\gamma(q)$ are close in ML .

As remarked earlier, $\partial\mathcal{C}^+(q_0)$ is just the pleated surface obtained from $F^+(q_0)$ under a pure bend by i along the measured lamination $pl^+(q_0)$ while $\mathbf{P}_\gamma(q)$ is obtained from $\phi_\gamma(q)$ by a pure bend by i along $b_\gamma(q)$. The result now follows from proposition 29. \square

We now prove theorem 26.

Proof: Suppose first that $\nu \in ML_Q$. In this case the result follows as in the discussion following proposition 27 above, using proposition 29 as a substitute for the condition $\Im\tau$ near 0 when the base point q_0 is not Fuchsian.

Suppose therefore that $\nu \notin ML_Q$, and pick $\nu_n \in ML_Q$, $\nu_n \rightarrow \nu$. Find neighborhoods U of q_0 in \mathcal{QF} and X of $[\nu]$ in PML satisfying the conclusion of theorem 31.

Assume $q \in U$ and $\lambda_\nu(q) \in \mathbf{R}^+$. Since $\lambda_{\nu_n} \rightarrow \lambda_\nu$ uniformly on U , and since λ_ν is non-constant on U , by Hurwitz's theorem we can find $q_n \in U$, $q_n \rightarrow q$, such that $\lambda_{\nu_n}(q_n) = \lambda_\nu(q)$, and in particular such that $\lambda_{\nu_n}(q_n) \in \mathbf{R}^+$. Applying theorem 31, we see that for sufficiently large n , $\mathbf{P}_{|\nu_n|}(q_n)$ is one of the components of $\partial\mathcal{C}^+(q_n)$ so that $q_n \in \mathcal{P}_{\nu_n}^+$. Hence, by the continuity theorem 11, we get $q \in \mathcal{P}_\nu^+$. This completes the proof of theorem 26. \square

Corollary 32 *Suppose $\mu, \nu \in ML$, $[\mu] \neq [\nu]$. Let $q_0 \in \mathcal{P}_{\mu, \nu}^+ \cup \mathcal{F}$ and let $\mathcal{Q}_\mu^{q_0}$ be the μ -quakebend plane based at q_0 . There exists a neighborhood U of q_0 in $\mathcal{Q}_\mu^{q_0}$ such that if $q \in U$ and $\lambda_\nu(q) \in \mathbf{R}^+$, then $q \in \mathcal{P}_{\mu, \nu} \cup \mathcal{P}_{\nu, \mu} \cup \mathcal{F}$.*

Proof: This is just theorem 26 applied in the quakebend plane $\mathcal{Q}_\mu^{q_0}$. We can prove it either by applying proposition 25 to see that for $q \in \mathcal{Q}_\mu^{q_0}$ near q_0 , we have $q \in \mathcal{P}_\mu \cup \mathcal{F}$, and then applying theorem 26 to ν ; or by noting that since λ_μ is constant on $\mathcal{Q}_\mu^{q_0}$ and real valued at q_0 , we can apply theorem 26 first to μ and then to ν . \square

Remark 33 The condition $\lambda_\gamma(q) \in \mathbf{R}^+$ is key in proposition 28 and in theorem 31. We can always find a pleated surface Π whose pleating locus σ contains the geodesic γ . In general, however, σ properly contains γ and has leaves spiralling into γ , and thus carries no transverse measure. Then, even though $[\delta_\gamma]$ is near $[pl^\pm]$ in PML , the pleated surface \mathbf{P}_γ realizing γ (see [7],[45]) is not necessarily embedded; moreover, even if it is, neither of the half spaces it bounds in \mathbf{H}^3 will be convex. The point is that the condition $\sigma = \gamma$ is equivalent to $\lambda_\gamma(q) \in \mathbf{R}^+$.

8.1 Consequences of theorem 26

From theorem 26 we obtain the following local extension of the picture of Fuchsian space described in section 3.

Theorem 34 *Let $\mu, \nu \in ML$, $i(\mu, \nu) > 0$, $p \in \mathcal{F}$. Then there is a neighborhood U of p in \mathcal{QF} such that*

- (1) *if $p \notin \mathcal{F}_{\mu, \nu}$ then $\mathcal{P}_{\mu, \nu} \cap U = \emptyset$, while*
- (2) *if $p \in \mathcal{F}_{\mu, \nu}$ then the intersection of the \mathbf{R} -loci $\lambda_\mu^{-1}(\mathbf{R}) \cap \lambda_\nu^{-1}(\mathbf{R})$ with U is exactly*

$$(\mathcal{P}_{\mu, \nu} \cup \mathcal{P}_{\nu, \mu} \cup \mathcal{F}) \cap U.$$

In the second case, let $p = p(\mu, \nu, c) \in \mathcal{F}_{\mu, \nu}$, let \mathcal{Q}_μ^p be the quakebend plane along μ based at p and let $V = U \cap \mathcal{Q}_\mu^p$. Then $\lambda_\nu|_V$ has a simple critical point at p and $\lambda_\nu^{-1}(\mathbf{R}^+) \cap (V - \mathcal{F})$ has exactly two components, one lying in $\mathcal{P}_{\mu, \nu}$ and the other in $\mathcal{P}_{\nu, \mu}$.

Proof: Part 1 follows since for $p \notin \mathcal{F}_{\mu, \nu}$, there exists a neighborhood U of p in the quakebend plane \mathcal{Q}_μ^p based at p such that $\lambda_\nu^{-1}(\mathbf{R}^+) \cap U \subset \mathcal{F}$. The first statement in part 2 is immediate from Theorem 26.

By Corollary 21, $\lambda_\nu|_{\mathcal{E}_{\mu, c}}$ has exactly one critical point at p and it is simple. Thus the second statement in part 2 is a restatement of corollary 32 with $q_0 = p$. \square

We note that this theorem provides an alternative proof of theorem 13.

We can also now prove the density theorem 4 of the introduction. First, we need a bound on the bending angle in a quakebend plane.

Proposition 35 *Suppose $\mu \in ML$, $q \in \mathcal{P}_\mu \cup \mathcal{F}$ and let \mathcal{Q}_μ^q be the quakebend plane along μ based at q with parameter $\tau = \tau_\mu$. Given $K > 0$, there exist $B_2 > B_1 > 0$ such that if $|\Re\tau| < K$ and $B_2 > \Im\tau > B_1$, then $\mathcal{Q}_\mu^q(\tau) \notin \mathcal{P}_\mu$.*

We need the restriction $B_1 < \Im\tau < B_2$ rather than simply $\Im\tau > B_1$ because of the periodicity of the twist parameter τ_μ for rational laminations. The period is $2\pi i$ when $\mu = \delta_\gamma, \gamma \in \mathcal{S}$. The statement $\mathcal{Q}_\mu^q(\tau) \notin \mathcal{P}_\mu$ means that either $\mathcal{Q}_\mu^q(\tau) \notin \mathcal{QF}$ or that $\mathcal{Q}_\mu^q(\tau) \in \mathcal{QF}$ but $[pl^+(\mathcal{Q}_\mu^q(\tau))] \neq [\mu]$. We show that, under the hypotheses of the proposition, $\mathcal{Q}_\mu^q(\tau)$ fails to be in \mathcal{P}_μ because the surface obtained by bending along μ is not embedded. In this situation, it may or may not be true that $\mathcal{Q}_\mu^q(\tau) \in \mathcal{QF}$. The proof is given in appendix 12.3, see also [31] theorem 6.2.

As an immediate corollary we have

Proposition 36 *Suppose $q \in \mathcal{QF}$, $q \in \mathcal{P}_{\mu,\nu} \cup \mathcal{F}$. Then the holomorphic function $\lambda_\nu(q)$ is non-constant on $\mathcal{Q}_\mu^q \cap \mathcal{QF}$.*

Proof: Since $q \in \mathcal{P}_\nu$ we know $\lambda_\nu(q) \in \mathbf{R}^+$. By construction $\lambda_\mu(q) = c > 0$ for all $q \in \mathcal{Q}_\mu^q$. Suppose that $\lambda_\nu(q) = d > 0$ for all $q \in \mathcal{Q}_\mu^q \cap \mathcal{QF}$. By theorem 26, $\mathcal{P}_{\mu,\nu}$ is open in \mathcal{Q}_μ^q .

Now suppose that $q_n = \mathcal{Q}_\mu^q(\tau_n) \in \mathcal{P}_{\mu,\nu}$ and that $\tau_n \rightarrow \tau_\infty$. Since $l_\mu(q_n) = c$ and $l_\nu(q_n) = d$ for all n , it follows from theorem 15 that $q_n \rightarrow q_\infty \in \mathcal{QF}$. By theorem 11, $q_\infty \in \mathcal{P}_{\mu,\nu} \cup \mathcal{F}$. Clearly, $q_\infty = \mathcal{Q}_\mu^q(\tau_\infty)$ and so $\mathcal{P}_{\mu,\nu}$ is closed in $\mathcal{Q}_\mu^q - \mathcal{F}$. Therefore $\mathcal{P}_{\mu,\nu}$ is a connected component of $\mathcal{Q}_\mu^q - \mathcal{F}$ and must be one of the half planes $\Im\tau_\mu > 0$ or $\Im\tau_\mu < 0$, contradicting proposition 35. \square

Finally we can prove theorem 4.

Theorem 4 *The rational pleating varieties $\mathcal{P}_{\mu,\nu}, \mu, \nu \in ML_Q$ are dense in \mathcal{QF} .*

Proof: Let $q \in \mathcal{QF}$ and let $\mu \in [pl^+(q)], \nu \in [pl^-(q)]$. By theorem 23, $\lambda_\mu(q), \lambda_\nu(q) \in \mathbf{R}^+$. Clearly, we may as well assume $\mu \notin ML_Q$. Find a sequence $\{\mu_n\} \in ML_Q, \mu_n \rightarrow \mu$. By Hurwitz's theorem in \mathcal{QF} , we can find points $q_n \rightarrow q$ with $\lambda_{\mu_n}(q_n) \in \mathbf{R}^+$ and so by theorem 31, $q_n \in \mathcal{P}_{\mu_n}$ for large enough n . If $\nu \in ML_Q$ we are done, otherwise find $\{\nu_n\} \in ML_Q, \nu_n \rightarrow \nu$. By proposition 36, λ_{ν_n} is non-constant on $\mathcal{Q}_{\mu_n}^{q_n} \cap \mathcal{QF}$ and we can apply Hurwitz's theorem again in $\mathcal{Q}_{\mu_n}^{q_n} \cap \mathcal{QF}$ to find q'_n near q_n , such that $q'_n \rightarrow q$ and such that $\lambda_{\nu_n}(q'_n) \in \mathbf{R}^+$. By theorem 31 again, $q'_n \in \mathcal{P}_{\mu_n, \nu_n}$ for large enough n . \square

9 Pleating rays and planes

In this section, we apply the local and limit pleating theorems to prove our main results theorems 6 and 2 of the introduction.

Recall from section 4.2 the definition of the pleating ray

$$\mathcal{P}_{\mu,\nu,c} = \{q \in \mathcal{P}_{\mu,\nu} : l_\mu(q) = c\},$$

where $(\mu, \nu) \in ML \times ML$, and $c > 0$. Pleating rays are the basic building blocks out of which we construct pleating planes and the BM -slices mentioned in the introduction. Notice that, because of theorem 23, we can equally well define

$$\mathcal{P}_{\mu,\nu,c} = \{q \in \mathcal{P}_{\mu,\nu} : \lambda_\mu(q) = c\}.$$

Our results will justify the names “rays” and “planes”.

The main work is in the study of the pleating rays. Our strategy is as follows. We begin by applying the limit pleating theorem and the local pleating theorem to obtain some general results about $\mathcal{P}_{\mu,\nu}$ for arbitrary $\mu, \nu \in ML$. We then prove theorem 6 in the case where $[\mu] = [\delta_\gamma], [\nu] = [\delta_{\gamma'}]$ and (γ, γ') is a marking for \mathcal{T}_1 . We show that in this case $\mathcal{P}_{\delta_\gamma, \delta_{\gamma'}, c}$, which we call an *integral pleating ray*, is a straight line segment in the quakebend plane $\mathcal{Q}_{\gamma,c}$. Using the integral rays we derive constraints on the rays $\mathcal{P}_{\delta_\gamma, \nu, c} \subset \mathcal{Q}_{|\mu|,c}$ for arbitrary ν ; using our general results we are then able to deduce theorem 6 in the general case. Finally, we apply theorem 6 to deduce theorem 2.

9.1 Pleating rays

In the four lemmas which follow, μ, ν are arbitrary laminations in ML and, as usual, \mathcal{Q}_μ^q denotes the μ -quakebend plane through $q \in \mathcal{P}_\mu \cup \mathcal{F}$.

Lemma 37 *Let $q \in \mathcal{P}_{\mu,\nu}$. The set $\mathcal{P}_{\mu,\nu} \cap \mathcal{Q}_\mu^q$ is a union of connected components of the \mathbf{R} -locus of λ_ν in $(\mathcal{Q}\mathcal{F} - \mathcal{F}) \cap \mathcal{Q}_\mu^q$.*

Proof: We have to show that $\mathcal{P}_{\mu,\nu} \cap \mathcal{Q}_\mu^q$ is open and closed in the \mathbf{R} -locus of λ_ν in $(\mathcal{Q}\mathcal{F} - \mathcal{F}) \cap \mathcal{Q}_\mu^q$. The openness is the local pleating theorem 26 and closure follows by the continuity theorem 11. \square

If $\nu = k\delta_\gamma, \gamma \in \mathcal{S}$, we obtain a stronger result. Let $V \in G$ represent γ . In this case, by proposition 24, trace $\text{Tr } V$ is defined and holomorphic on all of \mathcal{Q}_μ^q (including the part outside $\mathcal{Q}\mathcal{F}$), and we obtain a version of lemma 37 for

the \mathbf{R} -locus of λ_γ in \mathcal{Q}_μ^q . Define the hyperbolic locus of γ in \mathcal{Q}_μ^q as $\{q \in \mathcal{Q}_\mu^q : \text{Tr } V \in \mathbf{R}, |\text{Tr } V| > 2\}$.

Lemma 38 *Let $\nu = k\delta_\gamma \in ML_Q$ and let $q \in \mathcal{P}_{\mu,\nu}$. Let $V \in G$ represent γ . Then the set $\mathcal{P}_{\mu,\nu} \cap \mathcal{Q}_\mu^q$ is a union of connected components of the hyperbolic locus of $\text{Tr } V$ in $\mathcal{Q}_\mu^q - \mathcal{F}$.*

Proof: The openness follows as above, using the local pleating theorem 26. The closure follows from theorem 15. The point is first, that length and trace are related by the trace formula $\text{Tr } V = 2 \cosh(l_\gamma/2)$, and second, that if we reach a limit point at which $|\text{Tr } V| > 2$, then $l_\gamma > 0$ so that by the second part of theorem 15 we must still be in $\mathcal{Q}\mathcal{F}$. (See [18] proposition 5.4 for a more elementary proof without using theorem 15.) \square

This is a strong result. The point is, that starting from a point we know is in $\mathcal{Q}\mathcal{F}$, the lemma asserts that if we move along branches of the hyperbolic locus, then we stay in $\mathcal{Q}\mathcal{F}$ until we reach a boundary point of $\partial\mathcal{Q}\mathcal{F}$ at which $|\text{Tr } V| = 2$. This observation is what makes it possible to use the pleating invariants for computations of $\partial\mathcal{Q}\mathcal{F}$, see theorem 3 of the introduction.

With the notation of lemma 37, set $c = \lambda_\mu(q)$. Clearly, $\mathcal{P}_{\mu,\nu} \cap \mathcal{Q}_\mu^q = \mathcal{P}_{\mu,\nu,c}$. As usual, we let $p_{\mu,\nu,c} \in \mathcal{F}_{\mu,\nu}$ be the minimal point for the length function l_ν on the earthquake path $\mathcal{E}_{\mu,c}$. The following lemma makes essential use of theorem 15.

Lemma 39 *Let $q \in \mathcal{P}_{\mu,\nu}$ and let $c = \lambda_\mu(q)$. The image of each component of $\mathcal{P}_{\mu,\nu} \cap \mathcal{Q}_\mu^q$ under the map λ_ν is an interval of the form $(0, \infty)$, $(0, d)$ and (d, ∞) where $d = f_{\mu,\nu}(c) = l_\nu(p_{\mu,\nu,c})$. Moreover, there is at most one component of $\mathcal{P}_{\mu,\nu} \cap \mathcal{Q}_\mu^q$ whose image is $(0, d)$; the closure of such a component meets \mathcal{F} exactly in $p(\mu, \nu, c)$.*

Remark 40 As we shall see in theorem 6, in fact $\mathcal{P}_{\mu,\nu} \cap \mathcal{Q}_\mu^q$ has a unique component, and the image of this component is $(0, d)$.

Proof: Let K be a connected component of $\mathcal{P}_{\mu,\nu,c}$. By theorem 23, $\lambda_\nu|_K$ is real valued and, by proposition 36, it is non-constant on \mathcal{Q}_μ^q . Since it is holomorphic, it is not locally constant and thus not constant on K . Therefore by lemma 37 the image I_K of $\lambda_\nu|_K$ is an open interval in \mathbf{R}^+ .

Suppose that $r \in \mathbf{R}^+$ and that there is a sequence $\{q_n\} \in K$ such that $\lambda_\nu(q_n) \rightarrow r$. Since $\lambda_\mu(q_n) = c$, by theorem 15 a subsequence of $\{G(q_n)\}$ has an algebraic limit G_∞ . Furthermore, since $\lambda_\nu(q_n) \rightarrow r > 0$, the group G_∞ is represented by a point $q \in \mathcal{Q}\mathcal{F}$ such that $\lambda_\nu(q) = r$. If $q \in \mathcal{Q}\mathcal{F} - \mathcal{F}$ then by theorem 11, $q \in K$ so that $r \in I_K$. On the other hand, if $q \in \mathcal{F}$ then by

theorem 34, $q = p(\mu, \nu, c)$ and $r = \lambda_\nu(q) = f_{\mu, \nu}(c) = d$. Thus $\lambda_\nu(K)$ is open and closed in $(0, d) \cup (d, \infty)$. The result follows from theorem 34. \square

Lemma 41 *Let $q \in \mathcal{P}_{\mu, \nu}$ and let $c = \lambda_\mu(q)$. Let τ denote the quakebend parameter in the quakebend plane \mathcal{Q}_μ^q . Suppose that the points $q_n \in \mathcal{P}_{\mu, \nu, c}$ are represented by the quakebend parameter τ_n and that $\lambda_\nu(q_n) \rightarrow \infty$. Then $|\Re(\tau_n)| \rightarrow \infty$.*

Proof: Since $q_n \in \mathcal{P}_{\mu, \nu, c}$ we know $\lambda_\nu(q)$ is real. Moreover, $\lambda_\nu(q) \leq l_\nu(F^+(q_n))$; that is, $\lambda_\nu(q)$ is bounded above by the length of ν on the flat structure of $\partial\mathcal{C}^+/G(q_n)$. This flat structure is determined by the length of μ , which is fixed, and the earthquake parameter $\Re(\tau_n)$. Thus if $|\Re(\tau_n)|$ is bounded, so is $\lambda_\nu(q_n)$. \square

We can now start investigating the integral pleating rays. Suppose that $[\mu] = [\delta_\gamma], [\nu] = [\delta_{\gamma'}]$ and (γ, γ') is a marking for \mathcal{T}_1 . For simplicity, we write \mathcal{P}_γ for $\mathcal{P}_{\delta_\gamma}$ and so on. Let $(\lambda_V, \tau_{V,W}) \in \mathbf{C}^2$ be complex Fenchel Nielsen coordinates relative to a marked pair of generators (V, W) corresponding to (γ, γ') . As in section 7.3, we denote by $\mathcal{Q}_{\gamma, c}$ the slice $\{(c, \tau)\} \subset \mathbf{C}^2$; $\mathcal{Q}_{\gamma, c}$ is the quakebend plane along γ that meets \mathcal{F} along the earthquake path $\mathcal{E}_{\gamma, c}$. We denote points in this slice simply by the parameter $\tau = \tau_{\delta_\gamma}$. As usual, $\tau = 0$ corresponds to the point $p(\gamma, \gamma', c) \in \mathcal{F}$, while $\Im\tau = 0$ is the earthquake path $\mathcal{E}_{\gamma, c}$.

For $m \in \mathbf{Z}$, the pair $(\gamma, \gamma^m \gamma')$ is a pair of marked generators for $\pi_1(\mathcal{T}_1)$ corresponding to the pair of generators $V, V^m W$ for G . Clearly $\mathcal{P}_{\gamma, \gamma^m \gamma', c} \subset \mathcal{Q}_{\gamma, c}$. The generators V, VW are obtained from the pair V, W by the map induced by a Dehn twist about γ . The basepoint relative to which we measure the twist parameter changes and we find $\tau_{V, VW} = \tau_{V, W} + \lambda_V$; similarly, $\tau_{V, V^m W} = \tau_{V, W} + m\lambda_V$.

The following formula is derived in [39] for any pair (V, W) of marked generators for G :

$$\cosh \frac{\tau_{V, W}}{2} = \pm \cosh \frac{\lambda_W}{2} \tanh \frac{\lambda_V}{2} \quad (2)$$

By our conventions, $\Re\lambda_V, \Re\lambda_W > 0$, so that we should choose the $+$ sign on \mathcal{F} and hence everywhere in $\mathcal{Q}\mathcal{F}$.

Applying this formula to the generators $(V, V^{-m}W)$ we find

$$\cosh \frac{\tau - m\lambda_V}{2} = \cosh \frac{\lambda_{V^{-m}W}}{2} \tanh \frac{\lambda_V}{2}. \quad (3)$$

In particular, at $\tau = mc$ we have

$$1 = \cosh \frac{\lambda_{\gamma-m\gamma'}}{2} \tanh \frac{\lambda_{\gamma}}{2}. \quad (4)$$

or equivalently

$$\sinh \frac{c}{2} \sinh \frac{\lambda_{\gamma-m\gamma'}}{2} = 1. \quad (5)$$

Proposition 42 *Let (γ, γ') be a marked pair of generators for $\pi_1(\mathcal{T}_1)$ and let $c > 0$. Then for $m \in \mathbf{Z}$, $\mathcal{P}_{\gamma, \gamma-m\gamma', c}$ and $\mathcal{P}_{\gamma-m\gamma', \gamma, c}$ are the two line segments $\Re\tau = mc$, $|\Im\tau| < 2 \arccos \tanh \frac{c}{2}$ in $\mathcal{Q}_{\gamma, c}$. The two line segments $\Re\tau = mc$, $|\Im\tau| \geq 2 \arccos \tanh \frac{c}{2}$ in $\mathcal{Q}_{\gamma, c}$ have empty intersection with \mathcal{QF} .*

Remark 43 Which of the two segments corresponds to $\mathcal{P}_{\gamma, \gamma-m\gamma', c}$ and which to $\mathcal{P}_{\gamma-m\gamma', \gamma, c}$ depends on our convention for measuring τ and is not important here.

Proof: Because $\tau_{V, V-mW} = \tau_{V, W} - mc$, we may restrict ourselves to the case $m = 0$. From lemma 38, $\mathcal{P}_{\gamma, \gamma'}$ is a union of connected components of the hyperbolic locus of γ' in $\mathcal{Q}_{\gamma, c} - \mathcal{F}$, and by theorem 34 there is a unique component K whose closure meets the critical line $\mathcal{F}_{\gamma, \gamma'}$ in $p(\gamma, \gamma', c)$.

From equation (2) ,

$$\cosh \frac{\tau}{2} = \cosh \frac{\lambda_{\gamma'}}{2} \tanh \frac{\lambda_{\gamma}}{2}.$$

Thus the \mathbf{R} -locus of $\lambda_{\gamma'}$ in $\mathcal{Q}_{\gamma, c}$ is the set defined by $\cosh \frac{\tau}{2} \in \mathbf{R}$, or equivalently, $\{\Re\tau = 0\} \cup \{\Im\tau = 0\}$. The real axis $\Im\tau = 0$ corresponds to $\mathcal{E}_{\gamma, c} = \mathcal{Q}_{\gamma, c} \cap \mathcal{F}$ and we see easily (see lemma 38) that the connected components of the hyperbolic locus of γ' in $\mathcal{Q}_{\gamma, c} - \mathcal{F}$ which meet the real axis are the two segments $0 < |\Im\tau| < 2 \arccos \tanh \frac{c}{2}$. One of these segments must be the component K and the other is the corresponding component for $\mathcal{P}_{\gamma', \gamma}$. Each of these segments is mapped bijectively by $\lambda_{\gamma'}$ to $[0, 2 \arccos \tanh \frac{c}{2})$.

Now on the imaginary axis, we have $\cosh \frac{\lambda_{\gamma'}}{2} \leq (\tanh \frac{c}{2})^{-1}$, and hence by lemma 39, $\mathcal{P}_{\gamma, \gamma', c}$ and $\mathcal{P}_{\gamma', \gamma, c}$ have no other components.

Finally we have to show that no other points on the imaginary axis lie in \mathcal{QF} . Equation (3) holds for groups in $\mathcal{Q}_{\gamma, c}$ even when they are outside \mathcal{QF} . On this axis, therefore, we always have

$$-1 \leq \cosh \frac{\lambda_{\gamma'}}{2} \tanh \frac{c}{2} \leq 1.$$

In [39] proposition 6.2, it is shown by a direct argument that if $\lambda_{\gamma'} \in \mathbf{R}$ and the above inequality is strict, then the group generated by V, W is quasifuchsian

and contained in $\mathcal{P}_{\gamma, \gamma'}$. Moreover, in this situation, this group is determined by λ_γ and $\lambda_{\gamma'}$ up to conjugacy. If equality holds, the group represents the unique point $p(\gamma, \gamma', c) \in \mathcal{F}$. These are therefore the groups we have already discussed.

Since $\cosh \frac{\lambda_{\gamma'}}{2} \in \mathbf{R}$, the only other possibility is that $\lambda_{\gamma'}$ is purely imaginary. In this case the corresponding group element would have to be elliptic which is impossible in \mathcal{QF} . \square

We can now obtain a bound on the pleating rays $\mathcal{P}_{\gamma, \nu, c}$ for arbitrary $\nu \in ML$.

Corollary 44 *Let $\nu \in ML$, $i(\nu, \gamma) > 0$. Then $|\Re\tau|$ is bounded on each component of $\mathcal{P}_{\gamma, \nu, c}$, where τ denotes the quakebend parameter τ_{δ_γ} in $\mathcal{Q}_{\gamma, c}$.*

Proof: If along some component of $\mathcal{P}_{\gamma, \nu, c}$ in $\mathcal{Q}_{\gamma, c}$, $|\Re\tau| \rightarrow \infty$, the component would have to intersect infinitely many of the lines $\tau = mc + i\theta, \theta \in \mathbf{R}$. According to proposition 42, however, each such line is the union of the integral pleating rays $\mathcal{P}_{\gamma, \gamma^{-m}\gamma', c}$, $\mathcal{P}_{\gamma^{-m}\gamma', \gamma, c}$, the point $p(\gamma, \gamma^{-m}\gamma', c) \in \mathcal{F}$, and points not in \mathcal{QF} . This is impossible. \square

We can now prove theorem 6 on the structure of the pleating rays. Recall from section 7.3 that $\mathcal{Q}_{\mu, c}$ is the quakebend plane along μ which meets \mathcal{F} along the earthquake path $\mathcal{E}_{\mu, c}$.

Theorem 6 *Let μ, ν be measured laminations on \mathcal{T}_1 with $i(\mu, \nu) > 0$ and let $c > 0$. Then the set $\mathcal{P}_{\mu, \nu, c} \subset \mathcal{QF}$ on which $[pl^+] = [\mu]$, $[pl^-] = [\nu]$ and $l_\mu = c$, is a non-empty connected non-singular component of the \mathbf{R}^+ -locus of the restriction of λ_ν to $\mathcal{Q}_{\mu, c}$. The restriction of λ_ν to $\mathcal{P}_{\mu, \nu, c}$ is a diffeomorphism onto its image $(0, f_{\mu, \nu}(c)) \subset \mathbf{R}^+$.*

Proof: We assume first that $\mu \in ML_Q$; without loss of generality we may take $\mu = \delta_\gamma$, $\gamma \in \mathcal{S}$. Let $c > 0$ and let K be a component of $\mathcal{P}_{\gamma, \nu, c}$. By corollary 44, $|\Re\tau|$ is bounded on K . By lemma 41, $\lambda_\nu|_K$ is bounded and hence by lemma 39 the image is the interval $(0, d)$ where $d = f_{\gamma, \nu}(c)$. Moreover, there exist points $\tau_n \in K$, $\tau_n \rightarrow p(\gamma, \nu, c) \in \mathcal{F}_{\gamma, \nu}$.

Now by theorem 34, there is only one branch of $\lambda_\nu^{-1}(\mathbf{R}^+)$ near $p(\gamma, \nu, c)$; thus if the degree of $\lambda_\nu|_K$ were greater than one, there would be points $\tau'_n \in K$ with $\lambda_\nu(\tau'_n) \rightarrow d$, but with $\tau'_n \rightarrow q_\infty \in \mathcal{QF} - \mathcal{F}$. Then, by lemma 39, $\lambda_\nu(K) \supset (0, \infty)$, which is impossible.

Now we remove the restriction that $\mu \in ML_Q$. Suppose that $q \in \mathcal{P}_{\mu, \nu, c}$. We have to replace the plane $\mathcal{Q}_{\gamma, c}$ by the plane \mathcal{Q}_μ^q , in which we denote the quake-

bend parameter τ_μ by τ . Because there are no integral pleating rays if μ is irrational, we need another argument to bound $\Re\tau$.

Choose a sequence $\nu_n \in ML_Q$ such that $\nu_n \rightarrow \nu$. By theorem 20 the holomorphic function $\lambda_\nu(q)$ is continuous in ν and by proposition 36 it is nonconstant. Thus we can apply Hurwitz's theorem in \mathcal{Q}_μ^q to find $q_n \in \mathcal{Q}_\mu^q$ such that $q_n \rightarrow q$ and $\lambda_{\nu_n}(q_n) \in \mathbf{R}^+$. By theorem 31, for large enough n , $q_n \in \mathcal{P}_{\mu, \nu_n, c}$. Now because $\nu_n \in ML_Q$, we can apply the argument above with the roles of μ and ν_n reversed to deduce that $\lambda_\mu(q_n) < f_{\nu_n, \mu}(\lambda_{\nu_n}(q_n))$. Thus, since $f_{\nu_n, \mu}$ is monotonic decreasing, we have that $\lambda_{\nu_n}(q_n) < f_{\nu_n, \mu}^{-1}(\lambda_\mu(q_n))$. Since $f_{\nu_n, \mu}^{-1} = f_{\mu, \nu_n}$ (from the definition of $f_{\nu_n, \mu}$) we conclude that $\lambda_{\nu_n}(q_n) < f_{\mu, \nu_n}(c)$.

Because $\nu_n \rightarrow \nu$, by corollary 8 and theorem 20 we have $f_{\mu, \nu_n}(c) \rightarrow f_{\mu, \nu}(c)$ so that $\{\lambda_{\nu_n}(q_n)\}$ is bounded by a constant depending only on μ, ν and c . The remainder of the argument is as before. \square

As an immediate corollary we have

Corollary 45 *If $q \in \mathcal{P}_\mu$, then $G(q)$ is obtained from a group $G(p)$, $p \in \mathcal{F}$ by a quakebend $\mathcal{Q}_\mu^p(\tau^*)$ along μ . Moreover, there is a quakebend path $\sigma : [0, 1] \rightarrow \mathbf{C}$ in \mathcal{QF} from p to q , or, in the coordinate of $\mathcal{Q}_\mu^p(\tau)$, $\sigma(0) = 0$, $\sigma(1) = \tau^*$ and $\mathcal{Q}_\mu^p(\sigma(t)) \in \mathcal{QF}$, $0 \leq t \leq 1$.*

This settles the question about uniqueness of quakebend planes raised at the end of section 7.2.

Remark 46 In [18], we studied the *Maskit slice* for punctured tori in terms of pleating rays with a similar definition to the above. In particular, theorem 7.2 of [18], asserts a non-singularity result similar to that in theorem 6. It has been pointed out to us by Y. Komori that our proof in [18] in the case of rays $\nu \notin ML_Q$ is incorrect. In fact, we need an openness result like theorem 26 above. The methods above also prove the important result, omitted in [18], that the range of the length function on an irrational ray in the Maskit slice is $(0, \infty)$. We refer to [25] for a corrected version of the argument in [18].

9.2 Pleating planes

We are finally able to prove theorem 2 on the structure of the pleating varieties $\mathcal{P}_{\mu, \nu}$. As in the introduction, let $L_{\mu, \nu} : \mathcal{QF} \rightarrow \mathbf{C}^2$ be the map $q \mapsto (\lambda_\mu(q), \lambda_\nu(q))$.

Theorem 2 *Let (μ, ν) be measured laminations on \mathcal{T}_1 with $i(\mu, \nu) > 0$. Then the set $\mathcal{P}_{\mu, \nu} \subset \mathcal{QF}$ on which $[pl^+] = [\mu]$, $[pl^-] = [\nu]$ is a non-empty connected*

non-singular component of the \mathbf{R}^2 -locus in $\mathcal{QF} - \mathcal{F}$ of the function $L_{\mu,\nu}$. The restriction of $L_{\mu,\nu}$ to $\mathcal{P}_{\mu,\nu}$ is a diffeomorphism to the open region under the graph of the function $f_{\mu,\nu}$ in $\mathbf{R}^+ \times \mathbf{R}^+$.

Proof: By theorem 23, the map $L_{\mu,\nu}|_{\mathcal{P}_{\mu,\nu}}$ takes values in $\mathbf{R}^+ \times \mathbf{R}^+$. That $L_{\mu,\nu}$ restricted to $\mathcal{P}_{\mu,\nu}$ is injective follows immediately from the injectivity of λ_ν on each pleating ray $\mathcal{P}_{\mu,\nu,c}$. Hence, $\mathcal{P}_{\mu,\nu}$ is a non-singular \mathbf{R}^2 -locus in $\mathcal{QF} - \mathcal{F}$. The statement about the image of $L_{\mu,\nu}$ follows from theorem 6. \square

We remark that a similar proof shows that $\mathcal{P}_{\mu,\nu}$ and $\mathcal{P}_{\nu,\mu}$ are the unique connected components of the \mathbf{R} -locus of $L_{\mu,\nu}$ in $\mathcal{QF} - \mathcal{F}$ whose closure in \mathcal{QF} meets \mathcal{F} in $\mathcal{F}_{\mu,\nu}$.

We also remark that if in theorem 2 we replace μ, ν by $\mu' = s\mu, \nu' = t\nu, s, t, \in \mathbf{R}^+$, then $\mathcal{P}_{\mu,\nu}$ is unchanged and the length function $L_{\mu',\nu'}$ is simply a rescaling of $L_{\mu,\nu}$:

$$L_{\mu',\nu'}(q) = (s\lambda_\mu(q), t\lambda_\nu(q)).$$

Our main result, theorem 1, that a group in \mathcal{QF} is characterized by its pleating invariants, uniquely up to conjugation in $PSL(2, \mathbf{C})$, is an immediate consequence of theorem 2.

9.3 Relation to Otal's theorem

In [37] and later [5], Bonahon and Otal study spaces of various topological types of 3-manifolds with a hyperbolic structure \mathbf{H}^3/G such that $\partial\mathcal{C}(G)$ is a pleated surface with (in our terminology) a fixed rational pleating lamination. Translated to our situation, this means the study of a rational pleating plane $\mathcal{P}_{\gamma,\gamma'}$ for fixed $\gamma, \gamma' \in \mathcal{S}$. Write $pl^+ = \theta\delta_\gamma, pl^- = \theta'\delta_{\gamma'}, \theta, \theta' \in \mathbf{R}$. A special case of their results shows that the map $\Theta(q) = (\theta(q), \theta'(q))$ is a homeomorphism from $\mathcal{P}_{\gamma,\gamma'}$ to an open neighborhood of $(0, 0)$ in $(0, \pi) \times (0, \pi)$.

Our methods prove that the map Θ is open and proper; we have thus far however, been unable to derive injectivity by our methods. (For the special case $i(\gamma, \gamma') = 1$, see [39], theorem 3.6.)

Note however that if $q_n \in \mathcal{P}_{\gamma,\gamma'}, q_n \rightarrow p \in \mathcal{F}$, then $\Theta(q_n) \rightarrow (0, 0)$ so the whole critical line $\mathcal{F}_{\gamma,\gamma'}$ appears on the boundary of this Bonahon-Otal embedding as a single point.

10 BM-slices

In this section we study what happens when we fix the pleating invariants on one side of $\partial\mathcal{C}$. The slices thus defined turn out to be the complex extensions of the earthquake paths into \mathcal{QF} .

The space of marked conformal structures on \mathcal{T}_1 can be identified with the space \mathcal{F} . For $q \in \mathcal{QF}$, let $w^\pm(q)$ denote the marked conformal structures of $\Omega^\pm/G(q)$. Bers used the embedding $q \mapsto (w^+, w^-)$ of \mathcal{QF} into $\mathcal{F} \times \overline{\mathcal{F}}$ to find holomorphic coordinates for \mathcal{F} by fixing the second factor w^- and proving that w^+ varies over \mathcal{F} ; this is called the *Bers embedding* of \mathcal{F} . (Recall that the orientation and hence the marking on $\Omega^-/G(q)$ is reversed; this is why in the second factor we write $\overline{\mathcal{F}}$.) Maskit, on the other hand, fixed a curve γ on \mathcal{T}_1 and studied the family of groups on $\partial\mathcal{QF}$ for which $\lambda_\gamma = 0$ and the corresponding element $V \in G$ is an accidental parabolic. These groups are known as *cusps*. The conformal structure w^- is then fixed and represents a family of thrice punctured spheres; Maskit proved that the first coordinate w^+ varies so as to define an embedding of \mathcal{F} into \mathbf{C} . We studied the pleating invariants for this Maskit embedding of \mathcal{F} in detail in [18]. McMullen [31], defines coordinates for Bers embeddings of \mathcal{QF} that extend to Maskit and generalized Maskit embeddings on $\partial\mathcal{QF}$. On the Maskit embeddings his coordinates agree with the pleating invariants of [18].

In terms of Thurston's ending invariants [45], [35], both constructions correspond to holding the ending invariant of one side fixed and allowing the other to vary. It is thus natural to ask what happens when, instead of fixing an ending invariant, we fix the pleating invariants of one side.

Let $\mu \in ML$, $c \in \mathbf{R}^+$ and set

$$BM_{\mu,c}^+ = \{q \in \mathcal{P}_\mu^+ : \lambda_\mu(q) = c\}.$$

On $BM_{\mu,c}^+$, neither the conformal structure on Ω^+/G nor the flat structure on $\partial\mathcal{C}^+/G$ are fixed. They are, however, constrained by the condition $\lambda_\mu(q) = c$. We define

$$J : BM_{\mu,c}^+ \rightarrow (PML - \{[\mu]\}) \times \mathbf{R}^+,$$

by

$$J(q) = \left([pl^-(q)], \frac{l_{pl^-(q)}}{i(\mu, pl^-(q))} \right).$$

Since $[pl^-(q)] \neq [pl^+(q)]$, $i(\mu, pl^-(q)) > 0$. The map J is continuous by theorem 11. Since for fixed $\mu \in ML$, the functions l_ν and $i(\mu, \nu)$ scale in the same way as we vary ν in its projective class in PML , the entry in the second

coordinate of J depends only on $[pl^-]$; it can therefore be written in terms of our pleating invariants as $\lambda_\nu(q)/i(\mu, \nu)$ for any choice of $\nu \in [pl^-]$.

Set

$$\mathcal{X}(\mu, c) = \left\{ ([\nu], s) \in (PML - \{[\mu]\}) \times \mathbf{R}^+ : 0 < s < \frac{f_{\mu, \nu}(c)}{i(\mu, \nu)} \right\}.$$

Identifying $PML - \{[\mu]\}$ with \mathbf{R} as in section 2.2, we can think of $\mathcal{X}(\mu, c)$ as the region in $\mathbf{R} \times \mathbf{R}^+$ under the graph of the function $[\nu] \mapsto \frac{f_{\mu, \nu}(c)}{i(\mu, \nu)}$. As discussed above, this function is well defined and by corollary 8, it is continuous.

As before, we let $\mathcal{Q}_{\mu, c}$ denote the quakebend plane along μ that meets \mathcal{F} along $\mathcal{E}_{\mu, c}$. Clearly $\mathcal{Q}_{\mu, c} = \mathcal{Q}_\mu^p$ for all $p \in \mathcal{E}_{\mu, c}$.

Theorem 5 *Let $\mu \in ML$ and let $c > 0$. Then the closures in \mathcal{QF} of precisely two of the connected components of $\mathcal{Q}_{\mu, c} \cap (\mathcal{QF} - \mathcal{F})$ meet \mathcal{F} . These components are the slices $BM_{\mu, c}^\pm$. The intersection of the closure of each slice with \mathcal{F} is the earthquake path $\mathcal{E}_{\mu, c}$; furthermore each slice is simply connected and retracts onto $\mathcal{E}_{\mu, c}$ and the map $J : BM_{\mu, c}^\pm \rightarrow \mathcal{X}(\mu, c)$ is a homeomorphism.*

Proof: Noting that for $\nu \in ML$, the pleating ray $\mathcal{P}_{\mu, \nu, c}$ depends only on the projective class $[\nu]$ of ν , it is clear from the definitions that

$$BM_{\mu, c}^+ = \bigcup_{[\nu] \in PML - \{[\mu]\}} \mathcal{P}_{\mu, \nu, c}.$$

Since for $[\nu] \in PML - \{[\mu]\}$, the closure of the pleating ray $\mathcal{P}_{\mu, \nu, c}$ in \mathcal{QF} contains the point $p(\mu, \nu, c)$, the closure of $BM_{\mu, c}^+$ in \mathcal{QF} contains $\mathcal{E}_{\mu, c}$. It follows easily from theorems 15 and 26 that $BM_{\mu, c}^+$ is open and closed in $\mathcal{Q}_{\mu, c} \cap (\mathcal{QF} - \mathcal{F})$. By theorem 34 there are no other components of $\mathcal{Q}_{\mu, c}$ whose closure meets \mathcal{F} .

For $[\nu] \in PML - \{[\mu]\}$, by lemma 39, $\lambda_\nu|_{\mathcal{P}_{\mu, \nu, c}}$ is a homeomorphism to the interval $(0, f_{\mu, \nu}(c))$. This proves J is a homeomorphism onto $B_{\mu, c}^+$. Clearly therefore, $BM_{\mu, c}^+$ is simply connected and retracts to $\mathcal{E}_{\mu, c}$ along rays. \square

In analogy with theorem 4 we have

Theorem 47 *The rational pleating rays $\mathcal{P}_{\mu, \nu, c}$ are dense in $BM_{\mu, c}^+$.*

Remark 48 As discussed above, holding the ending invariant of one side fixed and letting the ending invariant of the other side vary over the full Teichmüller space \mathcal{F} , we obtain the Bers and Maskit slices. By contrast, the set of flat structures $F^-(q)$ for points $q \in BM_{\mu, c}^+$ cannot be the full image of \mathcal{F} . In fact, on each ray $\mathcal{P}_{\mu, \nu, c}$, the length λ_ν is bounded above by $f_{\mu, \nu}(c)$. Since

by a theorem of Sullivan, [9], lengths on $\partial\mathcal{C}^-$ and Ω^- are in bounded ratio, those points on the earthquake path $\mathcal{E}_{\mu,c}$ in \mathcal{F} at which λ_ν is very large will not occur as $F^-(q)$ for points $q \in BM_{\mu,c}^+$. See also [31] for related phenomena.

11 Rational pleating planes and computation

We can now easily prove theorem 3 of the introduction.

Theorem 3 *Let $\delta_\gamma, \delta_{\gamma'}$ be rational laminations represented by non-conjugate elements $V, V' \in G$. Then $\mathcal{P}_{\gamma,\gamma'}$ and $\mathcal{P}_{\gamma',\gamma}$ are the unique components of the \mathbf{R}^2 -locus of the function $\text{Tr } V \times \text{Tr } V'$ in $\mathcal{QF} - \mathcal{F}$ whose closures meet \mathcal{F} in $\mathcal{F}_{\gamma,\gamma'}$. On $\mathcal{P}_{\gamma,\gamma'} \cup \mathcal{P}_{\gamma',\gamma}$ the function $\text{Tr } V \times \text{Tr } V'$ is non-singular and the boundary of $\mathcal{P}_{\gamma,\gamma'} \cup \mathcal{P}_{\gamma',\gamma}$ can be computed by solving $\text{Tr } V = \pm 2$ and $\text{Tr } V' = \pm 2$ on this component.*

Proof: If $V, V' \in G$ represent γ, γ' in \mathcal{S} , then the \mathbf{R}^+ -loci in \mathcal{QF} of $\text{Tr } V, \text{Tr } V'$ and $\lambda_\gamma, \lambda_{\gamma'}$ agree. As a consequence of theorem 2, $\mathcal{P}_{\gamma,\gamma'}$ can be uniquely identified as the component of the $\mathbf{R}^+ \times \mathbf{R}^+$ -locus of $\text{Tr } V \times \text{Tr } V'$ which meets \mathcal{F} in the critical line $\mathcal{F}_{\gamma,\gamma'}$. \square

As a consequence of this theorem, given any embedding $\mathcal{QF} \rightarrow \mathbf{C}^2$, we can compute the position of $\mathcal{P}_{\gamma,\gamma'}$ and its boundary exactly, provided we can express $\text{Tr } V$ and $\text{Tr } W$ as holomorphic functions of the parameters and identify the critical line.

For the complex Fenchel Nielsen embedding this works as follows. We first note:

Proposition 49 *Let $(\lambda_V, \tau_{V,W})$ be complex Fenchel Nielsen coordinates for \mathcal{QF} relative to a marked pair of generators (V, W) . Suppose $\gamma' \in \mathcal{S}$ with corresponding element $V' \in G$. Then for fixed λ_V , the trace $\text{Tr } V' = \pm 2 \cosh \lambda_{\gamma'}$ is a polynomial in $\cosh \tau_{V,W}/2$ and $\sinh \tau_{V,W}/2$.*

Proof: From equation (3) we have

$$\cosh \frac{\lambda_W}{2} = \cosh \frac{\tau_{V,W}}{2} / \tanh \frac{\lambda_V}{2}$$

and

$$\cosh \frac{\lambda_{VW^{\pm 1}}}{2} = \cosh \frac{\tau_{V,W} \pm \lambda_V}{2} / \tanh \frac{\lambda_V}{2}.$$

Expanding $\cosh \frac{\tau_{V,W} \pm \lambda_V}{2}$, the result follows in the special cases $V' = W$ and

$V' = VW^{\pm 1}$. The results for general V' follow from the recursive scheme in [47], see also [18], which allows us to express $\text{Tr } V'$ as a polynomial (with integer coefficients) in $\text{Tr } V$, $\text{Tr } W$ and either $\text{Tr } VW$ or $\text{Tr } VW^{-1}$. \square

To find the critical line $\mathcal{F}_{\gamma, \gamma'}$ we proceed as follows. Fix $c > 0$ and consider the function $\text{Tr } V' = \text{Tr } V'(\lambda_V, \tau_{V,W})$. Along the earthquake path $\mathcal{E}_{\gamma, c}$, $t = \tau_{V,W}$ is real and varies over all of \mathbf{R} ; λ_V is fixed and equal to c . By Kerckhoff's theorem, the function $\lambda_{\gamma'}$ has a unique critical point $p = p(\gamma, \gamma', c) \in \mathcal{F}_{\gamma, \gamma'}$ along $\mathcal{E}_{\gamma, c}$; clearly the same is true of the trace function $\text{Tr } V'$. Using proposition 49, the position of this point can be computed as a function of t . Moreover there are exactly two branches σ^{\pm} of the \mathbf{R} -locus of $\text{Tr } V'$ in $\mathcal{QF} - \mathcal{F}$ whose closures meet \mathcal{F} at p .

By theorem 2, the pleating plane $\mathcal{P}_{\gamma, \gamma'}$ is the union of the pleating rays $\mathcal{P}_{\gamma, \gamma', c}$, $c \in \mathbf{R}^+$. By theorem 6, the pleating ray $\mathcal{P}_{\gamma, \gamma', c}$ is one of the two branches σ^{\pm} , each of which maps homeomorphically to $(0, 2 \cosh f_{\gamma, \gamma'}(c)/2)$ under $\text{Tr } V'$. Analytically continue $\text{Tr } V'$ along σ^{\pm} . Again by theorem 6, these branches are non-singular \mathbf{R} -loci and remain in \mathcal{QF} until they reach points τ^* such that $\text{Tr } V'(\tau^*) = \pm 2$. The groups corresponding to such τ^* are cusp groups on $\partial \mathcal{QF}$ for which γ' is pinched and V' is an accidental parabolic.

Drawing these rays for various c 's, we get a picture of the pleating planes $\mathcal{P}_{\gamma, \gamma'}$ and $\mathcal{P}_{\gamma', \gamma}$. Allowing γ' to vary with c fixed gives us the slices $BM_{\gamma, c}^{\pm}$. By theorems 4 and 47, we can build up an arbitrarily accurate picture of \mathcal{QF} . Pictures of various slices drawn this way have been obtained in [47] and [38].

In [25], similar ideas are used to draw a picture of the *Earle slice* of \mathcal{QF} . This slice is an embedding of the Teichmüller space of \mathcal{T}_1 into \mathcal{QF} consisting of groups for which the structures on Ω^+ and Ω^- are related by a conformal involution which induces the rhombus symmetry on $\pi_1(\mathcal{T}_1)$.

11.1 Examples

We give two examples in which it is especially easy to compute the pleating plane.

Example 1. Take γ, γ' to be generators of $\pi_1(\mathcal{T}_1)$, represented by the marked pair $V, W \in G$. By equation (2), $\cosh(\frac{\lambda_W}{2}) = \cosh(\frac{\tau_{V,W}}{2}) / \tanh(\frac{\lambda_V}{2})$, so that on the earthquake path $\mathcal{E}_{\gamma, c}$, $\cosh(\frac{\lambda_W}{2}) = \cosh(\frac{t}{2}) / \tanh(c)$, $t \in \mathbf{R}$. This function clearly has a unique critical point at the rectangular torus $t = 0$. Therefore the critical line $\mathcal{F}_{\gamma, \gamma'}$ is defined by the equation $\sinh(\frac{\lambda_V}{2}) \sinh(\frac{\lambda_W}{2}) = 1$ and the

range of $\lambda_V \times \lambda_W$ is the region

$$\{(c, t) \in \mathbf{R}^+ \times \mathbf{R}^+ : 0 < c < 2 \sinh^{-1}\left(\frac{1}{\sinh(\frac{c}{2})}\right)\}.$$

Notice that under the rectangular symmetry $(V, W) \rightarrow (V, W^{-1})$ the group is fixed but the marking is changed; clearly $\Omega^+(G(V, W)) = \Omega^-(G(V, W^{-1}))$. Thus $\mathcal{P}_{\gamma, \gamma'}$ maps bijectively to $\mathcal{P}_{\gamma', \gamma}$ while $\mathcal{F}_{\gamma, \gamma'} = \mathcal{F}_{\gamma', \gamma}$ is fixed. This implies $\cosh \frac{\lambda_{VW}}{2} = \cosh \frac{\lambda_{VW^{-1}}}{2}$ on $\mathcal{F}_{\gamma, \gamma'}$. Solving this equation in \mathcal{F} gives another way of finding the equation of the critical line.

Example 2. Let (V, W) be a marked pair of generators for G and let γ, γ' be the curves represented by VW and VW^{-1} . Since G is a punctured torus group, the condition that the commutator $[V, W]$ be parabolic is expressed by the well known Markov equation

$$\mathrm{Tr}^2 V + \mathrm{Tr}^2 W + \mathrm{Tr}^2 VW = \mathrm{Tr} V \mathrm{Tr} W \mathrm{Tr} VW. \quad (6)$$

Writing $x = \mathrm{Tr} V, y = \mathrm{Tr} W$, we can solve for $z = \mathrm{Tr} VW$ and $z' = \mathrm{Tr} VW^{-1}$. On the pleating plane $\mathcal{P}_{\gamma, \gamma'}$, both z and z' are real so that xy and $x^2 + y^2$ are real. It follows that $x = \bar{y}$. Further, on $\mathcal{P}_{\gamma, \gamma'}$, $x, y \in \mathbf{R}$ if and only if $G \in \mathcal{F}$. Thus in the real (z, z') plane, the critical line

$\mathcal{F}_{\gamma, \gamma'}$ has equation $zz' = 2(z+z')$; in other words the hyperbola $(z-2)(z'-2) = 4$. Rewriting in terms of the lengths $2 \cosh^{-1} \frac{z}{2}, 2 \cosh^{-1} \frac{z'}{2}$ we find the region $T_{\gamma, \gamma'}$ is of the shape claimed.

We note that in this case, the critical line $\mathcal{F}_{\gamma, \gamma'}$ is the fixed line of the rhombic symmetry $(A, B) \rightarrow (B, A)$ in \mathcal{F} , giving an alternative proof that on this line, $\lambda_A = \lambda_B$. It is also interesting to note in this example that the Earle slice studied in [25] is the holomorphic extension of the critical line $\mathcal{F}_{\gamma, \gamma'}$ into \mathcal{QF} .

12 Appendix

12.1 The convergence lemma

For the proof of the convergence lemma 1, we need to recall some general facts about laminations. Let Σ be a hyperbolic surface and let α be a geodesic lamination on Σ . We call a set $R \subset \Sigma$ a *flow box* for α if:

- (1) R is a closed hyperbolic rectangle embedded in Σ , with one pair of opposite sides called “horizontal” and the other pair “vertical”.
- (2) The horizontal sides T, T' of R are either disjoint from α or transversal to α . If a leaf γ of α intersects R then it intersects both T and T' .
- (3) The vertical sides of R are disjoint from α .

Label the sides of R in counterclockwise order 1, 2, 3, 4 so that 1, 3 are the horizontal sides and 2, 4 are the vertical ones. Suppose that $\beta \in ML$ is any measured lamination on Σ . The underlying lamination $|\beta|$ intersects R in a family of pairwise disjoint arcs. If such an arc joins a vertical to a horizontal side, we call it a corner arc; if it joins the two horizontal sides we call it a vertical arc and otherwise it is a horizontal arc. For $i, j \in \{1, \dots, 4\}$, let $\beta(i, j) = \beta(j, i)$ denote the total transverse measure of the arcs joining side i to side j . Clearly, $\alpha(1, 3) = \alpha(3, 1) = \alpha(T) = \alpha(T')$, the transverse measure of the transversal T , while $\alpha(i, j) = 0$ otherwise.

The following simple lemma applies to any hyperbolic surface Σ .

Lemma 11.1 *Let $\nu_0 \in ML$ and let R be a flow box for $|\nu_0|$. Suppose $\nu_0(T) \neq 0$. Then for $\nu \in ML$ sufficiently near ν_0 , the lamination $|\nu|$ has a vertical arc.*

Proof: Note that because $|\nu|$ consists of pairwise disjoint simple geodesics, it does not have both horizontal and vertical arcs. Let V, V' denote the vertical sides. Since $\nu_0(V) = \nu_0(V') = 0$, both $\nu(V)$ and $\nu(V')$ can be assumed arbitrarily small by taking ν sufficiently close to ν_0 in ML . We can write $\nu(V) = \nu(4, 1) + \nu(4, 2) + \nu(4, 3)$ and $\nu(V') = \nu(2, 1) + \nu(2, 4) + \nu(2, 3)$. All the terms on the right in these relations are non-negative so each is arbitrarily small.

If we assume $|\nu|$ has no vertical arc we have $\nu(T) = \nu(4, 1) + \nu(2, 1)$, $\nu(T') = \nu(3, 2) + \nu(3, 4)$ and by the above we deduce that both are arbitrarily small. But this is a contradiction because $\nu(T)$ and $\nu(T')$ are both near $\nu_0(T)$ which is a definite positive value. \square

Now we need some facts specific to laminations on a punctured torus (see [45], 9.5.2). Let $\alpha \in \mathcal{S}$ and cut \mathcal{T}_1 along α to obtain a punctured annulus A with boundary curves α_1 and α_2 . The leaves of any measured lamination ν , $|\nu| \neq \alpha$ intersect A in a union of arcs that either join α_1 to α_2 or join one of the boundary components to itself. It is easy to show, (see [45]), that the set of arcs joining a component α_i to itself has zero transverse measure. In particular, by minimality any transversal to any leaf of $|\nu|$ carries non-zero measure, so that all arcs of $|\nu|$ in A join α_1 to α_2 .

We also recall that on \mathcal{T}_1 , if $\nu \notin ML_Q$, the complement of $|\nu|$ is a punctured bigon B , and also that there is a horocyclic neighborhood of definite size about the cusp disjoint from the support of any measured lamination.

Now we can prove the convergence lemma 1.

Lemma 1 *Suppose that $\nu_0 \in ML - ML_Q$, and that ν and ν_0 are close in ML . Then $|\nu|$ and $|\nu_0|$ are close in the Hausdorff topology on GL .*

Proof: First we show that given a long arc in $|\nu_0|$ there exists a long nearby arc in $|\nu|$. Let $L, \epsilon > 0$ be given. Since $\nu_0 \in ML - ML_Q$, all leaves have infinite length. Thus, given $x \in |\nu_0|$, by choosing sufficiently short transversals we can find a flow box for which the leaf of $|\nu_0|$ through x is a vertical arc, the segments of length L on either side of x are contained in R , and the horizontal sides of R have length less than ϵ . We call a flow box of this kind, a good ϵ, L -flow box for x . Now standard hyperbolic geometry estimates show, that if two geodesics are a bounded distance apart over a long distance t , then in fact they are close to order e^{-t} along a large fraction of their length. Thus any vertical arc in a good ϵ, L -flow box is certainly close to leaves of $|\nu_0|$ over distance at least $2L$. Clearly, $|\nu_0|$ can be covered by a finite number of flow boxes of this kind.

Now suppose we are given a long arc λ of a leaf of $|\nu_0|$. Let x be the midpoint of λ and let R be a good ϵ, L -flow box for x . By lemma 12.1, we deduce that if $\nu \in ML$ is near ν_0 , then ν has a vertical arc in R so that by the above, $|\nu|$ has long arc of a leaf near λ as required.

Next we claim conversely, that given a long arc in $|\nu|$ there exists a long nearby arc in $|\nu_0|$. For a lamination λ , let $T_1(\lambda)$ denote the the set of unit tangent vectors to leaves pointing along leaves of λ . Since there is a horocyclic neighborhood of definite size about the cusp disjoint from the support of any measured lamination on \mathcal{T}_1 , the set $\cup_{\lambda \in GL} T_1(\lambda)$ is a compact subset of the unit tangent bundle $T_1(\mathcal{T}_1)$. Clearly, laminations λ and λ' are close in the Hausdorff topology on closed subsets of GL if and only if $T_1(\lambda)$ and $T_1(\lambda')$ are close in the Hausdorff topology on closed subsets of $T_1(\mathcal{T}_1)$.

If our claim is false, then there is a sequence of points $\bar{v}_n \in T_1(|\nu_n|)$, $\nu_n \in ML_Q$ with $\nu_n \rightarrow \nu_0$ in ML , for which there are no nearby points of $T_1(|\nu_0|)$. A geodesic β through a limit point of the vectors \bar{v}_n will be a limit of leaves of $|\nu_n|$, but will not be a leaf of $|\nu_0|$.

If $\beta \cap |\nu_0| \neq \emptyset$, we obtain a contradiction. For if $x \in \beta \cap |\nu_0|$, the tangent directions to β and $|\nu_0|$ at x are distinct. Therefore we can find a good $|\nu_0|$ flow box R for x , such that the arc of β through x is only close to the leaf of ν_0 through x for a short distance and thus cannot be either a vertical or a corner

arc in R . But then all laminations $|\nu|$ with leaves close to β also contain arcs which must intersect R in horizontal arcs, contradicting lemma 12.1.

To complete the proof we must show $\beta \cap |\nu_0| \neq \emptyset$. If not, then β is contained in the complement of $|\nu_0|$ in \mathcal{T}_1 . Since $\nu_0 \notin ML_Q$, the complement of $|\nu_0|$ is a punctured bigon B . If β enters B through one vertex and leaves through the other it is homotopic to, and therefore coincides with, a leaf of $|\nu_0|$; thus β must come in from one vertex of the bigon, go around the puncture and return back to the *same* vertex. Let α be a simple closed curve that intersects β and as above, cut \mathcal{T}_1 along α to obtain a punctured annulus A with two boundary curves α_1, α_2 . Since β goes around the puncture, it crosses one of the α_i and returns through the same side of α_i (see the figure in [45], 9.5.2). It follows that any closed simple geodesic sufficiently close in the Hausdorff topology to β would also have an arc entering and leaving A across the same α_i . But any arc of a simple closed geodesic carries a non-zero transverse measure, and by the fact stated above, must join α_1 to α_2 . Hence $\beta \cap |\nu_0| \neq \emptyset$. \square

12.2 Proof of proposition 28.

Before beginning the proof, we need to review the definitions of the bending measure and intrinsic metric for paths on $\partial\mathcal{C}$ as given in [19]. We suppose that $q \in \mathcal{QF}$, and that as usual $\partial\mathcal{C} = \partial\mathcal{C}(q)$ is the convex hull boundary of $\mathbf{H}^3/G(q)$. We shall only indicate the dependence on q when needed in the proof. In fact, we shall only need to apply what follows to the component $\partial\mathcal{C}^+$.

A *support plane* for $\partial\mathcal{C}$ at a point $x \in \partial\mathcal{C}$ is a hyperbolic plane P containing x such that \mathcal{C} is contained entirely in one of the two half spaces cut out by P . The *bending angle* between two intersecting support planes P_1, P_2 at points $x_1, x_2 \in \partial\mathcal{C}$ is the absolute value of the angle $\theta(P_1, P_2)$ between their outward normals from $\partial\mathcal{C}$.

Let $\Pi(x)$ denote the set of oriented support planes at $x \in \partial\mathcal{C}$ and let

$$Z = \{(x, P(x)) : x \in \partial\mathcal{C}, P(x) \in \Pi(x)\},$$

with topology induced from $\mathcal{G} = \mathbf{H}^3 \times \mathcal{G}_2(\mathbf{H}^3)$, where $\mathcal{G}_2(\mathbf{H}^3)$ is the Grassmannian of 2-planes in \mathbf{H}^3 . Let Z^+ be the obvious restriction of Z to $\partial\mathcal{C}^+$ and call the it *approximating set* for $\partial\mathcal{C}^+$.

To define the bending measure and intrinsic metric, it suffices to define the measure and length of any path $\bar{\omega}$ on $\partial\mathcal{C}$. Any such path lifts to a path $\omega: [0, 1] \rightarrow Z$ as follows. Suppose $x \in \bar{\omega}$. Either $\Pi(x)$ consists of a unique point, in which case there is nothing to do, or we add to the path an arc in which the first coordinate x is fixed but the second moves continuously on the

line in \mathcal{G} from the left to the right extreme support planes at x .

A *polygonal approximation* to ω is a sequence

$$\mathcal{P} = \{\omega(t_i) = (x_i, P_i) \in Z\}; 0 = t_0 < t_1 < \dots < t_n = 1,$$

such that $P_i \cap P_{i+1} \neq \emptyset, i = 0, \dots, n-1$.

Let $\theta_i = \theta(P_{i-1}, P_i)$ be the bending angle between P_{i-1} and P_i , $i = 1, \dots, n$ and let d_i be the hyperbolic length of the shortest path from x_{i-1} to x_i in the planes $P_{i-1} \cup P_i$.

The intrinsic metric on $\partial\mathcal{C}$ is given by

$$l(\omega) = \inf_{\mathcal{P}} \sum_{i=1}^n d_i \tag{7}$$

and the bending measure β on $\partial\mathcal{C}$ by

$$\beta(\omega) = \inf_{\mathcal{P}} \sum_{i=1}^n \theta_i \tag{8}$$

where \mathcal{P} runs over all polygonal approximations to ω .

In order to prove proposition 28, we shall also make similar polygonal approximations to the pleated surface $\mathbf{P}_\gamma(q)$. We shall prove the proposition by showing that polygonal approximations in $Z^+ = Z^+(q)$ to the convex hull boundary $\partial\mathcal{C}^+$ can be replaced by polygonal approximations to the pleated surface $\mathbf{P}_\gamma(q)$, and that the above approximating sums are simultaneously good approximations to the intrinsic metric of the flat structure $\phi_\gamma^+(q)$ and the transverse measure $b_\gamma(q)$. Thus we also need to discuss polygonal approximations for $\mathbf{P}_\gamma(q)$.

The surface $\mathbf{P}_\gamma(q)$ is made up of planar pieces, precisely two of which meet along each bending line α (which projects to γ on \mathcal{T}_1). Call a plane P a *pseudo-support plane* to $\mathbf{P}_\gamma(q)$ if either it is one of these planar pieces, or if it meets $\mathbf{P}_\gamma(q)$ along α and lies in the half space cut out by the planar pieces of $\mathbf{P}_\gamma(q)$ through α . The pseudo-support planes of $\mathbf{P}_\gamma(q)$ inherit natural orientations from the pleated surface map under which $\mathbf{P}_\gamma(q)$ is an immersed image of the hyperbolic disk \mathbf{D} in \mathbf{H}^3 .

Let $\tilde{\Pi}(x)$ denote the set of oriented pseudo-support planes at $x \in \mathbf{P}_\gamma(q)$ and let

$$W = W(q) = \{(x, P(x)) | x \in \mathbf{P}_\gamma(q), P(x) \in \tilde{\Pi}(x)\},$$

with topology induced from \mathcal{G} as before. We define polygonal approximations in $W(q)$ in the obvious way, and call $W(q)$ the approximating set for $\mathbf{P}_\gamma(q)$.

We claim that the flat metric $\phi_\gamma(q)$ and the measure b_γ on $\mathbf{P}_\gamma(q)$ are defined by sums similar to those in (7) and (8), where the infimum is taken now over polygonal approximations in $W(q)$.

Let ω be a path in W and let $\{(x_i, Q_i)\}$ be such a W -polygonal approximation. As in the proof of Proposition 4.8 of [19], we consider the segment of path ω_i in $W(q)$ between x_{i-1} and x_i , and we work in a hyperbolic plane H through x_{i-1} and x_i , such that the shortest path σ from x_{i-1} to x_i in the planes $Q_{i-1} \cup Q_i$ is contained in the intersections of these planes with H . Let the segments of σ in Q_{i-1} and Q_i have lengths a_1 and a_2 respectively, so that $a_1 + a_2$ is an upper bound for the contribution to the sum giving the length of ω_i . Notice that even though we do not know that $\mathbf{P}_\gamma(q)$ bounds a convex half space, it follows easily from Gauss-Bonnet that ω_i does not intersect σ . Thus it is easy to check that inserting an extra pair $(x, Q) \in W$ between x_{i-1} and x_i , the approximating sum for the length of ω_i decreases. Since by assumption $[\gamma] \in ML_Q$, there are in fact sufficiently fine polygonal approximations for which the sum in (7) actually *equals* the intrinsic metric on $\mathbf{P}_\gamma(q)$. A similar argument, on the lines of that in Proposition 4.8 of [19], shows that the sums (8) decrease on inserting extra support planes and that there are sufficiently fine sums which actually equal the measure b_γ .

We are now ready to prove proposition 28.

Proposition 28 *Given $\nu_0 \in ML - ML_Q$, and $q_0 \in \mathcal{P}_{\nu_0}^+ \cup \mathcal{F}$, let $F^+(q_0) \in \mathcal{F}$ and $pl^+(q_0)$ be the flat structure and bending measure of $\partial\mathcal{C}^+(q_0)$ respectively. (If $q_0 \in \mathcal{F}$, then $pl^+(q_0) = 0$ and $F^+(q_0)$ is the Fuchsian group representing q_0 .) Then, given neighborhoods V of $F^+(q_0)$ in \mathcal{F} and W of $pl^+(q_0)$ in ML , there exist neighborhoods U of q_0 in \mathcal{QF} and X of $[\nu_0]$ in PML such that if $q \in U$, $[\delta_\gamma] \in X \cap PML_Q$ and $\lambda_\gamma(q) \in \mathbf{R}^+$, then the flat structure $\phi_\gamma(q)$ of $\mathbf{P}_\gamma(q)$ is in V and the transverse measure $b_\gamma(q)$ is in W .*

Proof: Let ν_0, q_0 be as in the statement of the proposition. Suppose that for some q near q_0 and $[\delta_\gamma]$ near $[\nu_0]$, we have $\lambda_\gamma(q) \in \mathbf{R}^+$. Let $\mathbf{P}_\gamma(q)$ be the associated pleated surface with approximating set $W(q) \subset \mathcal{G}$ as above. Let $Z^+(q_0)$ and $Z^+(q)$ be the approximating sets for $\partial\mathcal{C}^+(q_0), \partial\mathcal{C}^+(q)$ respectively.

We claim that for every $(x, P(x)) \in Z^+(q_0)$ and $q \in \mathcal{QF}$ near q_0 , there is a nearby pair $(y, P(y)) \in W(q)$, and conversely. This will follow immediately if we can show that, for every geodesic in $|pl^+(q)|$, there is a geodesic in the bending locus of $\mathbf{P}_\gamma(q)$ with nearby endpoints in \mathbf{H}^3 , and vice versa. Now, the crucial condition $\lambda_\gamma(q) \in \mathbf{R}^+$ implies that the bending locus of $\mathbf{P}_\gamma(q)$ is *exactly* $\gamma = \gamma(q)$. Thus, applying lemma 1 to the laminations ν_0 and $k\delta_\gamma$ for

a suitable choice of $k > 0$ on the surface $\partial\mathcal{C}^+(q_0)$, we see that $|\nu_0(q_0)|$ and $\gamma(q_0)$ are close in the Hausdorff topology on closed subsets of $\partial\mathcal{C}^+(q_0)$. Lifting to \mathbf{H}^3 , this means that the endpoints x_0, x'_0 of any lift of a leaf of $|\nu_0(q_0)|$ are close to the endpoints x, x' of a lift of $\gamma(q_0)$ and vice versa since the geodesic representative of γ on $\partial\mathcal{C}^+$ has the same endpoints as the geodesic γ in \mathbf{H}^3 . It follows that the \mathbf{H}^3 geodesics with x_0, x'_0 and x, x' also have long close arcs. Finally, moving to a nearby point q in \mathcal{QF} , the endpoints of geodesics which project to the leaves of $|\nu_0(q)|$ are close to the endpoints of geodesics which project to $|\nu_0(q_0)|$, and similarly for endpoints of geodesics which project to $\gamma(q)$ and $\gamma(q_0)$. The claim follows.

We now consider the key estimates which were the basis of the continuity results proved in [19]. Call a polygonal approximation an (α, s) -approximation if

$$\max_{1 \leq i \leq n} \theta(P_{i-1}, P_i) < \alpha$$

and

$$\max d_\omega(x_{i-1}, x_i) < s$$

where d_ω is distance along ω measured in the intrinsic metric on $\partial\mathcal{C}$. We have

Proposition [19], Prop. 4.8 *There is a universal constant K , and a function $s(\alpha)$ with values in $(0, 1)$, such that if \mathcal{P} is an $(\alpha, s(\alpha))$ -approximation to a path ω in Z , where $\alpha < \pi/2$, then*

$$\left| \sum_{\mathcal{P}} d_i - l(\omega) \right| < K\alpha l(\omega)$$

and

$$\left| \sum_{\mathcal{P}} \theta_i - \beta(\omega) \right| < K\alpha l(\omega).$$

To complete the present proof, it suffices to check that similar estimates hold if polygonal approximations in $Z^+(q_0)$ are replaced by approximations in $W(q)$. The estimates work in exactly the same way; the only point to note is that we need the same local convexity property implied by Gauss Bonnet as above. \square

12.3 Proof of proposition 35

Proposition 35 *Suppose $\mu \in ML$, $q \in \mathcal{P}_\mu \cup \mathcal{F}$ and consider the quakebend plane \mathcal{Q}_μ^q along μ based at q with parameter τ_μ . Given $K > 0$, there exist $B_2 > B_1 > 0$ such that if $|\Re\tau| < K$ and $B_2 > |\Im\tau| > B_1$, then $\mathcal{Q}_\mu^q(\tau) \notin \mathcal{P}_\mu$.*

Proof: Our proof will show that if τ_μ is inside the range described the proposition, then the pleated surface obtained by bending by τ along μ cannot be

embedded and thus that $\mathcal{Q}_\mu^q(\tau_\mu) \notin \mathcal{P}_\mu$. The group $\mathcal{Q}_\mu^q(\tau_\mu)$ may or may not be in $\mathcal{Q}\mathcal{F}$.

We use the definitions of support planes and bending angles from the proof of proposition 12.2. From the definition, the bending angle between two intersecting support planes P_1, P_2 to $\partial\mathcal{C}$ at points x_1, x_2 is an upper bound for the bending measure of a transversal to $|\mu|$ joining x_1, x_2 which lies between the “roof” formed by P_1 and P_2 and the \mathbf{H}^3 geodesic from x_1 to x_2 .

We make the following claims.

- (1) There exists $\epsilon > 0$ such that if $x_1, x_2, x_3 \in \partial\mathcal{C}$ lie in a ball of radius ϵ in \mathbf{B}^3 , and if P_1, P_2, P_3 are support planes to $\partial\mathcal{C}$ at x_1, x_2, x_3 respectively, then either $P_1 \cap P_3 \neq \emptyset$, or both $P_1 \cap P_2 \neq \emptyset$ and $P_2 \cap P_3 \neq \emptyset$.
- (2) Given $\epsilon > 0$, $\mu \in ML$, $\mu \neq 0$, and a compact subset $V \subset \mathcal{F}$, there is a constant $a > 0$ such that if $\phi \in V$, then there is a transversal κ to $|\mu|$ with hyperbolic length $l(\kappa) < \epsilon$ in the structure ϕ and transverse measure $\mu(\kappa) > a$.

Proof of claim 1. A support plane P to $\partial\mathcal{C}$ meets $\hat{\mathbf{C}}$ in a circle which contains points of the limit set Λ and which bounds a disk $D(P)$ containing no points of Λ . Therefore if $P_1 \cap P_3 = \emptyset$, the discs $D(P_1)$ and $D(P_3)$ are disjoint. To prove the claim amounts to showing that in this case, both $D(P_1) \cap D(P_2)$ and $D(P_3) \cap D(P_2)$ are non-empty. Without loss of generality, we may suppose that x_1, x_2, x_3 are within hyperbolic distance ϵ of the origin O in \mathbf{B}^3 so that the planes P_i are close to equatorial planes through O . The result is then obvious.

Proof of claim 2. Choose $\gamma \in \mathcal{S}$ with $i(\gamma, \mu) > 0$. There are constants c_1, c_2, d_1, d_2 such that $c_1 < \mu(\gamma) < c_2$ and $d_1 < l(\gamma) < d_2$ for $\phi \in V$. Subdividing γ into N segments with $d_2/N < \epsilon$, the result is clear with $a = c_1/N$.

Now, working in the quakebend plane \mathcal{Q}_μ^q , with parameter $\tau = \tau_\mu$, consider the set of groups for which $|\Re\tau| < K$. The corresponding flat structures $F^+(\tau)$ are independent of $\Im\tau$ and thus lie in a compact set $V \subset \mathcal{F}$. Choose a transversal κ as in claim (2). Let x_1, x_3 be its initial and final points and x_2 its midpoint, and let P_i be a support plane at x_i . Using claim 1, either P_1, P_3 , or both pairs P_1, P_2 and P_2, P_3 , intersect. Thus at least one of the segments (x_1, x_3) , (x_1, x_2) or (x_2, x_3) of κ , for definiteness say the segment κ_1 joining (x_1, x_2) , has $\mu(\kappa_1) > a/2$.

Consider the point in \mathcal{Q}_μ^q with parameter τ . The bending measure $pl^+(\tau)(\kappa_1)$ of κ_1 is $k + \Im\tau\mu(\kappa_1)$, where $k = pl^+(q)(\kappa_1)$ is the bending measure of κ_1 at the base point q . The bending angle between P_1, P_2 is not in the range $(\pi, 2\pi)$. From the above, $a/2 < \mu(\kappa_1) < c_2$. Putting this together gives an upper bound for $pl^+(\tau)(\kappa_1)$, and we obtain the required bound on $|\Im\tau|$. \square

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