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On period doubling phenomena and Sharkovskii type ordering for the family $\lambda \tan(\mathbf{z})$

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Abstract. In this paper we prove that all the hyperbolic components of the family $\lambda \tan z$ except the punctured unit disk are simply connected. Then, restricting λ to the imaginary axis, We show that there is a "Sharkovskii-like" ordering of the periods of the hyperbolic components. Finally, again for λ in the imaginary axis, we show that period doubling phenomena occur.

1. Introduction

The tangent family $\lambda \tan z$ is the simplest one parameter family of meromorphic functions. In [**5, 6, 7, 8**] we studied the dynamics generated by functions in this family. In particular, we saw that in many respects this behavior was similar to what we find for the quadratic family $z^2 + c$: the only possible stable behavior is eventually periodic and every stable component falls onto a cycle of attracting, parabolic or simply connected rotation domains. There are also similarities with the parameter plane of the quadratic map: the interior of the unit disk is like the outside of the Mandelbrot set and other components are deployed with a well defined combinatorial structure. The essential singularity of the tangent however, creates some differences; for example, the infinite to one properties are reflected in the parameter plane. The analogue of a hyperbolic component for the quadratic family is a pair of hyperbolic components (Ω_p, Ω_p') with a common boundary point c_p. For $\lambda \in \Omega_p$ there are two symmetric attracting cycles of period p and for $\lambda \in \Omega_p'$ there is a single attracting cycle of period $2p$. The point c_p has properties analogous to the center of a component for the quadratic family and is called a virtual center. We prove the following theorem which was asserted in [**7**] and [**8**].

Theorem 1. *Let Ω be any hyperbolic component except Δ^*. Then there is a conformal homeomorphism $\tilde{M} : \Omega \to \mathbb{H}$ between Ω and the upper half plane $\mathbb{H}$.*

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Moreover, $\tilde{M}$ extends continuously to the boundary $\partial\Omega$ such that the point $i\infty$ is the unique boundary point of $\mathbb{H}$ corresponding to virtual center c of Ω. Thus the virtual center is unique.

The real axis of the parameter plane plays a special role for the quadratic family. It is here that we see period doubling and the limiting Feigenbaum point as well as the Sharkovskii ordering of the periods of the hyperbolic components. In this paper we see that the imaginary axis plays an analogous role for the tangent family: each of the points $\pm p_k i = \pm(k+\frac{1}{2})\pi i, k = 0, 1, \dots$ in the λ plane is analogous to the origin for the real axis. We obtain a "Sharkovskii-like" theorem: we get a combinatorial description of the ordering of the periods of the hyperbolic components on the imaginary axis.

THEOREM 2. *For each $k = 0, 1, 2, \dots$, the intervals $\pm I_k = \pm(p_k i, p_{k+1} i)$ of the imaginary axis contain hyperbolic component pairs of every even period. Furthermore, there is a lower bound on the number of such pairs in I_k that depends on k and the deployment of these components for all n can be described.*

The computer pictures created by Jiang (see [**8**]) indicate that the bifurcations that occur for real quadratics have an analog for the tangent family. *Standard period doubling* occurs at a parabolic point when an attracting cycle bifurcates to a cycle of double the period. *Non-standard period doubling* occurs at a parabolic point when an attracting cycle bifurcates to a pair of cycles of the same period. We show

THEOREM 3. *Both standard and non-standard period doubling occur in the tangent family.*

The organization of the paper is as follows. In section 2 we recall the basic theory of iteration for meromorphic functions and in section 3 the basic theory of the tangent family. In section 4 we describe the general properties of hyperbolic component pairs and prove Theorem 1. In section 5 we discuss the special properties of the hyperbolic components on the imaginary axis. In section 6 we prove the main lemma needed for Theorem 2 and in section 7 we prove Theorem 2. In section 8 we describe bifurcations on the imaginary axis and prove theorem 3.

2. Dynamics of meromorphic functions

The orbits of points under iteration by a meromorphic function fall into three categories: they may be infinite, they may become periodic and hence consist of a finite number of distinct points or they may terminate at a pole of the function. Points in last category are called *prepoles*. For transcendental meromorphic functions with more than one pole, it follows from Picard's theorem that there are infinitely many prepoles.

The *stable set* or *Fatou set* F_f for a function f is defined just as for rational functions: F_f is the set of $z \in \mathbb{C}$ such that the iterates are defined and form a normal family in a neighborhood of z. The *unstable set* or *Julia set* J_f is the complement of F_f in $\bar{\mathbb{C}}$. Thus, F_f is open, J_f is closed and contains the prepoles $\mathcal{P}$ and both sets are completely invariant. As for rational functions, J_f is the accumulation set of the backward orbit of almost any point so that $J_f = \overline{\mathcal{P}}$.

The *singular set* S_f of a meromorphic function of f consists of those values at which f is not a regular covering. These are either critical values (algebraic singularities) or asymptotic values (transcendental singularities). The *postsingular set* PS_f is the union of the forward orbits of the singular values; its accumulation set is the omega-limit set ω_f.

A *holomorphic family* of meromorphic maps with finite singular set over a complex manifold Λ is a map $f : \Lambda \times \mathbb{C} \to \hat{\mathbb{C}}$ holomorphic in Λ and meromorphic in $\mathbb{C}$ given by $(\lambda, z) \mapsto f_\lambda(z)$.

3. The family $\mathcal{F}$

The tangent family

$$\mathcal{F} = \{f_\lambda(z) = \lambda \tan z\}$$

is a holomorphic family over the punctured complex plane $\mathbb{C}^* = \mathbb{C} - \{0\}$. In [**6**] we proved that the family is topologically closed in the sense that if g is topologically conjugate to $f_\lambda \in \mathcal{F}$ and is meromorphic, then if the conjugacy fixes $\{0, \infty, \pi/2\}$ and if the two asymptotic values of g are symmetric with respect to the origin, it follows that $g = f_{\lambda'}$ for some $\lambda' \in \mathbb{C}^*$.

The singular sets S_{f_λ} contain exactly two asymptotic values, the omitted values $\pm\lambda i$. The preimage of a punctured neighborhood of λi (resp. $-\lambda i$) is an upper (resp. lower) half plane. It follows from the symmetry of the tangent function that the forward orbits of these asymptotic values are symmetric with respect to origin. It is these orbits that control the dynamics.

To simplify notation we write F_λ, J_λ, ω_λ for objects associated to functions in $\mathcal{F}$. All functions in the family have the same poles; we use the notation $p_k = \frac{\pi}{2} + k\pi$, $k \geq 0$ for the poles on the positive axis and $-p_k$ for the poles on the negative axis.

The symmetry of the maps with respect to 0 implies that the stable and unstable sets are symmetric with respect to the origin. Precisely

$$f_\lambda(-z) = -f_\lambda(z)$$

so for all $k \in \mathbb{N}$

$$f_\lambda^k(-z) = -f_\lambda^k(z) \quad \text{and} \quad [f_\lambda^k(z)]' = [f_\lambda^k(-z)]'.$$

It follows that if $f_\lambda^p(z) = z$, then $f_\lambda^p(-z) = -z$. Moreover if z_0 is a periodic point of period p then either $-z_0$ belongs to the same cycle and p must an even number (i.e. $f_\lambda^{p/2}(z_0) = -z_0$) or $-z_0$ belongs to the other cycle of period p and p may be odd or even. Thus there is the following symmetry:

Either there is a single cycle $z_0, \dots z_{2p-1}$ with period $2p$ and

$$z_p = -z_0,\ z_{p+1} = -z_1,\ z_{p+2} = -z_2, \dots, z_{2p-1} = -z_{p-1}$$

or there are two cycles of period p

$$z_0, z_1, \dots, z_{p-1}, \ \text{and} \ -z_0, -z_1, \dots, -z_{p-1}$$

In [**5**] it is proved that any stable behavior of functions in $\mathcal{F}$ is necessarily eventually periodic. In [**8**], the stable behavior of functions in $\mathcal{F}$ is completely characterized: if $F_\lambda \neq \emptyset$ then exactly one of the following occurs:

[-] f_λ has an attracting cycle
[-] f_λ has a parabolic cycle
[-] f_λ has a neutral periodic cycle contained in Siegel disks.

In all other cases, $J_\lambda = \hat{\mathbb{C}}$. When $F_\lambda \neq \emptyset$, ω_λ contains each attracting or parabolic cycle or contains the boundary of the Siegel disks. As above, each cycle is either symmetric with respect to the origin or there is a second cycle containing the symmetric points. For each such pair symmetric cycles, the multipliers $M(\lambda, z_i) = \frac{df_\lambda^p}{dz}(z_i)$ (which are independent of i by the chain rule) are equal by symmetry.

4. The parameter space: Virtual centers

The hyperbolic maps form a natural and important subset of $\mathcal{F}$. In this family these maps can be characterized as

$$\mathcal{H} = \{\lambda \in \mathbb{C}^* : f_\lambda \text{ has an attracting periodic cycle}\}.$$

We give an analytic description of the parameter space in terms of the connected components of $\mathcal{H}$.

We denote a connected component of $\mathcal{H}$ by Ω. The attracting periodic points $z_i(\lambda)$ and their multipliers $M(\lambda) = M(\lambda, z_i(\lambda))$ are holomorphic functions of λ in Ω.

We remark that the function $h(z) = \bar{z}$ topologically conjugates the function f_λ to the function $f_{\bar{\lambda}}$ and conjugates the periodic points $z(\lambda)$ to periodic points $\overline{z(\lambda)} = z(\bar{\lambda})$ of the same order. Therefore if Ω is a hyperbolic component so is $\bar{\Omega} = \{\lambda | \bar{\lambda} \in \Omega\}$.

The punctured unit disk Δ^* is a hyperbolic component. For all $\lambda \in \Delta^*$ the origin is the single attracting fixed point and the multiplier is λ itself. Other special properties of Δ^* are that for $\lambda \in \Delta^*$ the stable set is a single connected, infinitely connected invariant component and the Julia set is a Cantor set. These properties are discussed in detail in [**6**].

Although some of the discussion below holds for Δ^*, we will really be interested in components of $\mathcal{H} \setminus \Delta^*$. For readability we will use the term *hyperbolic component* to mean these components and we will denote them by Ω.

In [**8**] section 8, we enumerate the hyperbolic components in terms of the following set:

$$\mathcal{C}_0 = \{\infty\}, \quad \mathcal{C}_p = \{\lambda : f_\lambda^p(\lambda i) = \infty\}, p > 0, \quad \mathcal{C} = \bigcup_0^\infty \mathcal{C}_p.$$

Points in $\mathcal{C}_p$ are called *virtual centers* of order p. We prove the following ([**8**], proposition 8.11, theorem 8.12)

THEOREM 4.1. *The virtual centers $c_p \in \mathcal{C}_{p-1}$ are in one to one correspondence with pairs of hyperbolic components (Ω_p, Ω'_p). In Ω_p each function has a pair of periodic cycles of period p and each attracts the orbit of an asymptotic value whereas in Ω'_p each function has a single attracting cycle of period $2p$ which attracts both asymptotic values. The virtual center $c_p \in \mathcal{C}_{p-1}$ is a common boundary point of (Ω_p, Ω'_p). The virtual center c_0 is the point at infinity and it corresponds to the unique pair of components (Ω_1, Ω'_1); these are the only unbounded components and they are linked by hyperbolic components of period 2.*

REMARK 1. *We note that the periodic cycles are holomorphic functions of λ. They have algebraic singularities at parameters λ where the multiplier of the periodic cycle takes the value* 1 *and transcendental singularities at the virtual center of the hyperbolic component in which they are attracting. In fact, the hyperbolic component* *is an asymptotic tract for the multiplier function of the cycle: as $\lambda \to c_p$ inside Ω, $M(\lambda) \to 0$. If λ tends to any other boundary point of Ω, $|M(\lambda)| \to 1$.*

The computer pictures of the λ-plane drawn by W. Jiang, [**7**] show the component pairs bud off the unit circle at the endpoints of rational internal rays (see Figure 1 of [**8**]). That is, these endpoints are roots for a standard bifurcation.

The virtual center λ^* of any pair (Ω_2, Ω_2') satisfies $\lambda^* \tan(\lambda^* i) = \infty$ so that $\lambda^* i = p_k$, for some $k \in \mathbb{Z}$. It follows that λ^* is on the imaginary axis and by theorem 4.1 every point $p_k i$ is the virtual center of such a pair. Below, we will have occasion to enumerate these pairs and will denote them $(\Omega_{2,k}, \Omega_{2,k}')$ (see Figure 1).

We finish this section with the following theorem which was asserted in [**7**] and [**8**].

THEOREM 4.2. *Let Ω be any hyperbolic component except Δ^*. Then there is a conformal homeomorphism $\tilde{M} : \Omega \to \mathbb{H}$ between Ω and the upper half plane $\mathbb{H}$. Moreover, $\tilde{M}$ extends continuously to the boundary $\partial\Omega$ such that the point $i\infty$ is the unique boundary point of $\mathbb{H}$ corresponding to virtual center c of Ω. Thus the virtual center is unique.*

PROOF. Assume for argument's sake that Ω has 2 cycles of period p. Let λ_0 be an arbitrary point of Ω. Let $z_0 = z_0(\lambda_0)$ be the periodic point that contains $\lambda_0 i$ in its immediate basin of attraction D_0 and let $M(\lambda_0)$ be the multiplier of the cycle. Let N be a neighborhood of z_0 and $h : N \to \Delta$ a holomorphic homeomorphism such that $h(z_0) = 0$ and $h(f^p_{\lambda_0}(z)) = m(\lambda_0)h(z)$ for $z \in N$. Choose circles C_r and C_{Mr} centered at 0 in Δ where $r < 1$ and $M = |M(\lambda_0)|$ and let A be the annulus bounded by them. Identify the boundary points of A to obtain a torus T as follows: join the points $re^{i\theta}$ and $M(\lambda_0)re^{i\theta}$ by a spirals σ_θ parameterized by their polar coordinate ψ and choose σ_θ so that ψ doesn't change by more than π. Set $A_0 = h^{-1}(A)$.

Choose a branch $\tau_0 = \frac{1}{2\pi i} \log M(\lambda_0)$ so that $|\Re\tau_0| \leq 1/2$ and so that the covering group Γ_0 of the holomorphic projection of $\mathbb{C}$ onto T is generated by $\zeta \to \zeta + 1$, $\zeta \to \zeta + \tau_0$.

Set $\tilde{M}(\lambda_0) = \tau_0$; define $\tilde{M}(\lambda)$ in a neighborhood V of λ_0 contained in Ω such that $\tilde{M}(\lambda) = \frac{1}{2\pi i} \log M(\lambda)$ where the branch of the logarithm agrees with the branch chosen above so that $\tilde{M}$ is a conformal homeomorphism of V onto its image in $\mathbb{H}$. Note that if $V \cap \Im \neq \emptyset$, by symmetry we can infer $\Re\tilde{M} = 0$ on the imaginary axis.

Let $\tau \in \mathbb{H}$ and consider the group Γ_τ generated by $\zeta \to \zeta+1$, $\zeta \to \zeta+\tau$ acting on the plane $\mathbb{C}$. Let $\tilde{\phi}_\tau : \mathbb{C} \to \mathbb{C}$ be the affine plane map such that $\tilde{\phi}_\tau(0) = 0, \tilde{\phi}_\tau(1) = 1, \tilde{\phi}_\tau(\tau_0) = \tau$; then $\tilde{\phi}_\tau \circ \Gamma_0 = \Gamma_\tau \circ \tilde{\phi}_\tau$. An affine map is quasiconformal and its Beltrami differential $\tilde{\mu}_\tau$ is constant; in particular, we have $\tilde{\mu}_\tau(\zeta + n + m\tau_0) = \tilde{\mu}_\tau(\zeta)$ for any integers m, n.

The map $\pi_1 = \exp 2\pi i\zeta$ is a holomorphic universal cover of a round annulus A_τ by the strip $\{\zeta : 0 \leq \Im\zeta \leq \Im\tau\}$. Using h and π_1 we obtain a quasiconformal map $\phi_\tau : A_0 \to A_\tau$ with $\phi_\tau = \pi_1 \tilde{\phi}_\tau \pi_1^{-1} h$. There is an induced Beltrami differential μ_τ defined on A_0 independent of the choice of π_1^{-1}. We can extend μ_τ to the full grand orbit of A_0 so that it respects the dynamics as follows: For any $n > 0$ and $z^* \in f^n_{\lambda_0}(A_0)$ there is a well defined branch of the inverse, $g_n(z^*) = f^{-n}_{\lambda_0}(z^*) = z \in A_0$; set

$$\mu_\tau(z^*) = \mu_\tau(z) \frac{\overline{(g_n)'(z^*)}}{(g_n)'(z^*)}.$$

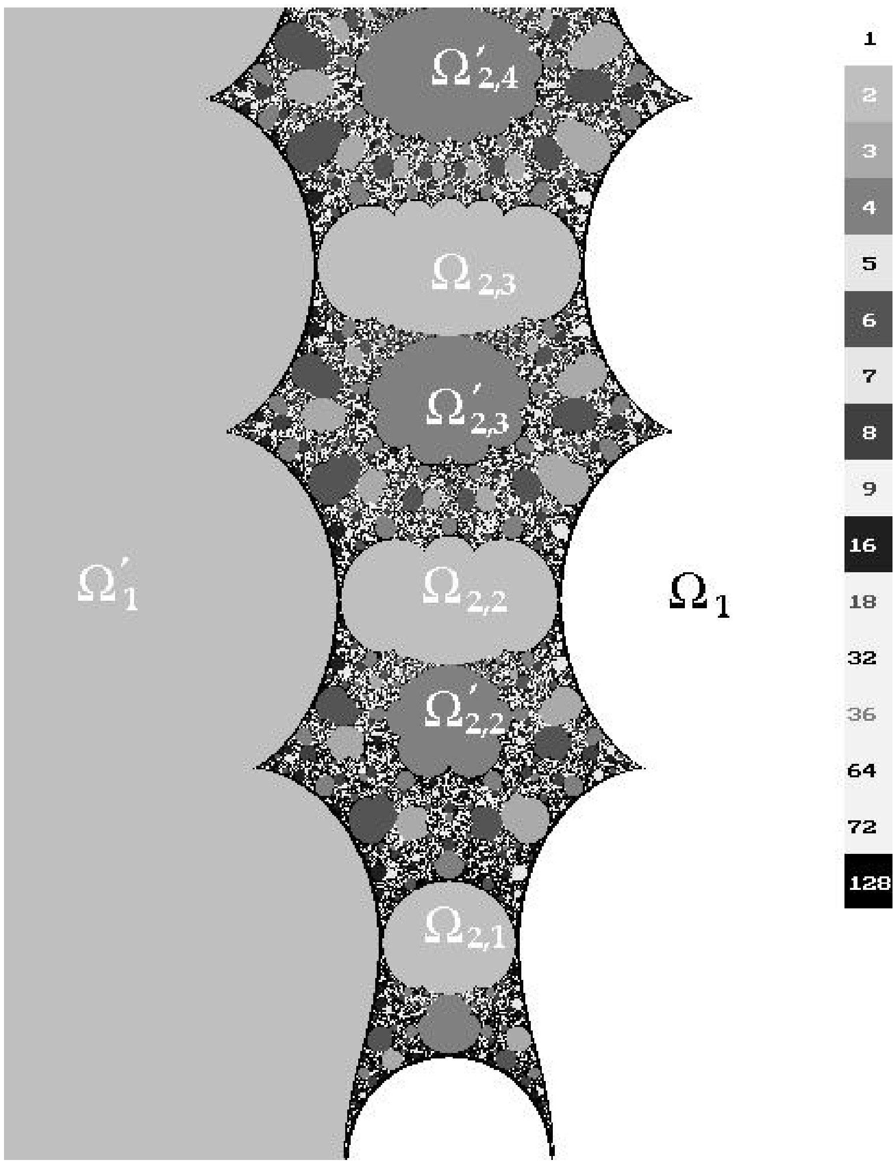

FIGURE 1. Hyperbolic components on the imaginary axis

Now suppose that $f^n_{\lambda_0}(z^*) = z \in A_0$; in this case we set

$$\mu_\tau(z^*) = \mu_\tau(z)\frac{\overline{(f^n_{\lambda_0})'(z^*)}}{(f^n_{\lambda_0})'(z^*)}.$$

Define μ_τ on the other cycle in the Fatou set by symmetry: set $\mu_\tau(-z) = \mu_\tau(z)$. Set $\mu_\tau \equiv 0$ on the complement of the Fatou set.

By the theorem of Ahlfors-Bers et al [**1, 3**] there is a quasiconformal homeomorphism Φ_τ of $\hat{\mathbb{C}}$, depending holomorphically on τ, with $\mu_{\Phi_\tau} = \mu_\tau$, such that $\Phi_\tau(0) = 0$, $\Phi_\tau(\pi/2) = \pi/2$ and $\Phi_\tau(\infty) = \infty$. By construction we conclude that $G_\tau = \Phi_\tau f_{\lambda_0}\Phi_\tau^{-1}$ is holomorphic in both z and τ and that $G_\tau(-z) = -G_\tau(z)$. It now follows from [**5**] that there is some λ_τ such that $G = f_{\lambda_\tau}$. Moreover, λ_τ depends holomorphically on τ since G does. Set $\tilde{M}(\lambda_\tau) = \tau$.

For $t \in [0, 1]$, set $\mu_{t\tau} = t\mu_\tau$ and apply the above. We obtain a path $\lambda_t \in \Omega$ with $\lambda_1 = \lambda_\tau$ and a corresponding path $\tau_t \in \mathbb{H}$ such that $\tau_t = \tilde{M}(\lambda_t)$ is the analytic continuation of $\tilde{M}(\lambda_\tau)$.

Choose any point $\lambda \in \Omega$ and draw a curve α from λ_0 to λ. We can lift the curve α by the above construction to see that there is some τ corresponding to λ.

Suppose now that $\tilde{M}(\lambda_1) = \tilde{M}(\lambda_2) = \tau$. By the construction of ϕ_τ

$$\phi_\tau f_{\lambda_0}\phi_\tau^{-1} = f_{\lambda_1} = f_{\lambda_2}$$

so that $\lambda_1 = \lambda_2$. We thus conclude Ω is conformally isomorphic to $\mathbb{H}$.

If there is one cycle of period $2p$ take the annulus A defined by C_r and C_{mr} where $m = |m(\lambda_0)|$, the modulus of the half multiplier. Let N be a neighborhood of z_0 and let $-N$ be the symmetric neighborhood of $-z_0$. There are holomorphic homeomorphisms $h : N \to \Delta$ with $h(z_0) = 0$ and $\bar{h} : -N \to \Delta$ with $\bar{h}(z) = h(-z)$ so that for $z \in -N$ $h(f^p_{\lambda_0}(z)) = m(\lambda_0)\bar{h}(z)$. Let $A_0 = h^{-1}(A)$ and $\bar{A}_0 = \bar{h}^{-1}(A)$. We proceed in the same manner as above. We obtain a torus T by identifying the boundary arcs of A. Next we choose a branch of the log so that $\tau_0 = \frac{1}{\pi i}\log m(\lambda_0) = \frac{1}{2\pi i}\log M(\lambda_0)$ and $|\Re\tau_0| \leq \frac{1}{2}$. Now, given τ define the maps $\tilde{\phi}_\tau$ and $\phi_\tau$as above. The induced Beltrami differential is symmetric: for $z \in A_0$, $\mu_\tau(-z) = \mu_\tau(z)$. Now every point in the whole Fatou set is in the grand orbit of a point in either A_0 or $\bar{A}_0$ so we may use the above equations to extend μ_τ to this set so that it respects the dynamics. Again we set μ_τ to be identically 0 off the Fatou set and proceed as above to obtain the map $\tilde{M}$.

We extend $\tilde{M}$ to $\partial\Omega$ in the obvious way so that it preserves prime ends. There is thus a unique point c on $\partial\Omega$ corresponding to the point at infinity of the upper half plane $\mathbb{H}$. By construction of $\tilde{M}$, for $it \in \mathbb{H}$, $\lim_{t\to\infty} M(\lambda(it)) = 0$ so c is the unique virtual center of Ω. □

5. The Parameter Space: Imaginary axis

For the family $\mathcal{F}$ the imaginary axis $\Im$ in the parameter plane plays the same role as the real axis does for the quadratic family. To justify this statement we will prove several basic facts.

PROPOSITION 5.1. *If $\lambda \in \Im$ and $z \in \mathbb{R}$ then for all $n \in \mathbb{N}$ we have $f^{2n}_\lambda(z) \in \mathbb{R}$ and $f^{2n+1}_\lambda(z) \in \Im$ or for some n, $f^{2n+1}_\lambda(z) = \infty$.*

PROOF. Suppose $\lambda \in \Im$. Then f_λ maps the imaginary axis univalently onto the subinterval $(-|\lambda|, |\lambda|)$ of the real axis. It also maps any period interval $(p_k, p_k + \pi)$

in the real axis univalently onto the full imaginary axis. Thus for $z \in \mathbb{R}$, $f_\lambda^{2n}(z) \in \mathbb{R}$ and either $f_\lambda^{2n}(z)$ is a pole or $f_\lambda^{2n+1}(z) \in \Im$. $\square$

PROPOSITION 5.2. *If $\lambda \in \Im$ belongs to a hyperbolic component, then the period of the attracting cycle (or cycles) is even.*

PROOF. For λ in any hyperbolic component, the orbits of the asymptotic values accumulate on the attracting periodic cycles. Under our hypothesis the asymptotic values are real and 0 is a repelling fixed point, and so by the above proposition, their orbits alternate between the real and imaginary axes. It follows that the periodic points of the attracting cycles alternate between the real and imaginary axes and the period must be even. $\square$

REMARK 2. *If Ω intersects the imaginary axis, then $\Omega = -\bar{\Omega}$. This follows directly from the topological conjugation by complex conjugation and the symmetry of the tangent family.*

PROPOSITION 5.3. *Suppose Ω intersects the imaginary axis $\Im$. Then it has a unique virtual center on the imaginary axis and $J = \Omega \cap \Im$ is an interval whose endpoints are the virtual center λ_0 and a parameter λ_1 such that $\lim_{\lambda \to \lambda_1} \tilde{M}(\lambda) = \pm 1$, where the limit is taken inside Ω.*

PROOF. By theorem 4.2, Ω is simply connected so that J is a union of intervals. Moreover, there is a unique point c of $\partial\Omega$ corresponding to the virtual center. By the remark above, $-\bar{c} = c$ are both in $\partial\Omega$ and so would both have to be centers. Thus, $c \in \Im$.

Suppose I_0, I_1 are distinct intervals contained in J. Let λ_0, λ_1 belong to I_0, I_1 respectively. Let $\lambda \notin \Im$ belong to Ω. Then we can draw a closed curve γ, entirely in Ω joining, in order, $\lambda_0, \lambda, \lambda_1, -\bar{\lambda}, \lambda_0$. There are points not in Ω on $\Im$ between I_0 and I_1 so γ is homotopically non-trivial. This, however, contradicts the simple connectivity of Ω so J is an interval.

Suppose $\lambda = iy \in J$, $y \in \mathbb{R}$. Whether there are one or two attractive cycles (of period $2n$), the periodic points satisfy $z_{2j}(\lambda) = x_{2j}$, $z_{2j+1}(\lambda) = x_{2j+1}i$, $j = 0, \ldots, n-1$ where $x_j \in \mathbb{R}$. Then

$$M(\lambda) = (iy)^{2n} \Pi_0^{n-1} \sec^2(x_{2j}) \Pi_0^{n-1} \mathrm{sech}^2(x_{2j+1}) \tag{1}$$

is real on J and at its endpoints. Thus J is a component of the pullback of a lift to $\mathbb{H}$ of either the positive or the negative real axis in Δ^* and $M(\lambda)$ either takes values in $(0,1)$ or in $(-1,0)$ on J. The endpoint of λ_0 where the limit value (taken inside Ω) is 0 is thus equal to c, the unique virtual center of Ω and the value at the other endpoint λ_1 is either 1 or -1. $\square$

It follows from this proposition and the construction in the proof of theorem 4.1 in [**8**] (see also the proof of proposition 5.4 below) that if a hyperbolic component intersects the imaginary axis, so does its paired component. This pair must have the form $(\Omega_{2n}, \Omega'_{2n})$ where for $\lambda \in \Omega_{2n}$ the function f_λ has two attractive cycles of period $2n$, and for $\lambda \in \Omega'_{2n}$ there is only one attractive periodic cycle of period $4n$.

We complete the characterization of these components with

PROPOSITION 5.4. *If $ix, x \in \mathbb{R}$ is a virtual center of a component pair (Ω, Ω'), then both Ω and Ω' intersect the imaginary axis non-trivially.*

Proof. By theorem 4.1 we know that a pair of hyperbolic components exists at each virtual center. We must show these components intersect the imaginary axis.

By symmetry, if $ix, x \in \mathbb{R}$ is a virtual center of a component pair (Ω, Ω') then so is $-ix$. Assume for argument's sake that $x > 0$. Then we know that for some p, $f^p_{-ix}(x) = \infty$ so that for some pole p_k, $f^{p-1}_{-ix}(x) = p_k \in \mathbb{R}$. By proposition 5.1 therefore, $p - 1 = 2n$ for some n.

Suppose $y \in \mathbb{R}$ is close to x. Then $f^{p-1}_{-iy}(y) \in \mathbb{R}$ and belongs to a preasymptotic tract of the asymptotic value y. Following the proofs of [**8**], proposition 8.11, theorem 8.12, we can show that as $y \to x$ from above or below, y belongs to a hyperbolic component Ω_{2n} or Ω'_{2n}. It follows that the components attached to ix intersect the imaginary axis. □

The intersection of a hyperbolic component with the imaginary axis is either completely contained in the positive or negative imaginary axis. There is therefore a natural order imposed by the order of their intersections with this axis that is symmetric with respect to reflection in the real axis. We will use the following notation which takes this symmetry into account.

In the discussion below of the ordering of component pairs we will also care about the order of the components in the pair. We will write (Ω_p, Ω'_p) if

$$sup_{\lambda \in \Omega_p \cap \Im} |\lambda| \leq inf_{\lambda \in \Omega'_p \cap \Im} |\lambda|$$

and (Ω'_p, Ω_p) if

$$sup_{\lambda \in \Omega'_p \cap \Im} |\lambda| \leq inf_{\lambda \in \Omega_p \cap \Im} |\lambda|.$$

We order the component pairs intersecting the imaginary axis in terms of their distance from the origin. We say

$$(\Omega_p, \Omega'_p) \lhd (\Omega_q, \Omega'_q)$$

- if both pairs intersect $\Im^+$ and

$$sup_{\lambda \in (\Omega_p \cup \Omega'_p) \cap \Im^+} |\lambda| \leq inf_{\lambda \in (\Omega_q \cup \Omega'_q) \cap \Im^+} |\lambda|$$

- or both pairs intersect $\Im^-$ and

$$sup_{\lambda \in (\Omega_p \cup \Omega'_p) \cap \Im^-} |\lambda| \leq inf_{\lambda \in (\Omega_q \cup \Omega'_q) \cap \Im^-} |\lambda|$$

6. The parameter space: Combinatorial structure

We now turn our attention to the orbits $\{f^n_\lambda(\pm\lambda i)\}$ and define another family of real functions. By the symmetry of the tangent function, these orbits are symmetric. For the sake of readability in the rest of the paper, we assume that $\Im\lambda < 0$. Let $x = \lambda i$ and set $g_n(x) = \{f^n_\lambda(\lambda i)\}$. Set $g_0(x) = x$ and write recursively,

$$g_{2n}(\lambda i) = g_{2n}(x) = -ix \tan(-ix \tan g_{2n-2}(x)) = -x \tanh(x \tan g_{2n-2}(x)).$$

Thus $g_{2n}(x)$ is a map from the real axis to itself.

We note that $g_1 = -ix\tan(x)$ has poles at $x = \lambda i = p_k$ and g_2 has essential singularities at these poles. We can compute the asymptotic values:

$$\lim_{x \to p_k^+} g_2(x) = p_k, \quad \lim_{x \to p_k^-} g_2(x) = -p_k.$$

In general the iterate $g_n(\lambda) = f^n_\lambda(\lambda i)$ is a holomorphic function of λ on the complement of the virtual centers of order strictly less than n. It has essential singularities at these virtual centers and it has poles at the virtual centers of order n.

We note that there are no poles of g_2 on the real axis. The singularities of $g_{2n}(x)$ are singularities of $g_{2n-2}(x)$ and poles and singularities of $g_{2n-1}(x)$. Thus the singularities of $g_{2n}(x)$ are all the points $s_{\pm k,j}$ such that $g_{2j}(s_{\pm k,j}) = \pm p_k$ for all integers $k > 0$ and $j < n$.

6.1. Branches of g_{2n}. In figure 2 we show graphs of $g_2(x)$ and $g_4(x)$ in the interval $(3\pi/2, 5\pi/2)$. We see how g_4 has singularities at the points where the value of $g_2(x)$ is a pole of $g_3(x)$. We also see that the shapes of each of the g_4 curves is similar to that of the g_2 curves but is inverted.

The figure leads us to define a *branch* of g_{2n} as g_{2n} restricted to an interval $I = (p, q)$ between singularities. We define a *full branch* for g_{2n} to be a branch defined between consecutive poles of g_{2n-1} on an interval $I = (p, q)$ such that for n odd $g_{2n}(p) > g_{2n}(q)$ and for even n, $g_{2n}(p) < g_{2n}(q)$. Notice that the virtual centers corresponding to the endpoints of a full branch are of the same order.

Computer experiments, shown in figure 2 for $k = 2$ and $n = 1, 2$ but carried out for other k and for n up to 8, indicate that each full branch has a unique turning point and that all the full branches of the next generation are to the right of the turning point. There are always exactly $2k$ full branches in these experiments. What we can prove is that there exist at least $2k$ full branches and at most finitely many. First we gather some facts:

LEMMA 6.1. *Let $I_k = (p_{k-1}, p_k)$ and let*

$$g_2(x) = -x \tanh(x \tan x),\ x \in I_k.$$

Then

i) $\lim_{x\to p_{k-1}^+} g_2(x) = p_{k-1}, \quad \lim_{x\to p_k^-} g_2(x) = -p_k$

ii) $\lim_{x\to p_{k-1}^+} \frac{dg_2}{dx}(x) = 1, \quad \lim_{x\to p_k^-} \frac{dg_2}{dx}(x) = -1$

iii) *there are an odd number of points in $x_j \in I,\ j = 1, \ldots, 2m+1$ such that $\frac{dg_2}{dx}(x_j) = 0$. Moreover $x_j < \hat{x}_k$ where $\hat{x}_k$ is the unique solution in (p_{k-1}, π) to $x \tan x = -1$.*

PROOF. Let $y_1 = x \tan x$ and write $g_2(x) = -x \tanh y_1$. Since $\Im\lambda < 0$, $x = \lambda i > 0$. We may assume that $k \geq 1$. Then $y_1 \leq 0$ for $x \in (p_{k-1}, k\pi]$ and $\lim_{x\to p_{k-1}^+} y_1(x) = -\infty$. Thus $\lim_{x\to p_{k-1}^+} g_2(x) = p_{k-1}$. Analogously $y_1 \geq 0$ for $x \in [k\pi, p_k)$ and $\lim_{x\to p_k^-} y_1(x) = +\infty$. Thus $\lim_{x\to p_k^-} g_2(x) = -p_k$. We note that $y_1' = \tan x + x\sec^2 x > \pi/2 - 1$ on I_k because for any integer j and $x \in (2j\pi - \pi/2, 2j\pi + \pi/2)$,

$$\tan x + x\sec^2 x = \frac{1}{\cos x}(\sin x + \frac{x}{\cos x}) > (-1 + x) > \pi/2 - 1$$

and for any j and $x \in (2j\pi + \pi/2, 2j\pi + 3\pi/2)$,

$$\tan x + x\sec^2 x = \frac{1}{\cos x}(\sin x + \frac{x}{\cos x}) > (-1)(1 - x) > -1 + \pi/2$$

Next compute

$$y_1'' = 2\sec^2(x)(1 + x\tan x)$$

and note that there is a unique solution $\hat{x}_k$ to $x \tan x = -1$ in (p_{k-1}, π) and so a single inflection point.

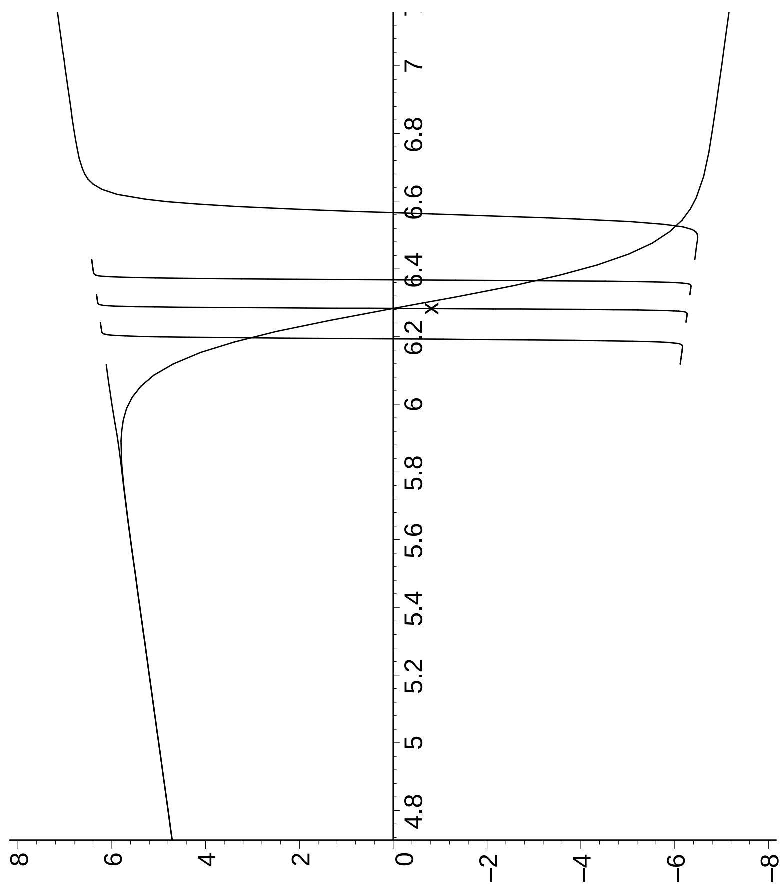

FIGURE 2. The curves $g_2(x)$ and $g_4(x)$

Now compute $\frac{dg_2}{dx} = -\tanh y_1 - x\,\mathrm{sech}^2\, y_1 y_1'$. Because $\lim_{x\to p_{k-1},p_k} \mathrm{sech}^2\, y_1 y_1' = 0$,

$$\lim_{x\to p_{k-1}^+} \frac{dg_2}{dx} = 1 \text{ and } \lim_{x\to p_k^-} \frac{dg_2}{dx} = -1$$

Moreover $\lim_{x\to k\pi} \frac{dg_2}{dx} = -(k\pi)^2$.

Thus for $x \in [k\pi, p_k)$ both summands in $\frac{dg_2}{dx}$ are negative and $\frac{dg_2}{dx} < 0$. We also we check that $\frac{dg_2}{dx}(\hat{x}_k) < 0$ and since y_1' increases for $x > \hat{x}_k$, $\frac{dg_2}{dx} < 0$ remains negative in the interval $[\hat{x}_k, p_k)$.

For $x \in (p_{k-1}, \hat{x}_k)$, by the above, $\lim_{x \to p_{k-1}^+} \frac{dg_2}{dx} = 1$ and $\lim_{x \to \hat{x}_k} \frac{dg_2}{dx} < 0$ so that $\frac{dg_2}{dx} = 0$ at least once and in general an odd number of times in this subinterval. There cannot be infinitely many solutions because $\frac{dg_2}{dx}$ is the restriction of a holomorphic function of a complex variable and its zeroes can only accumulate at a singularity. At the singularity p_{k-1} however, we have shown $\frac{dg_2}{dx} = 1$. □

We have similar but weaker results for full branches of g_{2n} for all n.

Lemma 6.2. *Assume that $I = (p, q)$ is an open interval on which a full branch of g_{2n+2} is defined, such that if n is odd $g_{2n}(p) = p_k$, $g_{2n}(q) = p_{k-1}$ while if n is even, $g_{2n}(p) = p_{k-1}$, $g_{2n}(q) = p_k$. Moreover, assume that neither p nor q is a turning point for g_{2n}. Then*

i) *In the odd case*

$$\lim_{x \to p^+} g_{2n+2}(x) = -p \qquad \lim_{x \to q^-} g_{2n+2}(x) = q$$

and

$$\lim_{x \to p^+} \frac{dg_{2n+2}}{dx}(x) = -1 \qquad \lim_{x \to q^-} \frac{dg_{2n+2}}{dx}(x) = 1$$

ii) *In the even case*

$$\lim_{x \to p^+} g_{2n+2}(x) = p \qquad \lim_{x \to q^-} g_{2n+2}(x) = -q$$

and

$$\lim_{x \to p^+} \frac{dg_{2n+2}}{dx}(x) = 1 \qquad \lim_{x \to q^-} \frac{dg_{2n+2}}{dx}(x) = -1$$

iii) *There are an odd number of points in $x_j \in I$, such that $\frac{dg_{2n+2}}{dx}(x_j) = 0$ and if t is the rightmost solution to $g_{2n}(t) = k\pi$, then $x_j < t, j = 1, \ldots, 2m+1$.*

Proof. Lemma 6.1 contains this lemma for $n = 1$. We proceed by induction. Let $y_{2n+1}(x) = x \tan g_{2n}(x)$ and set $I_0 = (p, t], I_1 = [t, q)$, where t is the rightmost solution to $g_{2n}(t) = k\pi$. Assume first that n is odd. By hypothesis, for x close to p in I_0, $\tan g_{2n}(x) > 0$ so that $y_{2n+1}(x) \geq 0$ and $\lim_{x \to p^+} y_{2n+1} = \infty$. Then $\lim_{x \to p^+} g_{2n+2}(x) = -p$. Analogously for $x \in I_1$, $y_{2n+1}(x) \leq 0$ so that $\lim_{x \to q^-} y_{2n+1} = -\infty$. Thus $\lim_{x \to q^-} g_{2n+2}(x) = q$. Moreover $g_{2n}(t) = k\pi$ so $y_{2n+1}(t) = 0$. Compute

$$\frac{dy_{2n+1}}{dx} = \tan g_{2n} + x \sec^2 g_{2n} \frac{dg_{2n}}{dx}.$$

Now apply the induction hypothesis that the lemma is true for n: neither p nor q is a turning point for g_{2n} so that close to p in I_0 and close to q in I_1, $\frac{dg_{2n}}{dx} < 0$. Then

$$\lim_{x \to p^+} \frac{dy_{2n+1}}{dx} = \lim_{x \to p^+} x \sec^2 g_{2n} \frac{dg_{2n}}{dx} = -\infty.$$

$$\lim_{x \to q^-} \frac{dy_{2n+1}}{dx} = \lim_{x \to q^-} x \sec^2 g_{2n} \frac{dg_{2n}}{dx} = -\infty.$$

Because on I_1 $\tan g_{2n} \leq 0$ by definition and $\frac{dg_{2n}}{dx} < 0$ by the induction hypothesis, it follows from the formula that $\frac{dy_{2n+1}}{dx} < 0$ on I_1. By definition $g_{2n+2}(x) = -x \tanh y_{2n+1}(x)$ and so

$$\frac{dg_{2n+2}}{dx} = -\tanh y_{2n+1} - x \,\mathrm{sech}^{\,2} y_{2n+1} \frac{dy_{2n+1}}{dx} > 0$$

Again, because $\frac{dg_{2n+2}}{dx}$ is the restriction of a holomorphic function whose only singularity in a neighborhood of I_0 is at the left endpoint, the number of zeroes is finite and by the sign considerations above, the number is odd.

Now assume n is even so that by definition g_{2n} is increasing on I_1. Changing signs where appropriate above we deduce that there again must be an odd number of zeros of the derivative. □

COROLLARY 6.3. *Let $I = (p, q)$ be an interval for a full branch of g_{2n} contained in the interval $I_k = (p_{k-1}, p_k)$. Then g_{2n+2} has at least $2k$ full branches in I.*

PROOF. Because $|g_{2n}(x)| < |x|$, the range of g_{2n} on I is contained in $(-p_k, p_k)$. Assume for argument's sake that n is even so that the asymptotic value of g_{2n} at p is $-p$. Then $g_{2n}(x)$ decreases, but not below $-p_k$, makes an odd number of turns, then increases and again takes the value $-p$ at $x_1 \in (p, q)$. Since it turns finitely often and finally continues increasing monotonically and has an asymptotic value q at q, the interval $(-p, q)$ covers $2k + 1$ poles; thus there are at least $2k$ full branches. □

COROLLARY 6.4. *Let $I_k = (p_{k-1}, p_k)$. Then g_{2n} has at least $(2k)^{n-1}$ full branches in I_k.*

The proof follows directly from corollary 6.3.

6.2. Branches of $h_{q,2n}$. We also define another family of maps $h_{q,2n}(x)$, $n \in \mathbb{N}$. As before we assume that $\Im\lambda < 0$. Set $\lambda = -iq, q \in \mathbb{R}^+$, then

$$h_{q,2}(x) = -iq \tan(-iq \tan(x)) = -q \tanh(q \tan(x)).$$

Let $h_{q,0}(x) = x$ and write recursively,

$$h_{q,2n}(x) = -iq \tan(-iq \tan h_{q,2n-2}(x)) = -q \tanh(q \tan h_{q,2n-2}(x)).$$

Thus $h_{q,2n}(x)$ is a map from the real axis to itself. We note that $h_{q,1}(x) = -iq \tan(x)$ has poles at $x = p_k$ and $h_{q,2}$ has essential singularities at these poles. We can compute the asymptotic values:

$$\lim_{x \to p_k^+} h_{q,2}(x) = q, \quad \lim_{x \to p_k^-} h_{q,2}(x) = -q.$$

We note that there are no poles of $h_{q,2}$ on the real axis. The singularities of $h_{q,2n}(x)$ are singularities of $h_{q,2n-2}(x)$ and poles and singularities of $h_{q,2n-1}(x)$. Thus the singularities of $h_{q,2n}(x)$ are all the points $s_{\pm k,j}$ such that $h_{q,2j}(s_{\pm k,j}) = \pm p_k$ for all integers $k > 0$ and $j < n$. First we gather some facts:

LEMMA 6.5. *Let $I_k = (p_{k-1}, p_k)$ and let*

$$h_{q,2}(x) = -q \tanh(q \tan x),\ x \in I_k.$$

Then

i) $\lim_{x \to p_{k-1}^+} h_{q,2}(x) = q, \quad \lim_{x \to p_k^-} h_{q,2}(x) = -q$

ii) $\frac{dh_{q,2}}{dx}(x) < 0, \quad x \in I_k,$

iii) $\lim_{x\to p_{k-1}^+} \frac{dh_{q,2}}{dx}(x) = 0, \quad \lim_{x\to p_k^-} \frac{dh_{q,2}}{dx}(x) = 0$

PROOF. Let $y_1 = q\tan x$ and write $h_{q,2}(x) = -q\tanh y_1$. Since $q > 0$. We may assume that $k \geq 1$. Then $y_1 \leq 0$ for $x \in (p_{k-1}, k\pi]$ and $\lim_{x\to p_{k-1}^+} y_1(x) = -\infty$. Thus $\lim_{x\to p_{k-1}^+} h_{q,2}(x) = q$. Analogously $y_1 \geq 0$ for $x \in [k\pi, p_k)$ and $\lim_{x\to p_k^-} y_1(x) = +\infty$. Thus $\lim_{x\to p_k^-} h_{q,2}(x) = -q$. We note that $y_1' = q\sec^2 x > 0$ on I_k, while $h_{q,2}'(x) = -q^2\operatorname{sech}^2(q\tan(x))\sec^2 x < 0$. □

The graphs of $h_{q,2}(x)$ and $g_2(x)$ have similar branching and similar shapes; but the graph of $h_{q,2}(x)$ is asymptotic to the horizontal at p_{k-1} and p_k. Also the shapes of the $h_{q,4}$ curves is similar to that of the $h_{q,2}(x)$ curves but is inverted. We define a *branch* of $h_{q,2n}$ as $h_{q,2n}$ restricted to an interval $I = (p, p')$ between singularities. We define a *full branch* for $h_{q,2n}$ to be a branch defined between consecutive poles of $h_{q,2n-1}$ on an interval $I = (p, p')$ such that for n odd $h_{q,2n}(p) > h_{q,2n}(p')$ and for even n, $h_{q,2n}(p) < g_{2n}(p')$.

LEMMA 6.6. *Assume that $I = (p, p')$ is an open interval on which a full branch of $h_{q,2n+2}$ is defined, such that if n is odd $h_{q,2n}(p) = p_k$, $h_{q,2n}(p') = p_{k-1}$ while if n is even, $h_{q,2n}(p) = p_{k-1}$, $h_{q,2n}(p') = p_k$. Then*

i) *In the odd case*

$$\lim_{x\to p_+} h_{q,2n+2}(x) = -q \qquad \lim_{x\to p'_-} h_{q,2n+2}(x) = q$$

$$\frac{dh_{q,2n+2}}{dx}(x) > 0, \quad x \in (p, p')$$

and

$$\lim_{x\to p_+} \frac{dh_{q,2n+2}}{dx}(x) = 0 \qquad \lim_{x\to p'_-} \frac{dh_{q,2n+2}}{dx}(x) = 0$$

ii) *In the even case*

$$\lim_{x\to p_+} h_{q,2n+2}(x) = q \qquad \lim_{x\to p'_-} h_{q,2n+2}(x) = -q$$

and

$$\frac{dh_{q,2n+2}}{dx}(x) < 0, \quad x \in (p, p')$$

$$\lim_{x\to p_+} \frac{dh_{q,2n+2}}{dx}(x) = 0 \qquad \lim_{x\to p'_-} \frac{dh_{q,2n+2}}{dx}(x) = 0$$

PROOF. Lemma 6.5 contains this lemma for $n = 1$. We proceed by induction. Let $y_{2n+1}(x) = q\tan h_{q,2n}(x)$ and set $I_0 = (p, t], I_1 = [t, p')$, where t is the unique solution to $h_{q,2n}(t) = k\pi$. Assume first that n is odd. By hypothesis, for x close to p in I_0, $\tan h_{q,2n}(x) > 0$ so that $y_{2n+1}(x) \geq 0$ and $\lim_{x\to p_+} y_{2n+1} = \infty$. Then $\lim_{x\to p_+} h_{q,2n+2}(x) = -q$. Analogously for $x \in I_1$, $y_{2n+1}(x) \leq 0$ so that $\lim_{x\to p'_-} y_{2n+1} = -\infty$. Compute

$$\frac{dh_{q,2n+2}}{dx} = -q^2 \operatorname{sech}^2(q\tan h_{q,2n})\sec^2(h_{q,2n})\frac{dh_{q,2n}}{dx}.$$

Since $-q^2\operatorname{sech}^2(q\tan h_{q,2n})\sec^2(h_{q,2n}) < 0$ we deduce from the induction hypothesis $\frac{dh_{q,2n}}{dx} < 0$, that $\frac{dh_{q,2n+2}}{dx} > 0$. Since $h_{q,2n}(p) = p_k$ and $h_{q,2n}(p') = p_{k-1}$ then $\lim_{x\to p} \frac{dh_{q,2n}}{dx}(x)$ and $\lim_{x\to p'} \frac{dh_{q,2n}}{dx}(x)$ are finite. But

$$\lim_{x\to p_+} -q^2 \operatorname{sech}^2(q\tan h_{q,2n})\sec^2(h_{q,2n})(x) =$$

$$\lim_{x\to p'} -q^2 \operatorname{sech}^2(q\tan h_{q,2n})\sec^2(h_{q,2n})(x) = 0$$

Thus $\lim_{x\to p+} \frac{dh_{q,2n+2}}{dx}(x) = 0$ and $\lim_{x\to p'_-} \frac{dh_{q,2n+2}}{dx}(x) = 0$. For n even the proof is analogous. □

7. Sharkovskii type ordering

We restrict our discussion to the negative imaginary axis. To get the analogous results for the positive imaginary axis we need only reflect in the real axis.

By theorem 4.1 we see that there is an ordered pair $(\Omega'_{2,k}, \Omega_{2,k})$ centered at each point $-p_k i$. We want to see what other periods occur for hyperbolic components between such pairs. To this end we have the following almost immediate corollary of proposition 5.4 and corollary 6.3. We use the notation $\Omega_{2n,q}$ (resp. $\Omega'_{2n,q}$) to denote a hyperbolic component with virtual center at $-iq$.

Proposition 7.1. *Assume that $k \geq 1$. Let $I = (p, q)$ be an interval for a full branch of g_{2n} contained in the interval $I_k = (p_{k-1}, p_k)$.*

- *For even n there exist hyperbolic components $\Omega'_{2n,p}$ and $\Omega_{2n,q}$ with virtual centers respectively at $-pi$ and $-qi$ such that $\Omega'_{2n,p} \cap \Im = (-pi, -ai)$ and $\Omega_{2n,q} \cap \Im = (-bi, -qi)$ where $p < a < b < q$.*
- *For odd n there exist hyperbolic components $\Omega_{2n,p}$ and $\Omega'_{2n,q}$ with virtual centers respectively at $-pi$ and $-qi$ such that $\Omega_{2n,p} \cap \Im = (-pi, -ai)$ and $\Omega'_{2n,q} \cap \Im = (-bi, -qi)$ where $p < a < b < q$.*

Proof. Since the endpoints p and q of I are prepoles of order $2n-1$ they are virtual centers of hyperbolic components Ω_{2n} and Ω'_{2n}. We want to see what order these components occur in.

Let x_1 be just to the right of p inside the hyperbolic component at p and consider the graph of $h_{x_1,2n}(x) = f^{2n}_{-x_1 i}(x)$ as a map of the real axis to itself. We first suppose n is even. Then $\lim_{x\to p^+} h_{x_1,2n}(x) = -x_1$ and the graph of $h_{x_1,2n}(x)$ intersects the line $y = -x$ at a point x_2 close to, but to the right of x_1. It follows that x_2 is part of a periodic cycle of period $4n$ of $f_{-x_1 i}(x)$ and that the hyperbolic component containing $-x_1 i$ is of the type Ω'_{2n}. A similar discussion shows that at a point x_1 in the hyperbolic component just to the left of q, the graphs of $h_{x_1,2n}(x)$ and $y = x$ intersect at a point x_2 near but left of q so that x_2 is a periodic point of period $2n$ and thus the component is of type Ω_{2n}.

Now suppose n is odd. We argue as above to see that for x_1 in the component to the right of p, the graph of $h_{x_1,2n}(x)$ intersects the line $y = x$ at a point x_2 close to, but to the right of p. It follows that the hyperbolic component is of type Ω_{2n}. Similarly, the component to the left of q is of type Ω_{2n}. □

Theorem 7.2. *There are hyperbolic component pairs of every even period on the imaginary axis. Precisely, fix $k > 1$ and the interval $I_k = (p_{k-1}, p_k)$. Choose $n > 0$. Then in $\Lambda_k = -iI_k$ there are at least $2k(2k-1)^{n-2}$ pairs $(\Omega_{2n}, \Omega'_{2n})$ (or $(\Omega'_{2n}, \Omega_{2n})$) whose virtual centers correspond to endpoints of full branches on g_{2n} in I_k. These are ordered inductively: to each of the perhaps more than $2k-1$ pairs of component pairs $(\Omega_{2n}, \Omega'_{2n})$ (or $(\Omega'_{2n}, \Omega_{2n})$) there are at least $2k$ component pairs $(\Omega'_{2n+2}, \Omega_{2n+2})$ (or $(\Omega_{2n+2}, \Omega'_{2n+2})$).*

Before we give the proof we assume $k = 2$ and look at the pairs in the interval $\Lambda_{-2} = [-\frac{3\pi i}{2}, -\frac{5\pi i}{2}]$. It follows from proposition 7.1 and corollary 6.3 that for $n = 1, 2$ and 3 we get (where there may be more components in-between):

$$\begin{gathered}
(\Omega'_{2,-2},\Omega_{2,-2}) \lhd \\
(\Omega_4,\Omega'_4) \lhd \\
(\Omega'_6,\Omega_6) \lhd (\Omega'_6,\Omega_6) \lhd (\Omega'_6,\Omega_6) \lhd (\Omega'_6,\Omega_6) \lhd \\
(\Omega_4,\Omega'_4) \lhd \\
(\Omega'_6,\Omega_6) \lhd (\Omega'_6,\Omega_6) \lhd (\Omega'_6,\Omega_6) \lhd (\Omega'_6,\Omega_6) \lhd \\
(\Omega_4,\Omega'_4) \lhd \\
(\Omega'_6,\Omega_6) \lhd (\Omega'_6,\Omega_6) \lhd (\Omega'_6,\Omega_6) \lhd (\Omega'_6,\Omega_6) \lhd \\
(\Omega_4,\Omega'_4) \\
\lhd (\Omega'_{2,-3},\Omega_{2,-3}) \qquad (2)
\end{gathered}$$

Note that the components $\Omega'_{2,-2}$ and $\Omega_{2,-3}$ are outside of Λ_{-2}. To get the order of hyperbolic components for the next integers $n = 4, 5$ we have to replace each piece of the form

$$\lhd (\Omega'_6,\Omega_6) \lhd (\Omega'_6,\Omega_6) \lhd (\Omega'_6,\Omega_6) \lhd (\Omega'_6,\Omega_6) \lhd$$

in (2) by

$$\begin{gathered}
\lhd (\Omega'_6,\Omega_6) \lhd \\
(\Omega_8,\Omega'_8) \lhd \\
(\Omega'_{10},\Omega_{10}) \lhd (\Omega'_{10},\Omega_{10}) \lhd (\Omega'_{10},\Omega_{10}) \lhd (\Omega'_{10},\Omega_{10}) \lhd \\
(\Omega_8,\Omega'_8) \lhd \\
(\Omega'_{10},\Omega_{10}) \lhd (\Omega'_{10},\Omega_{10}) \lhd (\Omega'_{10},\Omega_{10}) \lhd (\Omega'_{10},\Omega_{10}) \lhd \\
(\Omega_8,\Omega'_8) \lhd \\
(\Omega'_{10},\Omega_{10}) \lhd (\Omega'_{10},\Omega_{10}) \lhd (\Omega'_{10},\Omega_{10}) \lhd (\Omega'_{10},\Omega_{10}) \lhd \\
(\Omega_8,\Omega'_8) \lhd \\
(\Omega'_6,\Omega_6) \lhd \\
(\Omega_8,\Omega'_8) \lhd \\
(\Omega'_{10},\Omega_{10}) \lhd (\Omega'_{10},\Omega_{10}) \lhd (\Omega'_{10},\Omega_{10}) \lhd (\Omega'_{10},\Omega_{10}) \lhd \\
(\Omega_8,\Omega'_8) \lhd \\
(\Omega'_{10},\Omega_{10}) \lhd (\Omega'_{10},\Omega_{10}) \lhd (\Omega'_{10},\Omega_{10}) \lhd (\Omega'_{10},\Omega_{10}) \lhd \\
(\Omega_8,\Omega'_8) \lhd \\
(\Omega'_{10},\Omega_{10}) \lhd (\Omega'_{10},\Omega_{10}) \lhd (\Omega'_{10},\Omega_{10}) \lhd (\Omega'_{10},\Omega_{10}) \lhd \\
(\Omega_8,\Omega'_8) \lhd \\
(\Omega'_6,\Omega_6) \lhd \\
(\Omega_8,\Omega'_8) \lhd \\
(\Omega'_{10},\Omega_{10}) \lhd (\Omega'_{10},\Omega_{10}) \lhd (\Omega'_{10},\Omega_{10}) \lhd (\Omega'_{10},\Omega_{10}) \lhd \\
(\Omega_8,\Omega'_8) \lhd \\
(\Omega'_{10},\Omega_{10}) \lhd (\Omega'_{10},\Omega_{10}) \lhd (\Omega'_{10},\Omega_{10}) \lhd (\Omega'_{10},\Omega_{10}) \lhd \\
(\Omega_8,\Omega'_8) \lhd \\
(\Omega'_{10},\Omega_{10}) \lhd (\Omega'_{10},\Omega_{10}) \lhd (\Omega'_{10},\Omega_{10}) \lhd (\Omega'_{10},\Omega_{10}) \lhd \\
(\Omega_8,\Omega'_8) \lhd \\
(\Omega'_6,\Omega_6) \lhd
\end{gathered}$$

PROOF. Fix $k > 1$ and the interval $I_k = (p_{k-1}, p_k)$. Let $(p, q) \subset I_k$ be a full branch of g_{2n}. Then any component Ω_{2n} or Ω'_{2n} intersecting (p, q) has virtual center at either $-pi$ or $-qi$. The theorem follows by applying proposition 7.1 inductively and using corollary 6.3 to see that for every n and every full branch of g_{2n} defined on (p, q), there is a component pair with virtual center $-ip$ and another with virtual center $-iq$ and that these pairs are interleaved as asserted. Since the pairs are interleaved between pairs of pairs and there are at least $2k - 1$ pairs of pairs, we have at least $2k(2k - 1)^{n-2}$ pairs $(\Omega_{2n}, \Omega'_{2n})$ (or $\Omega'_{2n}, \Omega_{2n})$). □

8. Bifurcations

8.1. Periodic points: Sign of the multiplier. In propositions 5.1 and 5.2 we saw that if $\lambda \in \Im$ then any attractive or parabolic periodic cycle has even period $2n$ and the periodic points belong either to $\mathbb{R}$ or $\Im$. In the proof of proposition 5.3 we saw from equation (1) that the multiplier $M(\lambda)$ of such a cycle is always real. Using equation (1) again we have immediately

PROPOSITION 8.1. *If $\lambda \in \Im \cap \overline{\Omega}$, where Ω is a hyperbolic component and $\overline{\Omega}$ its closure, and if $M(\lambda)$ is the multiplier of the attractive or parabolic periodic cycle of period $2n$, then*

$$\operatorname{sgn}(M(\lambda)) = \operatorname{sgn}(\lambda^{2n}) = (-1)^n. \tag{3}$$

A direct corollary of equation (3) is

PROPOSITION 8.2. *Let $\lambda \in \Im$ belong to a hyperbolic component. If there is a single cycle of period $4n$ then $0 < M(\lambda) < 1$, while if there are two cycles of period $2n$ then either n is odd and $-1 < M(\lambda) < 0$ or n is even and $0 < M(\lambda) < 1$.*

In the components Ω'_{2n} with a single cycle of period $4n$ note that we can choose a particular square root of the multiplier as follows. We set

$$m(\lambda) = \frac{df_\lambda^{2n}}{dz}(z_i(\lambda)).$$

Another immediate corollary of equation (3) is

REMARK 3. *If $\lambda \in \Omega'_{2n} \cap \Im$ then* $\operatorname{sgn} m(\lambda) = (-1)^n$.

We call $m(\lambda)$ the *half multiplier*; note that it is well defined independent of i.

8.2. Orbits of the asymptotic values: Petals. In order to understand the bifurcation process we need to understand how the orbits of the asymptotic values approach the periodic points. For $\lambda \in \Im$ a periodic point is either real or imaginary and we can determine whether the orbit of a given asymptotic value approaches each periodic point from one or both sides.

PROPOSITION 8.3. *Assume $\lambda \in \Im$ and for readability omit the subscript λ.*

i) *Suppose $\lambda \in \Omega_{2n}$ and let $z_0, \ldots, z_{2n-1}$ be the attractive periodic cycle labelled such that $f^{2nk}(\lambda i) \to z_0$. If $M(\lambda) > 0$, then $f^{2nk}(\lambda i) - z_0$ always has the same sign while if $M(\lambda) < 0$ then $f^{2nk}(\lambda i) - z_0$ and $f^{2n(k+1)}(\lambda i) - z_0$ have opposite signs. The same is true for $-z_0$ and $-\lambda i$.*
ii) *Suppose $\lambda \in \Omega'_{2n}$ and let $z_0, \ldots, z_{2n-1}, -z_0, \ldots, -z_{2n-1}$ be the attractive periodic cycle labelled such that $f^{4nk}(\lambda i) \to z_0$ (and $f^{4nk+2n}(-\lambda i) \to z_0$). If $m(\lambda) > 0$ then $f^{4nk+2n}(-\lambda i) - z_0$ and $f^{4nk}(\lambda i) - z_0$ have opposite signs*

while if $m(\lambda) < 0$, then $f^{4nk+2n}(-\lambda i) - z_0$ and $f^{4nk}(\lambda i) - z_0$ have the same sign.

Proof. (i) Choose $\epsilon > 0$ so that the Taylor expansion

$$f^{2n}(z) = z_0 + M(\lambda)(z - z_0) + \mathrm{O}((z - z_0)^2)$$

is valid in $|z - z_0| < \epsilon$. Both z_0 and $f^{2nk}(\lambda i)$ belong to $\mathbb{R}$ so we can define $\operatorname{sgn}(f^{2nk}(\lambda i) - z_0)$. Since $f^{2nk}(\lambda i) \to z_0$ we can find k_0 so that for $k > k_0$, $|f^{2nk}(\lambda i) - z_0| < \epsilon$ and

$$f^{2nk+2n}(\lambda i) - z_0 = M(\lambda)(f^{2nk}(\lambda i) - z_0) + \mathrm{O}(\epsilon^2).$$

If $M(\lambda) > 0$ then $\operatorname{sgn}(f^{2nk+2n}(\lambda i) - z_0) = \operatorname{sgn}(f^{2nk}(\lambda i) - z_0)$; thus, for large k the sign of $f^{2nk}(\lambda i) - z_0$ is constant.

Analogously, if $M(\lambda) < 0$ then

$$\operatorname{sgn}(f^{2nk+2n}(\lambda i) - z_0) = -\operatorname{sgn}(f^{2nk}(\lambda i) - z_0).$$

(ii) Again expand $f^{4n}(z)$ in a Taylor series about z_0 for $|z - z_0| < \epsilon$:

$$\begin{aligned} f^{4n}(z) &= f^{4n}(z_0) + M(\lambda)(z - z_0) + \mathrm{O}((z - z_0)^2) \\ &= z_0 + M(\lambda)(z - z_0) + \mathrm{O}((z - z_0)^2). \end{aligned}$$

Here $M(\lambda) > 0$ so that $f^{4n}(z)$ is on the same side of z_0 as z for large k and

$$\operatorname{sgn}(f^{4n}(f^{4nk}(\lambda i)) - z_0) = \operatorname{sgn}(f^{4nk}(\lambda i) - z_0).$$

Now expand $f^{2n}(z)$ in a Taylor series about $-z_0$ for $|z + z_0| < \epsilon$:

$$\begin{gathered} f^{2n}(z) = f^{2n}(-z_0) + m(\lambda)(z + z_0) + \mathrm{O}((z + z_0)^2) = \\ z_0 + m(\lambda)(z + z_0) + \mathrm{O}((z + z_0)^2). \end{gathered}$$

Since $f^{4nk}(-\lambda i) \to -z_0$ we get

$$f^{2n}(f^{4nk}(-\lambda i)) = z_0 + m(\lambda)(f^{4nk}(-\lambda i) + z_0) + \mathrm{O}((f^{4nk}(-\lambda i) + z_0)^2),$$

or rewriting,

$$f^{2n+4nk}(-\lambda i)) - z_0 = -m(\lambda)(f^{4nk}(\lambda i) - z_0) + \mathrm{O}((f^{4nk}(-\lambda i) + z_0)^2).$$

So if $m(\lambda) > 0$ then $\operatorname{sgn}(f^{2n+4nk}(-\lambda i)) - z_0) = -\operatorname{sgn}(f^{4nk}(\lambda i) - z_0)$, while if $m(\lambda) < 0$ then $\operatorname{sgn}(f^{2n+4nk}(-\lambda i)) - z_0) = \operatorname{sgn}(f^{4nk}(\lambda i) - z_0)$. $\square$

At each periodic parabolic point there are domains that play the role of the attractive basin. The Fatou Flower Theorem describes these domains and shows that they contain petal shaped regions. Further, for each petal at a periodic point of a parabolic cycle of period p there is a foliation of curves invariant under $f^p_{\lambda_0}$ which are tangent to the stable direction inside the petal and tangent to the unstable directions outside of it.

In the following we describe the petals at the parabolic boundary points on the imaginary axis.

Corollary 8.4. i) *Suppose $\lambda_0 \in \partial\Omega_{2n} \cap \Im$. If $M(\lambda_0) = 1$ then there is exactly one petal at each periodic point in the parabolic cycle whereas if $M(\lambda_0) = -1$ then there are two petals at each periodic point in the parabolic cycle.*

ii) *Suppose $\lambda_0 \in \partial\Omega'_{2n} \cap \Im$. If $m(\lambda_0) = 1$ then there are two petals at each periodic point in the parabolic cycle, whereas if $m(\lambda_0) = -1$ then there is only one petal at each periodic point in the parabolic cycle.*

Proof. With the notation of proposition 8.3, for $\lambda \in \Im \cap \Omega$, $\Omega = \Omega_{2n}$ or $\Omega = \Omega'_{2n}$, f_λ^{2n} maps $I = (z_0 - \epsilon, z_0 + \epsilon) \subset \mathbb{R}$ into itself. The combinatorial picture of the orbits of the asymptotic values of f_λ persists as $\lambda \to \lambda_0 \in \overline{\Omega}$.
(i) If $\lambda_0 \in \partial\Omega_{2n}$ then $f_{\lambda_0}^{2nk}(\lambda_0 i) \in I$ and approaches z_0 either from one or both sides. By Proposition 8.3, if $M(\lambda_0) < 0$ then $f_{\lambda_0}^{2nk}(\lambda_0 i) - z_0$ alternately changes sign and the orbit approaches z_0 from both sides. This means there are two stable directions and f_{λ_0} has two petals. If $M(\lambda_0) > 0$ then $f_{\lambda_0}^{2nk}(\lambda_0 i) - z_0$ always has the same sign, so there is only one stable direction. Note that since there are two cycles the other asymptotic value $-\lambda_0 i$ must tend to the other periodic cycle. Thus at z_0 there is exactly one petal. The same is true for $-z_0$.
(ii) Now assume $\lambda_0 \in \partial\Omega'_{2n}$. Then $f^{4nk}(\lambda_0 i) \to z_0$ and $f^{4nk+2n}(-\lambda_0 i) \to z_0$. If $m(\lambda_0) > 0$ then $f^{4nk+2n}(-\lambda_0 i) - z_0$ and $f^{4nk}(\lambda_0 i) - z_0$ have opposite signs so that f_{λ_0} has two stable directions at z_0 and consequently two petals. On the other hand, if $m(\lambda_0) < 0$ then $f^{4nk+2n}(-\lambda_0 i) - z_0$ and $f^{4nk}(\lambda_0 i) - z_0$ have the same sign so that f_{λ_0} has exactly one stable direction at z_0 and consequently only one petal. □

8.3. Periodic points: Derivative with respect to the parameter. The functions $f_\lambda^k(z)$ are meromorphic in both z and λ for all positive integers k as are the periodic points. Thus we can consider the derivatives $\frac{df_\lambda^k}{d\lambda}$.

In the hyperbolic components Ω'_{2n} intersecting the imaginary axis we have the following relation between the derivatives of f^{4n} and f^{2n}.

Lemma 8.5. *Let $\lambda \in \Omega'_{2n}$ and let $z_0(\lambda), \dots z_{4n-1}(\lambda)$ be the corresponding attractive periodic cycle. Then*

$$\frac{df_\lambda^{4n}}{d\lambda}(z_i(\lambda)) = (-1 + m(\lambda))\,\frac{df_\lambda^{2n}}{d\lambda}(z_i(\lambda))$$

where addition of subscripts is modulo $4n$.

Proof. For any z set $z_k = f_\lambda^k(z)$; note that these are not necessarily the periodic points here. Then

$$\frac{df_\lambda^{4n}}{d\lambda}(z) = \tan z_{4n-1} + \lambda \sec^2 z_{4n-1} \tan z_{4n-2} + \dots + \lambda^{4n-1} \prod_{k=1}^{4n-1} \sec^2 z_k \tan z \tag{4}$$

If we now assume the z_k are in the attractive periodic cycle, by its symmetry we have $\tan z_{2n+k}(\lambda) = -\tan z_k(\lambda)$ and $\sec^2 z_{2n+k}(\lambda) = \sec^2 z_k(\lambda)$ for $k = 0, \dots, 2n-1$. Set $z = z_0$ and use the symmetry relations in equation (4) to obtain

$$\frac{df^{4n}}{d\lambda}(z_0(\lambda)) = (-1 + m(\lambda))\,\frac{df^{2n}}{d\lambda}(z_0(\lambda))$$

Choosing $z = z_i(\lambda)$ for any other periodic point gives the general formula. □

Recalling proposition 8.2 and remark 3 and applying continuity we deduce the following relation at the parabolic boundary points of $\overline{\Omega}'_{2n}$.

Corollary 8.6. *Assume $\lambda \in \Im \cup \partial\Omega'_{2n}$ let $z_0(\lambda), \dots, z_{4n-1}(\lambda)$ be the parabolic cycle. Then for any point in the cycle:*

i) *If n is even,* $\frac{df^{4n}}{d\lambda}(z_i)(\lambda) = 0$
ii) *If n is odd,* $\frac{df^{4n}}{d\lambda}(z_i)(\lambda) \neq 0$ *if and only if* $\frac{df^{2n}}{d\lambda}(z_i(\lambda)) \neq 0$

8.4. Period doubling bifurcation. We need the following definitions:

If λ_0 is a parabolic boundary point of a hyperbolic component Ω and if $\partial\Omega$ can be locally parameterized by $|1 + a(\lambda - \lambda_0)^r| = 1$ for some rational $0 < r < 1$ then λ_0 is called a *cusp of* Ω.

If λ_0 is a parabolic boundary point of a hyperbolic component Ω_p with attracting cycle(s) of period p and if there is a component Ω_q with attracting cycle(s) of period q such that $\partial\Omega_p \cap \partial\Omega_q = \{\lambda_0\}$, then if $p|q$ Ω_q is called a *bud* of Ω_p and if $q|p$ then Ω_q is called a *root* of Ω_p.

Recall that in a standard period doubling bifurcation each attractive cycle bifurcates to an attractive cycle of double the period. Here we will also encounter a *non-standard* period doubling bifurcation where a single attractive cycle bifurcates to two distinct attractive cycles of the same period.

Theorem 8.7 (**Period Doubling bifurcations**). *Assume $\lambda_0 \in \Im$ is a parabolic boundary point of a hyperbolic component Ω intersecting $\Im$ and let $z_0, \ldots, z_{2n-1}$, $-z_0, \ldots, -z_{2n-1}$ be the parabolic periodic points. Assume also that the multiplier of the cycle(s) at the parabolic cycle is monotonic on $\Im$.*

i) *Either $\Omega = \Omega_{2n}$, and n is odd. Then $M(\lambda) = -1$ and there are two petals at each periodic point in the parabolic cycles. In this case there is a standard period doubling bifurcation: there is a hyperbolic component Ω_{4n} with root at λ_0 with two attractive cycles of period $4n$.*

ii) *Or $\Omega = \Omega'_{2n}$ and n is even. Then the half multiplier $m(\lambda) = 1$ and there are two petals at each periodic point in the parabolic cycle. In this case there is a non-standard bifurcation; that is, there is a hyperbolic component Ω_{4n} with root at λ_0 with two attractive cycles of period $4n$. Then there is a unique component Ω_{4n} tangent to Ω'_{2n} at λ_0 with two attractive cycles of period $4n$.*

This situation is reflected on the positive imaginary axis.

Proof. We first show that the bifurcations exist. It follows respectively from Proposition 8.1 and Remark 3 that the multiplier $M(\lambda_0) = -1$ in the case i) and the half multipier $m(\lambda_0) = 1$ in the case ii). By Corollary 8.4 in both cases i) and ii) there are two petals at each periodic point in the parabolic cycle(s). In case i) where there are two cycles of period $2n$ and n is odd, we may assume we have labelled the cycles so that the cycle $z_0, \ldots, z_{2n-1}$ attracts $\lambda_0 i$. If $f^k_{\lambda_0}(\lambda_0 i)$ belongs to one petal P' at z_0, then $f^{k+2n}_{\lambda_0}(\lambda_0 i)$ belongs to the other petal P'' at z_0 and $f^{k+4n}_{\lambda_0}(\lambda_0 i)$ belongs to P' again. The analogous statement is true for the other cycle and $-\lambda_0 i$.

In case ii) where there is one cycle of period $4n$ and n is even, the cycle attracts both $\pm\lambda i$. If $f^k_{\lambda_0}(\lambda_0 i)$ belongs to one petal P' at z_0, then $f^{k+2n}_{\lambda_0}(-\lambda_0 i)$ belongs to the other petal P'' at z_0; moreover, $f^{k+4n}_{\lambda_0}(\lambda_0 i)$ belongs to P' and $f^{k+6n}_{\lambda_0}(-\lambda_0 i)$ belongs to P''.

Suppose $\lambda \in \Im \cap \Omega_{2n}$ with n odd or $\lambda \in \Im \cap \Omega'_{2n}$ with n even and let $\lambda \to \lambda_0$. Then either f_λ has two distinct attractive cycles, $z_0(\lambda), \ldots, z_{2n-1}(\lambda)$, and $-z_0(\lambda), \ldots, -z_{2n-1}(\lambda)$ each of period $2n$ or it has a single attractive cycle $z_0(\lambda), \ldots, z_{2n-1}(\lambda)$, $-z_0(\lambda), \ldots, -z_{2n-1}(\lambda)$ of period $4n$.

Let V be a small neighborhood of $z_0(\lambda)$ such that $f^{2n}_\lambda(z)$ or $f^{4n}_\lambda(z)$ restricted to V is holomorphically conjugate to its linear part, $m(\lambda)z$. Then, by the definition of V, $f^{2n}_\lambda(z)$ or f^{4n}_λ is a contraction on V mapping its closure inside itself. For $\lambda = \lambda_0$ we have two attracting petals P' and P'' attached to $z_0(\lambda_0)$; in either case, the map f_λ^{4n} is a contraction on the boundary of each petal P' except for the point z_0; that is, $\partial f_\lambda^{4n}(P') \setminus \{z_0(\lambda_0)\}$ is contained inside P' and similarly for P''.

Denote the intersection of V with $\mathbb{R}$ by $I(\lambda)$. For $\lambda \in \Im$ and close to λ_0 as above, the map f_λ^{4n} is a contraction on $I(\lambda)$. Set $I(\lambda_0) = (P' \cup \{z_0(\lambda_0)\} \cup P'') \cap \mathbb{R}$. This is an interval on which $f_{\lambda_0}^{4n}$ is a contraction. Perturbing λ beyond λ_0, perturbs the endpoints of $I(\lambda_0)$. On the resulting interval $I(\lambda)$, f_λ^{4n} is again a contraction; that is, $\overline{f_\lambda^{4n}(I(\lambda))} \subset I(\lambda)$.

Since we have assumed the multiplier is strictly monotonic in a neighborhood of λ_0 on $\Im$, $z_0(\lambda)$ becomes a repelling fixed point outside the hyperbolic component. Since there are only two asymptotic values, there are at most two cycles. To see whether there are one or two, we must look at the orbits of the asymptotic values. In case i), $z_0(\lambda)$ is now a repelling periodic point of period $2n$. The interval $I(\lambda)$ contains the iterates $f_\lambda^{k+2nj}(\lambda i)$ for some even k and all $j \geq 0$, but no iterates of $-\lambda i$. We deduce that the $4n$ attractive points in the intervals $f_\lambda^j(I(\lambda))$, $j = 0, \dots, 2n$ all belong to the same cycle. The same is true for $-z_0(\lambda)$ and $-I(\lambda)$; the $4n$ new attractive points in the intervals $f_\lambda^j(-I(\lambda))$, $j = 0, \dots, 2n$ all belong to the same cycle and there are two cycles of period $4n$.

In case ii), $z_0(\lambda)$ is now a repelling periodic point of period $4n$. Let L and R be the parts of the interval $I(\lambda)$ to the right and left of $z_0(\lambda)$ respectively. Then L contains the iterates $f_\lambda^{k+4nj}(\lambda i)$ and R contains $f_\lambda^{k+6nj}(-\lambda i)$ (or vice versa). It follows that the $4n$ attractive periodic points in the intervals $f_\lambda^j(L)$, $j = 0, \dots, 4n$ belong to one cycle and the $4n$ attractive periodic points in the intervals $f_\lambda^j(R)$, $j = 0, \dots, 4n$ belong to another cycle. Thus again there are two cycles of period $4n$. We conclude the bifurcation is standard or not as λ_0 is on the boundary of Ω_{2n} or Ω'_{2n}.

We now show that the component Ω_{4n} is tangent to Ω'_{2n} at λ_0. (See [**4**]). Expand $f_{\lambda_0}^{4n}(z)$ in a Taylor series about z_0:

$$f_{\lambda_0}^{4n}(z) = z_0 + a_1(z - z_0) + a_2(z - z_0)^2 + a_3(z - z_0)^3 + \dots \tag{5}$$

where $a_1 = M(\lambda_0) = m(\lambda_0)^2 = 1$. The index of the first non-zero term is equal to one plus the total number of petals associated to a point in the cycle; thus, $a_2 = 0, a_3 \neq 0$. There is a local change of coordinates (λ, z) in a neighborhood of (λ_0, z_0) such that

$$f_\lambda^{4n}(z) = z_0(\lambda) + a_1(\lambda)(z - z_0(\lambda)) + a_2(\lambda)(z - z_0(\lambda))^2 + a_3(\lambda)(z - z_0(\lambda))^3 + \dots \tag{6}$$

where $a_1(\lambda) \neq 0, a_2(\lambda) = 0, a_3(\lambda) \neq 0$ We can expand $a_1(\lambda)$ to get

$$a_1(\lambda) = a_1(\lambda_0) + \hat{a}_1(\lambda_0)(\lambda - \lambda_0) + \hat{a}_2(\lambda_0)(\lambda - \lambda_0)^2 + \hat{a}_3(\lambda_0)(\lambda - \lambda_0)^3 + ... \tag{7}$$

where $a_1(\lambda_0) = 1$. From the assumption that the multiplier is monotone at λ_0 the first non-zero term $\frac{d^m a_1(\lambda)}{d\lambda^m}(\lambda_0) = m!\hat{a}_m(\lambda_0) \neq 0$ occurs for m odd.

Set $F(z, \lambda) = f_\lambda^{4n}(z) - z$. We want to consider solutions of $F(z, \lambda) = 0$ in a neighborhood of (λ_0, z_0). Put $Z = z - z_0, \Lambda = \lambda - \lambda_0$ and write

$$0 = F(Z, \Lambda) = \sum_{j=0,k=0}^{\infty} A_{j,k} Z^j \Lambda^k \tag{8}$$

Then:

$$A_{0,0} = F(z_0, \lambda_0) = 0, \tag{9}$$

$$A_{0,1} = \frac{\partial F}{\partial \lambda}(z_0, \lambda_0) = 0, \tag{10}$$

$$A_{1,0} = \frac{\partial F}{\partial z}(z_0, \lambda_0) = 0 \tag{11}$$

$$A_{2,0} = \frac{\partial^2 F}{\partial z^2}(z_0, \lambda_0) = 0 \tag{12}$$

$$A_{3,0} = \frac{\partial^3 F}{\partial z^3}(z_0, \lambda_0) = 6a_3(\lambda_0) \neq 0 \tag{13}$$

$$A_{1,j} = \frac{\partial^{j+1} F}{\partial \lambda \partial z}(z_0, \lambda_0) = \hat{a}_j(\lambda_0) = 0, j = 1, \dots, m-1 \tag{14}$$

$$A_{1,m} = \frac{\partial^{m+1} F}{\partial \lambda \partial z}(z_0, \lambda_0) = \hat{a}_m(\lambda_0) \neq 0 \tag{15}$$

We claim that $\hat{a}_j(\lambda_0) = 0$ for $j = 1, \dots, m-1$ implies $A_{0,j+1} = 0$.

To prove the claim note first that $a_1(\lambda) = M(\lambda)$ so that $\hat{a}_i(\lambda) = \frac{1}{i!}\frac{d^i M(\lambda)}{d\lambda^i}$ Recall lemma 8.5

$$\frac{\partial f^{4n}}{\partial \lambda}(z_0) = (-1 + m(\lambda_0))\,\frac{\partial f^{2n}}{\partial \lambda}(z_0) = 0.$$

Write $m = m(\lambda), M = M(\lambda)$ and $\phi = f^{2n}(\lambda)$. It is easy to compute that

$$\frac{\partial^i F}{\partial \lambda^i} = (-1+m)\frac{d^i \phi}{d\lambda^i} + \frac{dm}{d\lambda}\frac{d^{i-1}\phi}{d\lambda} + \dots + \left(\begin{array}{c} i-2 \\ r \end{array} \right) \frac{d^r m}{d\lambda^r}\frac{d^{i-r}\phi}{d\lambda^{i-r}} + \dots + \frac{dm^{i-1}}{d\lambda}\phi$$

where $\left(\begin{array}{c} i-2 \\ r \end{array} \right)$ is "$i-2$ choose r".

Since $M = m^2$ we have for i odd

$$\frac{d^i M}{d\lambda^i} = 2(m\frac{d^j m}{d\lambda} + \dots + \left(\begin{array}{c} i \\ r \end{array} \right) \frac{d^r m}{d\lambda^r}\frac{d^{i-r} m}{d\lambda^{i-r}} + \dots + \left(\begin{array}{c} i \\ \frac{i-1}{2} \end{array} \right) \frac{d^{\frac{i-1}{2}} m}{d\lambda^{\frac{i-1}{2}}}\frac{d^{\frac{i+1}{2}}}{d\lambda^{\frac{i+1}{2}}})$$

and for i even

$$\frac{d^i M}{d\lambda^i} = 2(m\frac{d^j m}{d\lambda} + \dots + \left(\begin{array}{c} i \\ r \end{array} \right) \frac{d^r m}{d\lambda^r}\frac{d^{i-r} m}{d\lambda^{i-r}} + \dots + \frac{1}{2}\left(\begin{array}{c} i \\ \frac{i}{2} \end{array} \right) \frac{d^{\frac{i}{2}} m}{d\lambda^{\frac{i}{2}}}^2)$$

Since $m(\lambda_0) = 1$, if $\frac{d^j M(\lambda)}{d\lambda^j}(\lambda_0) = 0$ for $j = 1, \dots, m-1$ we see, by induction on j, that, for all j, $\frac{d^j m(\lambda)}{d\lambda^j}(\lambda_0) = 0$. Thus we have $A_{0,j} = 0$ for $j = 2, \dots, m$ as claimed. Thus

$$0 = A_{3,0}Z^3 + \dots + A_{1,m}Z\Lambda^m + \dots$$

We want to solve for Z in terms of Λ near 0. One solution is $Z = 0$; thus the term $\sum_2^\infty A_{0,j}\Lambda^j = 0$. The other solutions are the roots of

$$Z = c_1\Lambda^{m/2} + c_2\Lambda + \dots \tag{16}$$

where

$$c_1 = \left(-\frac{A_{1,m}}{A_{3,0}}\right)^{m/2} \tag{17}$$

We plug (17) into (16) to see that the roots

$$Z = z - z_0 \approx \left(-\frac{A_{1,m}}{A_{3,0}}\right)^{1/2} (\lambda - \lambda_0)^{m/2} \tag{18}$$

are the solutions of the equation

$$f_\lambda^{4n}(z_0(\lambda)) = z_0(\lambda). \tag{19}$$

We compute the multiplier

$$\frac{\partial f_\lambda^{4n}}{\partial z}(z_0(\lambda)) = 1 + \frac{\partial F}{\partial Z} = 1 + 3A_{3,0}Z^2 + A_{1,m}\Lambda + \dots$$

If we substitute (18) in this equation we see that

$$\frac{\partial f^{4n}{}_\lambda}{\partial z}(z(\lambda)) = 1 + \frac{\partial F}{\partial Z} = 1 - 2A_{1,1}(\lambda - \lambda_0)^m + \dots.$$

Thus λ_0 is on the boundary of a domain $\hat{\Omega}$ where the multiplier of the other solutions of (19) has absolute value less than 1. Therefore, $\hat{\Omega} = \Omega_{4n}$ is a hyperbolic component tangent to Ω'_{2n} at the parabolic point λ_0. $\square$

THEOREM 8.8. *For each $k \geq 1$ the parameter $-p_k i$ is the virtual center of a hyperbolic component pair $(\Omega'_{2,-k}, \Omega_{2,-k})$.*

1) *Let $\lambda_0 \in \partial\Omega'_{2,-k} \cap \Im$ be the parabolic boundary point. Then $\Omega'_{2,-k}$ has a cusp at λ_0.*
2) *Suppose $\lambda_1 \in \partial\Omega_{2,-k} \cap \Im$ is the parabolic boundary point. Then there is a standard period doubling bifurcation i.e. there is a pair of hyperbolic components (Ω_4, Ω'_4) intersecting $\Im$ with root at λ_1*

Thus

$$(\Omega'_{2,-k}, \Omega_{2,-k}) \lhd (\Omega_4, \Omega'_4).$$

and this situation is reflected on the positive imaginary axis.

PROOF. At the end of section 4 we saw that for each pole p_k, the parameter $p_k i$ is the virtual center of a pair $(\Omega'_{2,k}, \Omega_{2,k})$ intersecting $\Im$. By Proposition 5.3 each component of this pair intersects $\Im$ in an interval. We fix $k \geq 1$.
Part 1. Let $z_0(\lambda), \dots, z_3(\lambda)$ be the cycle of period 4 which is attractive for $\lambda \in \Omega'_{2,-k}$. It follows from Proposition 8.1 that $M(\lambda_0) = 1$. By Remark 3 the half multipier $m(\lambda_0) = -1$, thus by Corollary 8.4 there is one petal at each periodic point in the parabolic cycle. Again we expand

$$f_{\lambda_0}^4(z) = z_0 + a_1(z - z_0) + a_2(z - z_0)^2 + a_3(z - z_0)^3 + \dots \tag{20}$$

where $a_1 = M(\lambda_0) = m(\lambda_0)^2 = 1$. Now $a_2 \neq 0$. Again there is a local change of coordinates (λ, z) in a neighborhood of (λ_0, z_0) such that

$$f_\lambda^{4n}(z) = z_0(\lambda) + a_1(\lambda)(z - z_0(\lambda)) + a_2(\lambda)(z - z_0(\lambda))^2 + a_3(\lambda)(z - z_0(\lambda))^3 + \dots \tag{21}$$

where $a_1(\lambda) \neq 0$, $a_2(\lambda) \neq 0$

Label the cycle such that $z_0(\lambda_0) \in \mathbb{R}$, $z_1(\lambda_0) = f_{\lambda_0}(z_0) \in \Im$. Then

$$\frac{\partial f_\lambda^2}{\partial \lambda}(z_0, \lambda_0) = \tan(z_1) + \lambda_0 \sec^2 z_1 \tan z_0 + \lambda_0^2 \sec^2 z_1 \sec^2 z_0 =$$

$$= \frac{z_0}{\lambda_0} + \left[1 + \frac{z_0^2}{\lambda_0^2}\right] z_1 - 1.$$

The real parts of the first and second terms are zero so their sum cannot be equal to -1. Consequently $\frac{\partial f_\lambda^2}{\partial \lambda}(z_0, \lambda_0) \neq 0$. Since

$$\frac{\partial f_\lambda^4}{\partial \lambda}(z_0) = (-1 + m(\lambda)) \frac{\partial f_\lambda^2}{\partial \lambda}(z_0)$$

it follows that $\frac{\partial f_\lambda^4}{\partial \lambda}(z_0) \neq 0$.

Again, set $F(z, \lambda) = f_\lambda^4(z) - z$, put $Z = z - z_0, \Lambda = \lambda - \lambda_0$ and write

$$0 = F(Z, \Lambda) = \sum_{j=0,k=0}^{\infty} A_{j,k} Z^j \Lambda^k. \tag{22}$$

We want to consider solutions of $F(z, \lambda) = 0$ in a neighborhood of (λ_0, z_0). Now we have

$$A_{0,0} = F(z_0, \lambda_0) = 0, \tag{23}$$

$$A_{1,0} = \frac{\partial F}{\partial z}(z_0, \lambda_0) = 0 \tag{24}$$

$$A_{2,0} = \frac{\partial^2 F}{\partial z^2}(z_0, \lambda_0) = 2a_2(\lambda_0) \neq 0 \tag{25}$$

$$A_{0,1} = \frac{\partial F}{\partial \lambda}(z_0, \lambda_0) \neq 0. \tag{26}$$

Thus (22) has the form

$$0 = A_{2,0} Z^2 + A_{0,1} \Lambda + A_{1,1} Z \Lambda + \ldots$$

We want to solve for Z in terms of Λ near 0. Then

$$Z = c_1 \Lambda^{1/2} + c_2 \Lambda + \ldots \tag{27}$$

where

$$c_1 = \left(-\frac{A_{0,1}}{A_{2,0}}\right)^{1/2} \tag{28}$$

This implies that

$$\frac{\partial f^4}{\partial z}(z_0(\lambda)) = 1 + \frac{\partial F}{\partial Z} = 1 + 2A_{2,0}\left(-\frac{A_{0,1}}{A_{2,0}}\right)^{\frac{1}{2}} (\lambda - \lambda_0)^{\frac{1}{2}} + \ldots.$$

Thus the boundary of $\Omega_{2,-k}$ near λ_0 has the same form as

$$|1 + 2A_{2,0}\left(-\frac{A_{0,1}}{A_{2,0}}\right)^{\frac{1}{2}} (\lambda - \lambda_0)^{\frac{1}{2}} + \ldots| = 1.$$

This means that $\partial\Omega_{2,-k}$ has a cusp at λ_0.

Part 2. We prove that there is a standard period doubling bifurcation at the parabolic boundary point $\lambda_1 \in \Omega_{2,-k} \cap \Im$. For $\lambda \in \Omega_{2,-k}$ f_λ has two attractive cycles of period 2. We claim that the multiplier of these cycles is stricly monotonic on the imaginary axis at λ_1. Otherwise there is another hyperbolic component Ω with parabolic boundary point λ_1 such that Ω intersects the imaginary axis and has two attractive cycles of period 2. By Proposition 5.3, Ω has a unique virtual center which is on the imaginary axis. Thus Ω must be some $\Omega_{2,-j}$ for $j \neq k$. It must thus contain the full interval $(\lambda_1, -p_{k+1}i)$. This interval contains the point $-k\pi i$; at this point however, the image of the asymptotic values is the fixed point 0 and so it cannot belong to a hyperbolic component. Consequently the multiplier is monotonic at λ_1 and by Theorem 8.7 there is a standard period doubling bifurcation. $\square$

Finally we show the existence of a non-standard bifurcation.

THEOREM 8.9. *Suppose q_1 is the rightmost solution of $g_2(x) = p_k$ in (p_k, p_{k+1}). Then $\lambda_1 = -q_1 i$ is the center of an ordered pair (Ω_4, Ω_4'). Let $\lambda^* = -q^* i$ be the parabolic boundary point of Ω_4' on $\Im$. Then there is a non-standard bifurcation at λ^*; that is, there is a pair of hyperbolic components (Ω_8, Ω_8') intersecting $\Im$ such that the parabolic boundary point of Ω_8 is λ^* and*

$$(\Omega_{2,-k}', \Omega_{2,-k}) \vartriangle \ldots \lhd (\Omega_4, \Omega_4') \lhd (\Omega_8, \Omega_8')$$

The ellipsis above indicates the possibility of other component pairs (Ω_4, Ω_4'), occurring there but none of higher order.

PROOF. By lemma 6.1 there are at most finitely many turning points of $g_2(x)$ so there is a unique q_1 such that $g_2(q_1) = p_k$ and $g_2(x) < p_k$ for all $q_1 < x < p_{k+1}$. Since $\lim_{x \to p_{k+1}^-} g_2(x) = -p_{k+1}$ and $g_2(x)$ is continuous, there is a first q_2 such that $q_1 < q_2 < p_{k+1}$, $g_2(q_2) = p_{k-1}$ and $p_k > g_2(x) > p_{k-1}$ for all $q_1 < x < q_2$.

It follows now that there is a full branch of $g_4(x)$ in the interval (q_1, q_2) such that $\lim_{x \to q^+} g_4(x) = -q_1$ and $\lim_{x \to q_2^-} g_4(x) = q_2$. Now there is a leftmost q_{11} such that $g_4(q_{11}) = -p_k$ and $-p_{k+1} < g_4(x) < -p_k$ for all $q_1 < x < q_{11}$ with $\lim_{x \to q_1^+} g_4(x) = -q_1$ and $\lim_{x \to q_{11}^-} g_4(x) = -p_k$. Consider $g_6(x)$ on $(q_1, q_{11}))$; it has no singularities on this interval and $\lim_{x \to q_1^+} g_6(x) = -p_k$ and $\lim_{x \to q_{11}^-} g_6(x) = -q_{11}$. We deduce there is a point r, $q_1 < r < q_{11}$ such that $g_4(r) = g_6(r)$.

Now consider the pairs $(\Omega_4, \Omega_4')_{q_1}$ and $(\Omega_4, \Omega_4')_{q_{11}}$ centered respectively at q_1 and q_{11}. By construction, there are no other pairs with these periods between them. Let $\lambda^* = -q^* i$ be the parabolic boundary point of the Ω_4' centered at q_1 and suppose the multiplier of the parabolic cycle is not monotonic in q at q^*. Then there must be another component Ω_4' with parabolic boundary point λ^*, but a different center, in which the same cycle is attracting. The center of this component must therefore be q_{11}. This means there are only hyperbolic or parabolic points in (q_1, q_{11}). The point r, however, is neither hyperbolic nor parabolic since the asymptotic value is preperiodic. We therefore deduce that the multiplier must be monotonic at λ^* and the theorem follows directly from theorem 8.7. $\square$

The same proof works to show that there is an non-standard bifurcation from each of the pairs enumerated in theorem 7.2.

8.5. Final Remarks. Computer experiments indicate that each full branch of the maps g_{2n} defined in section 6.1 has a unique turning point.

Question *Is it true that each full branch of g_{2n} has a unique turning point?*

If the answer to this question is yes, then the statements and proofs of our theorems simplify. In particular, the number of full branches can be counted precisely. This gives a cleaner statement for theorem 7.2. The proof of theorem 8.9 also simplifies because we need not worry about taking the rightmost or leftmost point where g_{2n} is a prepole. In fact, one could carry out induction in this proof to prove that there is an infinite cascade of non-standard period doubling bifurcations on the imaginary axis; that is,

For each $k \geq 1$ *the parameter* $-p_k i$ *is the virtual center of a hyperbolic component pair* $(\Omega'_{2,-k}, \Omega_{2,-k})$ *and there is an infinite sequence of pairs of hyperbolic components* $(\Omega_{2^m}, \Omega'_{2^m})$ *such that for each* $m > 1$, Ω'_{2^m} *has a common boundary point with* $\Omega_{2^{m+1}}$ *and*

$$(\Omega'_{2,-k}, \Omega_{2,-k}) \lhd (\Omega_4, \Omega'_4) \lhd (\Omega_8, \Omega'_8) \lhd (\Omega_{16}, \Omega'_{16}) \lhd \ldots \lhd (\Omega_{2^m}, \Omega'_{2^m})$$
$$\lhd (\Omega_{2^{m+1}}, \Omega'_{2^{m+1}}) \lhd \ldots (\Omega'_{2,-(k+1)}, \Omega_{2,-(k+1)})$$

In addition, the parabolic endpoints of $\Omega'_{2,-k}$ *and* $\Omega'_{2,-k-1}$ *are cusps. This situation is reflected on the positive imaginary axis.*

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