Accumulation points of iterated function systems

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1. Introduction

In a recent series of papers, [**4, 5, 6**] we studied the limit functions of iterated function systems. To define an *Iterated Function System*, or IFS, we begin with the family of holomorphic functions $\mathcal{H}ol(\Omega, X)$ from a hyperbolic domain $\Omega \subset \mathbb{C}$ to a subdomain $X \subset \Omega$; we choose an arbitrary sequence of functions $\{f_n\}$, $f_n \in \mathcal{H}ol(\Omega, X)$; then we form the sequence

$$F_n = f_1 \circ f_2 \circ \ldots f_{n-1} \circ f_n.$$

The sequence $\{F_n\}$ is called the *iterated function system corresponding to* $\{f_n\}$. For readability, we denote by $\mathcal{F}(\Omega, X)$ the set of all iterated function systems made from functions in $\mathcal{H}ol(\Omega, X)$.

By Montel's theorem (see for example [**2**]), the sequence F_n is a normal family, and every convergent subsequence converges uniformly on compact subsets of Ω to a holomorphic function $F \in \mathcal{H}ol(\Omega, X)$. The accumulation functions F are called accumulation points of the IFS and are either open or constant maps. The constant accumulation points may be located either inside X or on the boundary of X.

Whether there are non-constant accumulation points and whether accumulation points may take values on the boundary of X depends on the geometry of X. In [**1**], the authors introduced the concept of a Bloch subdomain as follows:

Definition 1. *Let $R(X, \Omega)$ be the supremum of all radii of hyperbolic disks contained in X, where the radii are measured with respect to the hyperbolic metric on Ω. A subdomain X of Ω is called a Bloch subdomain of Ω if*

$$R(X, \Omega) < \infty.$$

Note that by a compactness argument, any relatively compact subdomain is a Bloch subdomain.

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In [**1**] the authors proved that if X is a Bloch subdomain then every accumulation point of an IFS in $\mathcal{F}(\Omega, X)$ is constant. In [**5**] we proved that this condition is also necessary; that is, if X is a non-Bloch subdomain there are non-constant accumulation points. In [**3, 7**], it was proved that if X is relatively compact in Δ, the accumulation point is unique. In our recent paper, [**6**], we showed that if X is Bloch and non-relatively compact, any finite number of arbitrary points in X can be prescribed as the set of accumulation points of an IFS in $\mathcal{F}(\Omega, X)$.

In this paper we turn our attention to iterated function systems where X is a non-Bloch subdomain of Δ and there are accumulation points that are non-constant. Any accumulation point can be represented as an infinite composition of maps in $\mathcal{H}ol(\Omega, X)$. To see this, suppose $F = \lim_{j\to\infty} F_{n_j}$ is an accumulation point of some IFS corresponding to $\{f_n\}$. Let $g_1 = f_1 \circ \ldots \circ f_{n_1}$, $g_2 = f_{n_1+1} \circ \ldots \circ f_{n_2}$, $g_k = f_{n_{k-1}+1} \circ \ldots \circ f_{n_{k+1}}$. Then $g_n \in \mathcal{H}ol(\Omega, X)$ and F is the infinite composition $F = g_1 \circ g_2 \circ \ldots \circ g_k \ldots$.

We now reverse this and ask whether a given holomorphic map in $\mathcal{H}ol(\Omega, \overline{X})$ is an accumulation point for some IFS in $\mathcal{F}(\Omega, X)$, or equivalently, whether it has a representation as an infinite composition of maps in $\mathcal{H}ol(\Omega, X)$.

In this paper we answer this question when the source domain Ω is the unit disk Δ. We prove

THEOREM 1. *If X is a non-Bloch subdomain of X then given any function $g \in \mathcal{H}ol(\Delta, \overline{X})$ there is an IFS in $\mathcal{F}(\Omega, X)$ with g as an accumulation point.*

and

THEOREM 2. *Let X be a non-Bloch subdomain of Δ. Let K be either an empty set or a compact subset of X and let C be any closed subset of $\overline{X}$. Let G be the set of covering maps $g \in \mathcal{H}ol(\Delta, X)$ such that $g(0) \in K$. Then there is a single IFS in $\mathcal{F}(\Delta, X)$ whose full set of accumulation points consists exactly of the open maps in G and the constants in C.*

The paper is organized as follows. In section 2, we prove two lemmas that are at the heart of the proof of the theorems and in section 3 we prove the theorems. In section 4 we derive two corollaries: the first says that any map in $\mathcal{H}ol(\Delta, \overline{X})$ can be an accumulation point of an IFS in $\mathcal{F}(\Delta, X)$ if X is a non-Bloch subdomain and any constant in X can be an accumulation point if X is Bloch; the second says that if $X \neq \Delta$ then any $g \in \mathcal{H}ol(\Delta, X)$ *cannot* be represented by a non-trivial finite composition of functions $f_i \in \mathcal{H}ol(\Delta, X)$. Finally, in the last section we show that the first corollary does not generalize to $\mathcal{F}(\Omega, X)$ for $\Omega \neq \Delta$.

2. Preparatory Lemmas

DEFINITION 2. *Let X be a subdomain of Ω and let a be a point of X. Let $C = R(X, \Omega, a)$ be the radius measured with respect to the hyperbolic metric ρ_Ω of Ω of the largest disk with center at a contained in X. Then C is called the* ρ-Bloch *radius at a.*

Using this definition we can restate the definition of a Bloch subdomain.

DEFINITION 3. *A subdomain X of Ω is a* Bloch subdomain *if the supremum of the ρ-Bloch radii, taken over all points in X, is finite.*

We now assume Ω is the unit disk Δ. The first lemma gives an estimate depending on the ρ-Bloch radius that relates the distances between relatively close points in the $\rho = \rho_\Delta$ and ρ_X metrics.

LEMMA 2.1. *Let a be a point in a subdomain X of Δ. Let $C = R(X, \Delta, a)$ be the ρ-Bloch radius at a. If z is another point in X, with*

$$\rho(a, z) < 1 < C,$$

then

$$\rho_X(a, z) \leq (1 + \epsilon)\rho(a, z)$$

where

$$\epsilon = \epsilon(C) \to 0 \text{ as } C \to \infty.$$

PROOF. By applying the Möbius transformation

$$A(z) = \frac{z - a}{1 - \overline{a}z},$$

we may assume that $a = 0$. Suppose that $C = R(X, \Delta, 0) > 1 > \rho(0, z)$ for some point z in X. Let D be a disk in Δ with center at 0 and ρ-radius C. Then $D \subset X$, so that $\rho_X \leq \rho_D$. Therefore, an easy calculation shows

$$\rho_X(0, z) \leq \rho_D(0, z) = \int_0^{|z|} \frac{c}{c^2 - t^2} dt$$

where c is the Euclidean radius of D. Obviously $c \to 1$ as $C \to \infty$. Therefore,

$$\rho_X(0, z) \leq \int_0^{|z|} \frac{1}{1 - t^2}(c + \frac{c(1 - c^2)}{c^2 - t^2})dt \leq$$

$$\int_0^{|z|} \frac{1}{1 - t^2}(c + \frac{c(1 - c^2)}{c^2 - |z|^2})dt \to \int_0^{|z|} \frac{1}{1 - t^2} dt = \rho(0, z)$$

as $c \to 1$, and the lemma follows. □

In the next series of lemmas we prove that from an arbitrary sequence of points $c_1, c_2, \ldots$ in the unit disk such that $|c_n| \to 1$, we can extract a subsequence and estimate the contraction properties of the infinite Blaschke product whose zeros are at the points of the subsequence.

Let $a_1, a_2, a_3, \ldots$ be a sequence of points in Δ and let k be a positive integer. Then by Montel's theorem, the sequence of finite Blaschke products

$$A_n(z) = z^k \frac{z - a_1}{1 - \overline{a_1}z} \frac{z - a_2}{1 - \overline{a_2}z} \cdots \frac{z - a_n}{1 - \overline{a_n}z},$$

is a normal family. Thus, some subsequence $A_{n(l)}$ of A_n converges locally uniformly to an infinite Blashke product

$$A(z) = z^k \frac{z - a_1}{1 - \overline{a_1}z} \frac{z - a_2}{1 - \overline{a_2}z} \cdots$$

Note that if the sequence of numbers $|a_1||a_2||a_3| \ldots |a_n|$ converges to a positive number, then the k^{th} derivative at zero of any accumulation point A of A_n is nonzero. This in turn implies that A is an open map from Δ to Δ.

In the following lemmas the notation $D(a, r)$ stands for a hyperbolic disk in Δ with center a and radius r.

LEMMA 2.2. *Let $b_1, b_2, b_3, \dots$ be a sequence of points in the unit disk Δ such that $|b_n| \to 1$. If $C > 1$ then there exists a subsequence $a_1, a_2, a_3, \dots$ chosen from the b_n such that for all $n \geq 1$ and all z in $D(a_n, \frac{n+4}{2})$, the finite Blaschke products*

$$A_n(z) = z^k \frac{z - a_1}{1 - \overline{a_1} z} \frac{z - a_2}{1 - \overline{a_2} z} \cdots \frac{z - a_n}{1 - \overline{a_n} z}$$

satisfy

$$|A_n(z)| \geq e^{\frac{-1}{2^{n-2}C}} |\frac{z - a_n}{1 - \overline{a_n} z}||z|^{k-1} \tag{1}$$

PROOF. Assume first that $k = 1$. Pick a constant $C > 1$. Let $a_0 = 0$. Choose a_1 from the sequence $\{b_n\}$ such that $D(a_1, 4C)$ is disjoint from $D(0, C)$. Continue by choosing $a_2, a_3, \dots$ such that the hyperbolic disks $D(a_n, 4^n C)$ are disjoint for all $n = 0, 1, 2, \dots$. Let z be a point in $D(a_n, \frac{n+4}{2})$. Then for $0 \leq l < n$, we have

$$\rho(a_l, z) \geq \rho(a_l, a_n) - \rho(a_n, z) > (4^n + 4^l)C - \frac{n+4}{2} \geq 2^n 2^l C.$$

Since $\ln(x) \leq x$ for all positive x, for all l, $1 \leq l < n$, we have

$$2^n 2^l C \leq \rho(a_l, z) = \frac{1}{2} \ln \frac{1 + |\frac{z - a_l}{1 - \overline{a_l} z}|}{1 - |\frac{z - a_l}{1 - \overline{a_l} z}|} \leq \frac{1}{2} \ln \frac{2}{1 - |\frac{z - a_l}{1 - \overline{a_l} z}|} \leq \frac{1}{1 - |\frac{z - a_l}{1 - \overline{a_l} z}|}$$

Therefore

$$|\frac{z - a_l}{1 - \overline{a_l} z}| \geq 1 - \frac{1}{2^n 2^l C}$$

Note that

$$2x \geq \ln \frac{1}{1 - x} \text{ whenever } 0 < x < \frac{1}{2}.$$

Therefore,

$$\ln |\frac{A_n(z)}{\frac{z - a_n}{1 - \overline{a_n} z}}| = \ln |z| + \ln |\frac{z - a_1}{1 - \overline{a_1} z}| + \ldots + \ln |\frac{z - a_{n-1}}{1 - \overline{a_{n-1}} z}| \geq$$
$$\sum_{l=0}^{n-1} \ln(1 - \frac{1}{2^n 2^l C}) \geq -\sum_{l=0}^{n-1} \frac{1}{2^{n-1} 2^l C} \geq -\frac{1}{2^{n-2} C}.$$

This is equivalent to

$$|A_n(z)| \geq e^{-\frac{1}{2^{n-2}C}} |\frac{z - a_n}{1 - \overline{a_n} z}|$$

as claimed.

If $k > 1$ we can apply the above argument to $\tilde{A}_n(z) = \frac{|A_n(z)|}{|z^{k-1}|}$. Multiplying the result for $\tilde{A}_n$ through by $|z|^{k-1}$ we obtain the statement of the lemma. □

We turn now to the infinite product $A(z)$. We prove

LEMMA 2.3. *Let A be an accumulation point of the finite Blaschke products $A_n(z)$ of lemma 2.2. Then for all n and all $z \in D(a_n, \frac{n+4}{2})$ we have*

$$|A(z)| \geq e^{-\frac{1}{2^{n-2}C}} |\frac{z - a_n}{1 - \overline{a_n} z}||z|^{k-1} \tag{2}$$

and the weaker inequality

$$|A(z)| \geq e^{-\frac{k}{2^{n-2}C}} |\frac{z - a_n}{1 - \overline{a_n} z}| \tag{3}$$

Moreover, $A'(a_n) \neq 0$ for all $n \neq 0$ and $A(z)$ is univalent in a neighborhood of each a_n. In addition, if $k = 1$, $A'(0) \neq 0$ and A is univalent in a neighborhood of zero.

In particular, for $n = 0$ this says

$$|A(z)| \geq e^{-\frac{4}{C}}|z|^k \text{ whenever } \rho(0, z) < 2 \tag{4}$$

Proof. We need to estimate the tail of the right side of

$$\ln|\frac{A(z)}{\frac{z-a_n}{1-\overline{a_n}z}}| = k\ln|z| + \ln|\frac{z-a_1}{1-\overline{a_1}z}| + \dots$$

$$+\ln|\frac{z-a_{n-1}}{1-\overline{a_{n-1}}z}| + \ln|\frac{z-a_{n+1}}{1-\overline{a_{n+1}}z}| + \dots \tag{5}$$

(Notice that if $n = 0$ we replace k by $k-1$.) To this end we note that for $l > n$ we have, with an argument similar to the one in lemma 2.2,

$$\rho(a_l, z) \geq \rho(a_l, a_n) - \rho(a_n, z) > 4^l C - \frac{n+4}{2} \geq 2^n 2^{l-1} C$$

so that

$$2^n 2^{l-1} C \leq \rho(a_l, z) = \frac{1}{2}\ln\frac{1+|\frac{z-a_l}{1-\overline{a_l}z}|}{1-|\frac{z-a_l}{1-\overline{a_l}z}|} \leq \frac{1}{2}\ln\frac{2}{1-|\frac{z-a_l}{1-\overline{a_l}z}|} \leq \frac{1}{1-|\frac{z-a_l}{1-\overline{a_l}z}|}$$

Therefore

$$|\frac{z-a_l}{1-\overline{a_l}z}| \geq 1 - \frac{1}{2^n 2^{l-1} C} \text{ for } l > n$$

and again arguing as in lemma 2.2,

$$\ln|\frac{z-a_{n+1}}{1-\overline{a_{n+1}}z}| + \dots = \sum_{l=n+1}^{\infty} \ln|\frac{z-a_l}{1-\overline{a_l}z}| \geq \sum_{l=n}^{\infty} \frac{-1}{2^{n-1}2^l C}$$

This together with the proof of lemma 2.2 implies,

$$\ln|\frac{A(z)}{\frac{z-a_n}{1-\overline{a_n}z}}| \geq \frac{-1}{2^{n-2}C} + (k-1)\ln|z|$$

or equivalently,

$$|A(z)| \geq e^{-\frac{1}{2^{n-2}C}}|\frac{z-a_n}{1-\overline{a_n}z}||z|^{k-1}$$

We obtain the second inequality by noting that we trivially have

$$\ln|\frac{A(z)}{\frac{z-a_n}{1-\overline{a_n}z}}| \geq k\ln|z| + k\ln|\frac{z-a_1}{1-\overline{a_1}z}| + \dots$$

$$+k\ln|\frac{z-a_{n-1}}{1-\overline{a_{n-1}}z}| + k\ln|\frac{z-a_{n+1}}{1-\overline{a_{n+1}}z}| + \dots \tag{6}$$

Then from the arguments above we have

$$\ln|\frac{A(z)}{\frac{z-a_n}{1-\overline{a_n}z}}| \geq$$

$$k\sum_{l=0}^{\infty}\ln(1-\frac{1}{2^n 2^l C}) \geq -k\sum_{l=0}^{\infty}\frac{1}{2^{n-1}2^l C} = -\frac{k}{2^{n-2}C}$$

□

The next step is an estimate for $k = 1$.

LEMMA 2.4. *Suppose $k = 1$. For any $\epsilon > 0$, there exists a $C > 1$ such that the infinite Blaschke product A of lemma 2.3 has the following property: for every point w in Δ with $\rho(0, w) < 1$, there exists a point z in Δ such that $A(z) = w$ and $\rho(0, z) \leq (1 + \epsilon)\rho(0, w)$.*

PROOF. We claim that we may choose C sufficiently large so that

$$\overline{D(0,1)} \subset A(D(0,2)) \tag{7}$$

To prove this, let w be a point in $\overline{D(0,1)} \setminus A(D(0,2))$ closest to 0. Since $k = 1$, $A(0) = 0$ and A is univalent in a neighborhood of 0 so that $|w| > 0$. Thus, there exists a sequence $w_n = A(z_n)$ such that $\rho(0, z_n) < 2$ and $w_n \to w$. Passing to a subsequence we may assume $z_n \to z$ with $\rho(0, z) = 2$. The inequality (4) implies that $|w_n| \geq e^{-\frac{4}{C}}|z_n|$. Passing to the limit, we obtain $|w| \geq |z|e^{-\frac{4}{C}}$. Because $\rho(0, w) \leq 1$ and $\rho(0, z) = 2$, $|z| > |w|$. Now if we choose C so large that $|z|e^{-\frac{4}{C}} > |w|$, we have a contradiction that proves the relation (7). Therefore, for every w in $D(0, 1)$ we conclude that there exists a point z in $D(0, 2)$ such that $A(z) = w$. The inequality (4) implies $\rho(0, z) \leq (1 + \epsilon)\rho(0, w)$ for the large C we have chosen. □

When $k > 1$ we have

LEMMA 2.5. *Suppose $k > 1$. Then any accumulation point $A(z)$ is an open map that satisfies*

$$\rho(0, A(z)) \leq \rho(0, z^k)$$

PROOF. The lemma follows from noting that $|A_n(z)| \leq |z|^k$ for all n. □

The last lemma of this group is

LEMMA 2.6. *For any two points p and q in Δ with $\rho(p, q) < 1$, and any $\sigma > 0$, for n sufficiently large there exist points p_n and q_n in $D(a_n, \frac{n+4}{2})$ such that $A(p_n) = p$, $A(q_n) = q$, and $\rho(p_n, q_n) \leq (1 + \sigma)\rho(p, q)$.*

PROOF. Let p and q be any two points in Δ with $\rho(p, q) < 1$ and let $\sigma > 0$. Let $N \geq 2$ be the smallest integer n such that both p and q belong to $D(0, \frac{n}{8}+1)$. Let $\tilde{C}$ be the value of C chosen in lemma 2.4 and now set $C = k\tilde{C}$. We claim that

$$\overline{D(0, \frac{n+4}{4})} \subset A(D(a_n, \frac{n+4}{2})) \tag{8}$$

for all sufficiently large n. To prove this claim, let $n \geq N$ and let w be a point in $\overline{D(0, \frac{n+4}{4})} \setminus A(D(a_n, \frac{n+4}{2}))$ closest to 0. Since $A(a_n) = 0$ and $A'(a_n) \neq 0$, we have $|w| > 0$. Thus, there exists a sequence $w_j = A(z_j)$ such that $z_j \in D(a_n, \frac{n+4}{2})$, $\rho(z_j, a_n) \to \frac{n+4}{2}$ and $w_j \to w$. Let $\tilde{z_j} \to \tilde{z}$ be a sequence such that $\rho(0, \tilde{z_j}) = \rho(a_n, z_j) \to \frac{n+4}{2}$. The inequality (3) implies that $|w_j| \geq e^{-\frac{k}{2^{n-2}C}}|\tilde{z}_j|$. Passing to the limit, we obtain $|w| \geq e^{-\frac{k}{2^{n-2}C}}|\tilde{z}|$ where $\rho(0, \tilde{z}) = \frac{n+4}{2}$. Therefore,

$$\frac{|w|}{|\tilde{z}|} < \frac{\frac{e^{\frac{n+4}{2}}-1}{e^{\frac{n+4}{2}}+1}}{\frac{e^{n+4}-1}{e^{n+4}+1}} < e^{-\frac{k}{2^{n-2}}},$$

which is a contradiction for all sufficiently large n. This proves the claim (8).

It now follows that there exist points p_n and q_n in $D(a_n, \frac{n+4}{2})$ such that $A(p_n) = p$ and $A(q_n) = q$. If $p = 0$, then we choose $p_n = a_n$ and the lemma follows directly from inequality (4). Thus, we may assume that both p and q are nonzero. By inequality (4), the points $\tilde{p}_n = \frac{p_n - a_n}{1-\overline{a}_n p_n}$ and $\tilde{q}_n = \frac{q_n - a_n}{1-\overline{a}_n q_n}$ are uniformly bounded away from both 0 and the unit circle. Furthermore, the inequality (3) implies that the functions $\frac{A(\frac{z+a_n}{1+\overline{a}_n z})}{z}$ converge locally uniformly to a point a on the unit circle as n approaches infinity. Therefore, $\frac{p}{\tilde{p}_n}$ and $\frac{q}{\tilde{q}_n}$ both converge to a, and since $\rho(p_n, q_n) = \rho(\tilde{p}_n, \tilde{q}_n)$, the lemma follows.

□

3. The main theorem

In this section we prove theorems 1 and 2.

Theorem 1 *If X is a non-Bloch subdomain of X then given any function $g \in \mathcal{H}ol(\Delta, \overline{X})$ there is an IFS in $\mathcal{F}(\Omega, X)$ with g as an accumulation point.*

Proof. If X is a non-Bloch subdomain of Δ there exists a sequence of points $a_1, a_2, a_3, \ldots$ in X such that the $\rho-$hyperbolic disks $D(a_n, n)$ with center at a_n and radius n satisfy

$$D(a_n, n) \subset X \tag{9}$$

Let z_0 be a point in the unit disk with $0 < \rho(0, z_0) < 1$. Choose a sequence $\epsilon_n > 0$ such that $(1 + \epsilon_n)^{2n-1} \to 1$.

Let $k = 1$. By lemma 2.4, there exists a subsequence a_{n_k} of the a_n determining an infinite Blashke product

$$A_1(z) = z\frac{z - a_{n_1}}{1 - \overline{a_{n_1}}z}\frac{z - a_{n_2}}{1 - \overline{a_{n_2}}z}\frac{z - a_{n_3}}{1 - \overline{a_{n_3}}z}\cdots$$

and there is a point z_1 in Δ satisfying

$$A_1(z_1) = z_0 \text{ and } \rho(0, z_1) \le (1 + \epsilon_1)\rho(0, z_0) \tag{10}$$

Now lemma 2.6 with $p = 0$, $q = z_0$ and $\sigma = \epsilon_2$ implies that for any $n(1)$ sufficiently large there are points p_1 and q_1 in $D(a_{n(1)}, \frac{n(1)+4}{2})$ such that

$$A_1(p_1) = 0,\ A_1(q_1) = z_0, \text{ and } \rho(p_1, q_1) \le (1 + \epsilon_2)\rho(0, z_0) \tag{11}$$

Observe that A_1 maps X onto Δ.

Choose $n(1)$ so large that by lemma 2.1 and formula (9), we have

$$\rho_X(p_1, q_1) \le (1 + \epsilon_2)\rho(p_1, q_1).$$

Next, choose a holomorphic covering map π_1 from the unit disk Δ onto X such that $\pi_1(0) = p_1$ and $\pi_1(w_1) = q_1$ for some point w_1 with $\rho(0, w_1) = \rho_X(p_1, q_1)$.

Again by lemma 2.4, there exists a subsequence of the a_{n_i} that determines another infinite Blashke product $A_2(z)$ such that $A_2(0) = 0$ and there is a point z_2 in Δ such that

$$A_2(z_2) = w_1 \text{ and } \rho(0, z_2) \le (1 + \epsilon_2)\rho(0, w_1) \tag{12}$$

Thus

$$\rho(0, z_2) \le (1 + \epsilon_2)^3\rho(0, z_0).$$

Let $f_2 = \pi_1 \circ A_2$.

Now lemmas 2.6 and 2.1 again, with $p = 0$, $q = z_0$ but $\sigma = \epsilon_3$, imply that for any $n(2)$ sufficiently large there are points p_{21} and q_{21} in $D(a_{n(2)}, \frac{n(2)+4}{2})$ such that

$$A_1(p_{21}) = 0,\ A_1(q_{21}) = z_0, \text{ and}$$

$$\rho_X(p_{21}, q_{21}) \leq (1 + \epsilon_3)\rho(p_{21}, q_{21}) \leq (1 + \epsilon_3)^2 \rho(0, z_0). \tag{13}$$

Since π_1 is a covering map, there exist points $\tilde{p}_{21}$ and $\tilde{q}_{21}$ such that $\pi_1(\tilde{p}_{21}) = p_{21}$, $\pi_1(\tilde{q}_{21}) = q_{21}$, and

$$\rho(\tilde{p}_{21}, \tilde{q}_{21}) = \rho_X(p_{21}, q_{21}) \leq (1 + \epsilon_3)^2 \rho(0, z_0).$$

Again choosing $n(2) > n(1)$ sufficiently large and applying lemmas 2.6 and 2.1, we obtain points p_2 and q_2 such that $A_2(p_2) = \tilde{p}_{21}$, $A_2(q_2) = \tilde{q}_{21}$, and

$$\rho_X(p_2, q_2) \leq (1 + \epsilon_3)\rho(p_2, q_2) \leq (1 + \epsilon_3)^2 \rho(\tilde{p}_{21}, \tilde{q}_{21}) \leq (1 + \epsilon_3)^4 \rho(0, z_0).$$

Next, choose a holomorphic covering map π_2 from the unit disk Δ onto X such that $\pi_2(0) = p_2$ and $\pi_2(w_2) = q_2$ for some point w_2 with $\rho(0, w_2) = \rho_X(p_2, q_2)$.

Repeating the arguments above, we find an infinite Blaschke product $A_3(z)$ such that $A_3(0) = 0$ and such that there is a point z_3 in Δ with

$$A_3(z_3) = w_2 \text{ and } \rho(0, z_3) \leq (1 + \epsilon_3)\rho(0, w_2).$$

Then

$$\rho(0, z_3) \leq (1 + \epsilon_3)^5 \rho(0, z_0).$$

Let $f_3 = \pi_2 \circ A_3$.

Continuing in this way we obtain a sequence of maps $f_n = \pi_{n-1} \circ A_n$ and a sequence of points z_n with $\rho(0, z_n) \to \rho(0, z_0)$ such that $\rho(0, z_n) \leq (1 + \epsilon_n)^{2n+1} \rho(0, z_0)$. Moreover, the maps

$$G_n = A_1 \circ f_2 \circ f_3 \ldots \circ f_n$$

are holomorphic self maps of the unit disk that satisfy $G_n(0) = 0$ and $G_n(z_n) = z_0$. Therefore any accumulation point g of G_n is a holomorphic self map of the unit disk that maps an accumulation point of z_n to z_0. Thus, $g(z) = e^{i\theta} z$.

For any given $f \in \mathcal{H}ol(\Delta, X)$ set $f_1(z) = f(e^{-i\theta} A_1(z))$. Then

$$f_1 \circ f_2 \circ f_3 \ldots = f(e^{-i\theta} A_1(f_2(f_3 \ldots)))$$

has f as an accumulation point.

If f is a constant map whose image is a point on the boundary of X the proof follows from the proof of the next theorem. □

Now we are ready to prove theorem 2

Theorem 2 *Let X be a non-Bloch subdomain of Δ. Let K be either an empty set or a compact subset of X and let C be any closed subset of $\overline{X}$. Let G be the set of covering maps $g \in \mathcal{H}ol(\Delta, X)$ such that $g(0) \in K$. Then there is a single IFS in $\mathcal{F}(\Delta, X)$ whose full set of accumulation points consists exactly of the open maps in G and the constants in C.*

Observe that in the first theorem we were able to obtain any holomorphic map as a limit function. In this theorem we can get any collection of maps at either end of the contraction spectrum; that is any collection of constants and hyperbolic isometries.

PROOF. Let X be a non-Bloch subdomain of Δ. Pick a countable dense subset $C_0 = \{c_1, c_2, c_3, \ldots\}$ of points in X such that $\overline{C_0} = C$. Similarly, pick a countable subset $X_0 = \{x_1, x_2, x_3, \ldots\}$ of points in X such that $\overline{X_0} = K$.

By the compactness of K there exists a point $x_0 \neq 0$ in Δ such that $x + x_0 \in X$ for every x in K. Furthermore, by rescaling, we may assume $\rho_X(x, x+x_0) < 1$ for every x in K.

We proceed much as we did in the proof above. We start with a sequence $\epsilon_n > 0$ such that $(1+\epsilon_n)^{4n-1} \to 1$. Then, since X is non-Bloch, we find a sequence of points $a_1, a_2, a_3, \ldots$ in X such that the disks $D(a_n, n)$ satisfy

$$D(a_n, n) \subset X \tag{14}$$

Using the preparatory lemmas, we use appropriate subsequences of the points a_n to construct the following sequences: infinite Blaschke products A_n, covering maps π_n from Δ to X and points $z_{2n} \in \Delta$ such that the maps $f_n = \pi_n \circ A_n$ have the following properties:

$$\begin{gathered} f_1 f_2 \ldots f_{2n}(z_{2n}) = x_n + x_0, \\ f_1 f_2 \ldots f_{2n}(0) = x_n, \\ \rho(0, z_{2n}) \leq (1+\epsilon_n)^{4n-1} \rho_X(x_n, x_n + x_0), \\ f_1 f_2 \ldots f_{2n-1}(0) = c_n, \end{gathered} \tag{15}$$

and for every $z \in \Delta$.

$$\rho(f_1 f_2 \ldots f_{2n-1}(0), f_1 f_2 \ldots f_{2n-1}(z)) \leq \rho(0, z^n).$$

Once we have these maps it is easy to check that the even subsequences $f_1 f_2 \ldots f_{2n}$ converge to covering maps that map 0 to points in K and that the odd subsequences $f_1 f_2 \ldots f_{2n-1}$ converge to constants in C proving the theorem. Notice that the construction of the even indexed functions is more complicated that the construction for the odd indexed one.

For the first map, choose a covering map π_1 from Δ onto X such that $\pi_1(0) = c_1$. Using the preparatory lemmas, we take a subsequence a_{n_i} and find an infinite Blashke product

$$A_1(z) = z^k \frac{z - a_{n_1}}{1 - \overline{a_{n_1}} z} \frac{z - a_{n_2}}{1 - \overline{a_{n_2}} z} \frac{z - a_{n_3}}{1 - \overline{a_{n_3}} z} \cdots.$$

Since A_1 is holomorphic, for all $z \in \Delta$

$$\rho(0, A_1(z)) \leq \rho(0, z)$$

Let $f_1 = \pi_1 \circ A_1$. Then we have $f_1(0) = c_1$ and $\rho(f_1(0), f_1(z)) \leq \rho(0, z)$.

Now we design f_2. Choose a point $y_1 \in \Delta$ such that $\pi_1(y_1) = x_1$. Then there exists a point $\tilde{y}_1$ such that $\pi_1(\tilde{y}_1) = x_1 + x_0$ and $\rho(y_1, \tilde{y}_1) = \rho_X(x_1, x_1 + x_0)$. Lemma 2.6 applied to A_1 with $\sigma = \epsilon_1$ implies that for all sufficiently large $n(1)$ there are points p_1 and q_1 in $D(a_{n(1)}, \frac{n(1)+4}{2})$ such that $A_1(p_1) = y_1$, $A_1(q_1) = \tilde{y}_1$, and $\rho(p_1, q_1) \le (1+\epsilon_1)\rho(y_1, \tilde{y}_1)$. Furthermore, if we choose $n(1)$ sufficiently large, by lemma 2.1, and formula (14), we also have

$$\rho_X(p_1, q_1) \le (1+\epsilon_1)\rho(p_1, q_1).$$

Pick $w_2 \in \Delta$ with $\rho(0, w_2) = \rho_X(p_1, q_1)$. Choose a holomorphic covering map π_2 from the unit disk Δ onto X such that $\pi_2(0) = p_1$ and $\pi_2(w_2) = q_1$. Using the preparatory lemmas we can find an infinite Blaschke product A_2 and a point z_2 in Δ such that $A_2(z_2) = w_2$ and $\rho(0, z_2) \le (1+\epsilon_1)\rho(0, w_2)$.

Thus

$$\rho(0, z_2) \le (1+\epsilon_1)^3 \rho_X(x_1, x_1 + x_0).$$

Let $f_2 = \pi_2 \circ A_2$. Obviously $f_1 f_2(0) = x_1$ and $f_1 f_2(z_2) = x_1 + x_0$.

Next, we design f_3. By lemma 2.6 the maps A_1 and A_2 map X onto Δ. Therefore the composition $f_1 \circ f_2$ maps X onto X and we can find a point d_2 in X such that $f_1 f_2(d_2) = c_2$. Choose a holomorphic covering map π_3 from Δ onto X such that $\pi_3(0) = d_2$. Now form the infinite Blashke product $A_3(z)$ such that $A_3(0) = 0$ and by lemma 2.5

$$\rho(0, A_3(z)) \le \rho(0, z^2)$$

Let $f_3 = \pi_3 \circ A_3$. Obviously $f_1 f_2 f_3(0) = c_2$ and

$$\rho(c_2, f_1 f_2 f_3(z)) \le \rho(0, A_3(z)) \le \rho(0, z^2).$$

The design of f_4 takes four steps. First choose a point $y_2 \in \Delta$ such that $\pi_1(y_2) = x_2$. Then there exists a point $\tilde{y}_2$ such that $\pi_1(\tilde{y}_2) = x_2 + x_0$ and $\rho(y_2, \tilde{y}_2) = \rho_X(x_2, x_2 + x_0)$. Lemma 2.6 applied to A_1 again implies that for all sufficiently large $n(2)$ there are points p_2 and q_2 in $D(a_{n(2)}, \frac{n(2)+4}{2})$ such that $A_1(p_2) = y_2$, $A_1(q_2) = \tilde{y}_2$, and

$$\rho(p_2, q_2) \le (1+\epsilon_2)\rho(y_2, \tilde{y}_2).$$

Furthermore, if we choose $n(2) > n(1)$ sufficiently large, by lemma 2.1, and formula (14) we have,

$$\rho_X(p_2, q_2) \le (1+\epsilon_2)\rho(p_2, q_2).$$

Second, since π_2 is a covering map, there exist points p_3 and q_3 in the unit disk such that $\pi_2(p_3) = p_2$, $\pi_2(q_3) = q_2$ and $\rho_X(p_2, q_2) = \rho(p_3, q_3)$. Lemma 2.6 applied to A_2 implies that for sufficiently large $n(3)$ there are points p_4 and q_4 in $D(a_{n(3)}, \frac{n(3)+4}{2})$ such that $A_2(p_4) = p_3$, $A_2(q_4) = q_3$, and

$$\rho(p_4, q_4) \le (1+\epsilon_2)\rho(p_3, q_3).$$

Furthermore, if we choose $n(3) > n(2)$ sufficiently large, by lemma 2.1, and formula (14) we have,

$$\rho_X(p_4, q_4) \le (1+\epsilon_2)\rho(p_4, q_4).$$

Third, since π_3 is a covering map, there exist points p_5 and q_5 in the unit disk such that $\pi_3(p_5) = p_4$, $\pi_3(q_5) = q_4$ and $\rho_X(p_4, q_4) = \rho(p_5, q_5)$. Lemma 2.6 applied to A_3 implies that for sufficiently large $n(4)$ there are points p_6 and q_6 in $D(a_{n(4)}, \frac{n(4)+4}{2})$ such that $A_3(p_6) = p_5$, $A_3(q_6) = q_5$, and

$$\rho(p_6, q_6) \le (1+\epsilon_2)\rho(p_5, q_5).$$

Furthermore, if we choose $n(4) > n(3)$ sufficiently large, by lemma 2.1, and formula (14) we have,

$$\rho_X(p_6, q_6) \leq (1 + \epsilon_2)\rho(p_6, q_6).$$

Finally, pick a point $w_4 \in \Delta$ with $\rho(0, w_4) = \rho_X(p_6, q_6)$ and choose a holomorphic covering map π_4 from the unit disk Δ onto X such that $\pi_2(0) = p_6$ and $\pi_4(w_4) = q_6$.

Then find an infinite Blashke product $A_4(z)$ and a a point z_4 in Δ such that $A_4(0) = 0$, $A_4(z_4) = w_4$ and

$$\rho(0, z_4) \leq (1 + \epsilon_2)\rho(0, w_4).$$

Putting these inequalities together we have

$$\rho(0, z_4) \leq (1 + \epsilon_2)^7 \rho_X(x_2, x_2 + x_0).$$

Let $f_4 = \pi_4 \circ A_4$. Obviously $f_1 f_2 f_3 f_4(0) = x_2$ and $f_1 f_2 f_3 f_4(z_4) = x_2 + x_0$.

We continue in this way to construct the full sequence of maps f_n; by construction they have the properties listed in equations (15). A properly chosen subsequence of the odd maps $f_1 \dots f_{2n-1}(0)$ will converge to any constant in C and by the last of equations (15) the limit function is just this constant. This finishes the last part of the proof of theorem 1.

Now note that the points z_{2n} are bounded in Δ by the third of equations (15). Given any $k \in K$ we can choose a subsequence $x_{n_i} \to k$. Then $f_1 \dots f_{2n_i}(0) = x_{n_i} \to k$ and $f_1 \dots f_{2n_i}(z_{2n_i}) = x_0 + x_{n_i} \to x_0 + k$ so that the limit function g is open. To see that g is a covering map set $\lim z_{n_i} = z_\infty$ and note that

$$\rho_X(g(0), g(z_\infty)) \leq \rho(0, z_\infty) \leq x_0 + k$$

so that g is an isometry and hence a covering map.

Hence we obtain an IFS whose set of accumulation functions contains all of the constants in C together with a set of covering maps in G that map 0 onto every point in K. To see that this is the set of all covering maps recall that a covering map is defined up to rotation by where it sends the origin. Therefore repeating this construction, replacing the set $\{x_1, x_2, x_3, \dots\}$ with the set $\{x_1, x_2, x_1, x_2, x_3, x_1, x_2, x_3, x_4, \dots\}$ we get infinitely many covering maps that send the origin to each x_i. If we pre-compose each of the corresponding f_{2n} with an appropriately chosen rational rotation about the point z_{2n}, we get all the covering maps. Thus we have an IFS whose limit functions contain the whole set G as well as the set C of constants. □

4. Corollaries

Corollary 1. *Let X be a subdomain of the unit disk Δ. If X is a Bloch subdomain then every constant in X is an accumulation point of some $\mathcal{F}(\Delta, X)$. On the other hand, if X is non-Bloch, then every holomorphic map from the unit disk to $\overline{X}$ is an accumulation point of some $\mathcal{F}(\Delta, X)$.*

It is an open question whether every point on ∂X of a Bloch subdomain can be an accumulation point. In [**6**] we prove that the answer is yes for two special classes: those with locally connected boundary and those obtained by removing an infinite set of points from the disk.

Proof. If X is non-Bloch, it follows directly from Theorem 1 that any point in $\bar{X}$ is an accumulation point of some $\mathcal{F}(\Delta, X)$.

If X is Bloch, it is shown in [**1**] that every accumulation point is a constant map. To obtain any given constant $c \in X$, take the first map of the IFS to be the map $f_1 \equiv c$. $\square$

Corollary 2. *Let X be a proper subdomain of the unit disk Δ and let f be a holomorphic covering map from Δ onto X. Then f can not be represented as a* finite *composition of two or more holomorphic maps from the Δ to X. Furthermore, f may be represented as an infinite composition $f_1 \circ f_2 \circ \dots$ of holomorphic maps f_n from Δ to X if and only if X is non-Bloch. In particular, if X is simply connected, then a Riemman map from the unit disk onto X may be represented as a non-trivial composition of holomorphic maps from Δ to X if and only if X is non-Bloch.*

Proof. The only part left to prove is the first part. Suppose $f = h \circ g$ is a covering map from Δ to X where g and h are holomorphic maps from Δ to X. Then by Schwarz's lemma

$$\begin{aligned} 1 = \rho_X(f(0))|f'(0)| &= \rho_X(h(g(0)))|h'(g(0))||g'(0)| \\ &\leq \rho(g(0))|g'(0)| = |(A \circ g)'(0)|, \end{aligned}$$

where

$$A(z) = \frac{z - g(0)}{1 - \overline{g(0)}z}.$$

Thus, $A \circ g$ must be a rotation and so the image of g must be all of Δ. $\square$

5. More general source domains

Our discussion above is restricted to the situation where the source domain is Δ. The Bloch condition is defined, however, for any source domain Ω and target subdomain X. Moreover, the proof in [**1**] extends to iterated function systems in $\mathcal{F}(\Omega, X)$; that is, if X is a Bloch subdomain of Ω then all limit functions are constant.

One could ask, in this more general situation if, when X is a non-Bloch subdomain of Ω, whether any function in $\mathcal{H}ol(\Omega, X)$ can be realized as the limit of an IFS in $\mathcal{F}(\Omega, X)$. We show the answer is not always yes.

Consider the following explicit set S:

Denote the Euclidean disk of radius a that is centered and punctured at the origin by Δ^*_a and set $\Delta^* = \Delta^*_1$. Let z_0 be the point $1/2$ on the real axis in Δ^* and, for $n = 1, 2, \dots$, let z_n be the point on the real axis between 0 and z_0 defined by the equation $\rho_{\Delta^*}(z_n, z_0) = 2^n$. Let I_n be a closed interval on the real line with center at z_n and length 2 measured with respect to the hyperbolic metric ρ_{Δ^*}. We set

$$S = \Delta^*_{1/2} \setminus \cup_n I_n.$$

First, note that S is relatively compact in Δ. Thus, S is a Bloch subdomain of Δ. Suppose f is a holomorphic map from Δ^* to S. Then since f is bounded and 0 is an isolated singularity, f extends to a holomorphic map $\tilde{f}$ from Δ to $\overline{S}$. Because S contains no punctures, $\tilde{f}$ maps Δ to S. Therefore, any IFS in $\mathcal{F}(\Delta^*, S)$

extends to an IFS in $\mathcal{F}(\Delta, S)$ and thus can only have constant accumulation points.

Second, note that large disks in Δ^* are contained in annuli around the puncture. The complement of the intervals I_n in $[0, 1/2]$ consists of intervals G_n between I_n and I_{n+1} and by construction the ρ_{Δ^*} length of G_n is at least $2(2^{n-1} - 1)$. Thus there are arbitrarily large disks in S and S is not a Bloch subdomain of Δ^*.

Therefore, S is a non-Bloch subdomain of Δ^* and any IFS in $\mathcal{F}(\Delta, S)$ has only constant limits.

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