Ergod. Th. & Dynam. Sys. (2008), **0**, 1–28
doi:10.1017/S0143385708000205

Bounded-type Siegel disks of a one dimensional family of entire functions

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(*Received* 1 *November* 2006 *and accepted in revised form* 10 *March* 2008)

Abstract. Let $0 < \theta < 1$ be an irrational number of bounded type. We prove that for any map in the family $(e^{2\pi i\theta} z + \alpha z^2)e^z$, $\alpha \in \mathbb{C}$, the boundary of the Siegel disk, with fixed point at the origin, is a quasi-circle passing through one or both of the critical points.

1. *Introduction*

Let $\mathcal{F}$ be a family of holomorphic functions fixing the origin. If $f \in \mathcal{F}$ is holomorphically conjugate on a neighborhood of the origin to an irrational rotation then the largest domain on which this conjugation is defined is called the Siegel Disk of f. The Siegel disk Δ_f is contained in the Fatou set and the boundary of Δ_f is contained in the Julia set of f. Two natural questions about the boundary of Δ_f are as follows.

(1) When is it a Jordan curve?

(2) When does it contain a critical point of f?

Both these questions are far from solved for general families. Many authors have made contributions to these problems for various families. The reader is referred to [**4, 5, 9, 15**], and [**20**] for more details.

Suppose the multiplier of $f \in \mathcal{F}$ at the origin is $\lambda = e^{2\pi i\theta}$. It is well known, [**5, 19**], that a sufficient condition for f to have a Siegel disk at the origin is that θ be of bounded type. Under this condition, it was proved in [**9**] that the boundary of the Siegel disk must contain a critical point. An interesting question is to find conditions under which the boundary of a Siegel disk is a Jordan curve. Douady [**5**], using the work of Herman and Swiatek, proved that bounded-type Siegel disks are quasi-disks for quadratic polynomials and then Zakeri [**20**] generalized this result to cubic polynomials. Later, using a somewhat different argument, Shishikura [**17**], proved that bounded-type Siegel disks are quasi-disks for polynomials of every degree.

THEOREM. (Douady–Zakeri–Shishikura) *Let θ be a bounded-type irrational number and let $n \geq 2$ be an integer. Then the boundary of the Siegel disk of any polynomial map*

$$P : z \mapsto e^{2\pi i\theta} z + a_2 z^2 + \cdots + a_n z^n, \quad a_n \neq 0,$$

centered at the origin, is a quasi-circle passing through at least one critical point of P.

It would be extremely interesting if the the above theorem generalized to large families of entire functions. In this paper we restrict our attention to a narrow class of entire functions, namely, those functions which have the following form

$$P(z)e^z = (e^{2\pi i\theta} z + a_2 z^2 + \cdots + a_n z^n)e^z.$$

The reason that we consider such functions is that they are a rather simple class of entire functions of 'finite type'; that is functions with finitely many critical and asymptotic values. In fact, they seem relatively close to polynomials in that they have only finitely many critical points and finitely many zeros. For this class we ask the following question.

QUESTION. *Let θ be an irrational number of bounded type and let $n \geq 2$ be an integer. Then is the boundary of the Siegel disk of the entire map*

$$f : z \mapsto (e^{2\pi i\theta} z + a_2 z^2 + \cdots + a_n z^n)e^z,$$

centered at the origin, a quasi-circle passing through at least one critical point of f?

In the case that $a_2 = \cdots = a_n = 0$ the answer was shown to be positive by Geyer.

THEOREM. (Geyer [**7**]) *Let θ be a bounded-type irrational number. Then the boundary of the Siegel disk of the entire map $e^{2\pi i\theta} z e^z$, centered at the origin, is a quasi-circle passing through the unique critical point.*

The main purpose of this paper is to prove a similar theorem for entire maps with $P(z)$ quadratic.

MAIN THEOREM. *Let θ be a bounded-type irrational number. Then for any entire map,*

$$f_a : z \mapsto (e^{2\pi i\theta} z + a z^2)e^z, \quad a \in \mathbb{C} - \{0\},$$

the boundary of the Siegel disk centered at the origin is a quasi-circle passing through one or both the critical points of f_a.

The main tool of the proof is to use techniques of quasi-conformal mappings presented in [**21**] (see also §3) to construct a function with a Siegel disk from a function with an attracting fixed point. This construction is similar in spirit to the one introduced by Cheritat [**4**] where he uses a Blaschke product model. Our construction has the advantage that it automatically induces a surgery map **S** defined on a one-dimensional parameter space of functions with an attracting fixed point. Using an argument of Zakeri [**20**], we prove that the surgery map **S** is continuous. The proof of the Main Theorem is then completed by showing that the surgery map **S** is surjective.

Now let us sketch the proof. We fix a θ of bounded type once and for all and set $\lambda = e^{2\pi i\theta}$. In §2, for each fixed $t \in \mathbb{C} - \{0\}$, we introduce the one complex dimensional parameter space Σ_t as follows:

$$\Sigma_t = \{f(z) = (tz + \alpha z^2)e^{\beta z} \mid f'(1) = 0, \alpha\beta \neq 0\}.$$

We mark the critical points and show that each Σ_t can be parameterized by the value β, and that, under this parametrization, Σ_t is homeomorphic to the punctured sphere $S^2 - \{0, \infty, -1, -2\}$ (Lemma 2.1).

We will be interested in two particular spaces: $\Sigma_{1/2}$ containing functions with an attracting fixed point; and Σ_λ which is the space of functions in our Main theorem conjugated by the map $z \to \beta z$. To differentiate between functions in these spaces we will denote those in $\Sigma_{1/2}$ by f_β and those in Σ_λ by g_β. It turns out that the two critical points of f_β and g_β are the same. We mark them and denote them by 1 and c_β.

For each $f_\beta \in \Sigma_{1/2}$ we introduce a geometric object D_β, which is a simply connected domain containing the origin (Definition 2.1). The key property of D_β is the following.

THEOREM 2.1. *∂D_β is a K-quasi-circle that passes through at least one of the critical points of f_β. Moreover, K is independent of β.*

In §3 we study the topological structure of the parameter space $\Sigma_{1/2}$. The main purpose of that section is to prove the Structure theorem for $t = 1/2$.

THEOREM 3.1. (Structure theorem for Σ_t) *There is a simple closed curve γ which separates $\{-2, \infty\}$ and $\{0, -1\}$ such that if β lies in the component of $S^2 \setminus \gamma$ containing $\{-2, \infty\}$ then ∂D_β passes through the critical point c_β but not the critical point* 1*, and if β lies in the other component, ∂D_β passes through the critical point* 1 *but not the critical point c_β. Moreover, γ is invariant under the involution $\sigma : \beta \to -(\beta+2)/(\beta+1)$ which interchanges the marked critical points.*

The curve γ separates $\Sigma_{1/2}$ into two components. We use Ω_{int} to denote the bounded component and Ω_{ext} the unbounded one.

In §4, we construct a surgery map $\mathbf{S} : \Omega_{\text{int}} \to \Sigma_\lambda$. In §5, adapting an argument of Zakeri [**20**], we show that the map $\mathbf{S}$ can be continuously extended to $\overline{\Omega_{\text{int}}}$ such that $\mathbf{S}(0) = 0$ and $\mathbf{S}(-1) = -1$.

In §6, we prove that the image of γ under the map $\mathbf{S}$ is a simple closed curve $\Gamma \subset \Sigma_\lambda$ which consists of all the maps for which the boundaries of the Siegel disks are quasi-circles passing through both of the critical points (Lemma 6.3). We use Θ_{int} to denote the bounded component of $\Sigma_\lambda - \Gamma$ and Θ_{ext} the unbounded one. We prove that the space Σ_λ is symmetric about the curve Γ under the map $\sigma : \beta \to -(\beta+2)/(\beta+1)$ induced by the linear conjugation map $z \mapsto z/c_\beta$ and that the map $\mathbf{S} : \gamma \to \Gamma$ has topological degree one (Lemmas 6.3 and 6.4). It follows that $\mathbf{S} : \Omega_{\text{int}} \to \Theta_{\text{int}}$ is surjective, which in turn implies the Main theorem and the Structure theorem for Σ_λ.

2. *The maximal linearization domain D_β*

2.1. *The parameterization of Σ_t.* For fixed $t \neq 0, \infty$, we use Σ_t to denote the space of all entire maps of the form

$$f(z) = (tz + \alpha z^2)e^{\beta z}$$

such that $f'(1) = 0$ and $\alpha\beta \neq 0$. This normalization marks the critical points. For $f \in \Sigma_t$, to simplify the notation, we suppress the dependence of f on t and the dependence of α on β.

LEMMA 2.1. *The space Σ_t is homeomorphic to the punctured sphere $S^2 \setminus \{-1, -2, 0, \infty\}$.*

Proof. For each $f \in \Sigma_t$, by definition, $f'(1) = 0$. By a simple calculation, this is equivalent to

$$\alpha = -t\frac{\beta + 1}{\beta + 2}. \tag{1}$$

Thus α is uniquely determined by β and it follows that the map $\rho : f \to \beta$ is a homeomorphism from Σ_t to $S^2 \setminus \{-1, -2, 0, \infty\}$. □

Note that the functions f fix the origin. Moreover, straightforward computations show that for each function there are exactly two asymptotic values, the origin and infinity. There are only two zeros, the origin and $(\beta + 2)/(\beta + 1)$. Every other point has infinitely many pre-images. Unless $\beta = -1 \pm i$, there are two distinct marked critical points, 1 and $c_\beta = -(\beta + 2)/\beta(\beta + 1)$ and two distinct critical values.

We will be interested in Σ_t for two specific values of t, $t = 1/2$ and $t = \lambda = e^{2\pi i\theta}$ where θ is the irrational of bounded type fixed in the introduction.

Remark 2.1. The functions in these spaces are of finite type; they have only finitely many singular values and in fact only finitely many critical points. The classification of their Fatou components is thus fairly simple. It is known (see for example, [**8**]) that there are no wandering domains and no Baker domains for such entire functions. There is one grand orbit of components in the Fatou set with a forward invariant component containing the origin. For $t = 1/2$ it is attracting and contains at least one critical point and for $t = \lambda$ it is a Siegel disk whose boundary contains the closure of the forward orbit of at least one critical point. In both cases, the forward invariant component contains the asymptotic value at the origin.

There can be at most one other grand orbit of components and it will contain the orbit of the 'other critical point'. This cycle can only be attracting, super-attracting, parabolic or contain another cycle of Siegel disks. In this paper, this potential second cycle will not play a role.

2.2. *The maximal linearization domain D_β.* Let us fix $t = 1/2$ throughout this section. From now on, we will identify the space $\Sigma_{1/2}$ with the parameter space $S^2 \setminus \{-1, -2, 0, \infty\}$. For each $\beta \in \Sigma_{1/2}$, let us denote

$$f_\beta(z) = (z/2 + \alpha z^2)e^{\beta z},$$

where α is given by formula (1) with $t = 1/2$.

Now for each β we define a domain D_β as follows. Let Δ denote the unit disk and $L_{1/2} : \Delta \to \Delta$ denote the contraction map defined by $z \to z/2$. Because the origin is an attracting fixed point with multiplier $1/2$, f_β is holomorphically conjugate to $L_{1/2}$ in a neighborhood of the origin.

Definition 2.1. For each $\beta \in \Sigma_{1/2}$ we define D_β to be the maximal subdomain of the immediate attracting basin of the origin on which f_β is holomorphically conjugate to the linear map $L_{1/2} : \Delta \to \Delta$.

The main purpose of this section is to prove the following theorem.

THEOREM 2.1. *There is a constant $K > 1$ such that for all $\beta \in \Sigma_{1/2}$, ∂D_β is a K-quasi-circle that passes through at least one of the critical points of f_β.*

We break the proof into a series of lemmas. In these we always have $\beta \in \Sigma_{1/2}$ and the map $h_\beta : \Delta \to D_\beta$ is always the unique holomorphic isomorphism such that $h_\beta(0) = 0$, $h'_\beta(0) > 0$ and $h_\beta^{-1} \circ f_\beta \circ h_\beta(z) = L_{1/2}(z)$ for all $z \in \Delta$.

LEMMA 2.2. *∂D_β is a quasi-circle passing through one or both of the critical points of f_β.*

Proof. Since the origin is an attracting fixed point of f_β, there must be a critical point in its immediate basin of attraction. By the maximality of D_β, it follows that ∂D_β must pass through at least one critical point of f_β.

By the definition of h_β we have

$$f_\beta(D_\beta) = f_\beta \circ h_\beta(\Delta) = h_\beta \circ L_{1/2}(\Delta).$$

Let $\mathbb{T}_{1/2} = \{z \mid |z| = 1/2\}$. It follows that $\partial(f_\beta(D_\beta)) = \partial h_\beta \circ L_{1/2}(\Delta) = h_\beta(\mathbb{T}_{1/2})$ is a real-analytic curve. Since f_β has exactly one finite asymptotic value which is at the origin and the origin is contained in the interior of $f_\beta(D_\beta)$, there are no asymptotic values of f_β on $\partial f_\beta(D_\beta)$. Thus ∂D_β is a bounded component of the lift of the real analytic curve $\partial f_\beta(D_\beta)$ by f_β^{-1} and is therefore a piecewise analytic curve with at most two corners at the critical points. It follows that D_β is actually a quasi-circle with finite Euclidean length. □

For any set $X \subset \mathbb{C}$, define the Euclidean diameter of X by

$$\mathrm{Diam}(X) = \sup_{a,b \in X} |a - b|.$$

For a piecewise smooth arc segment $I \subset \mathbb{C}$, let $|I|$ denote the Euclidean length of I.

We will need to estimate the relative diameters and lengths of quantities defined for each β. For simplicity, and to avoid the need for many constants, we introduce the following notation. For two quantities $X = X(\beta)$ and $Y = Y(\beta)$, we use the notation $X \preccurlyeq Y$ to mean that there is a constant $C > 0$, independent of β, such that $X \leq CY$.

The next lemma is technical. Recall that $\Delta_{1/2} = \{z \mid |z| < 1/2\}$ and that $\mathbb{T}_{1/2} = \partial \Delta_{1/2}$. For readability we drop the subscript β.

LEMMA 2.3. *Let $h : \Delta \to D$ be a univalent map such that $h(0) = 0$. Suppose that x and y are two distinct points on $h(\mathbb{T}_{1/2})$ which separate $h(\mathbb{T}_{1/2})$ into two disjoint arc segments I and J and suppose that I is the shorter arc, $|I| \leq |J|$. Then $|I| \preccurlyeq |x - y|$ where the constant is independent of β and the chosen points x, y.*

Proof. Let L be the straight segment which connects x and y. We now have two cases to consider. In the first case, $L \subset D$. Then $L' = h^{-1}(L) \subset \Delta$ is a smooth curve segment connecting two points x' and y' on $\mathbb{T}_{1/2}$. Suppose x' and y' separate $\mathbb{T}_{1/2}$ into two arc segments I' and J' such that $h(I') = I$ and $h(J') = J$. By the Köbe distortion theorem and the assumption that $|I| \leq |J|$, we have $|I'| \preccurlyeq |J'|$ and hence $|I'| \preccurlyeq |L'|$. Note the distortion theorem implies that the constant is independent of β and the points x, y.

Now there are two subcases. In the first subcase, there is an r, $1/2 < r < 1$ such that L' is contained in Δ_r. By Köbe's theorem and the fact that $|I'| \preccurlyeq |L'|$, we deduce that $|I| \preccurlyeq |L|$. Here the constant depends on r but not on β. In the second subcase there is no such r. Choose r_0, $1/2 < r_0 < 1$ and let $L'' \subset L' \cap \Delta_{r_0}$ be the component of L' that contains one of the end points of L', say x'. Again we have $|I'| \preccurlyeq |L''|$ and applying Köbe's theorem once more, we get $|I| \preccurlyeq |h(L'')| \preccurlyeq |L|$. Here the constant depends on the choice of r_0 but not on β or the points x, y.

In the second case, L is not contained in D. Again choose r_0, $1/2 < r_0 < 1$, and let L_0 be the component of $L \cap D$ that contains one of the end points of L, say x. Then $h^{-1}(L_0) \subset \Delta$ and intersects $\mathbb{T}_{r_0}$. Since $h^{-1}(x) \in \mathbb{T}_{1/2}$, it follows that $|I'| \preccurlyeq |h^{-1}(L_0)|$ and therefore by Köbe's theorem again, we get $|I| \preccurlyeq |L_0| \preccurlyeq |L|$. Here again the constant depends on r_0 but not on β or the points x, y. □

By Lemma 2.2, each ∂D_β is a quasi-circle for some K_β. We now claim we can use the same constant for all β in a compact subset of $\Sigma_{1/2}$.

LEMMA 2.4. *For any compact set $\Lambda \subset \Sigma_{1/2}$ there is a $K > 1$, depending only on Λ, such that for every $\beta \in \Lambda$, ∂D_β is a K-quasi-circle.*

Proof. Let $\mathbb{C}$ be the complex plane. First we claim that there is a compact set $E \subset \mathbb{C}$ depending only on Λ such that $\overline{D_\beta} \subset E$ for every $\beta \in \Lambda$. If the claim were not true there would be a sequence $\{\beta_n\} \subset \Lambda$ such that $\beta_n \to \beta \in \Lambda$ and such that $\mathrm{Diam}(\partial D_{\beta_n}) \to \infty$. Set $h_n = h_{\beta_n}$ and $h = h_\beta$. Then $h_n \to h$ uniformly on compact subsets of Δ. Therefore, there is some compact set $W \subset \mathbb{C}$ such that $f_{\beta_n}(\partial D_{\beta_n}) = h_n(\mathbb{T}_{1/2}) \subset W$.

Now since the Euclidean diameter of ∂D_{β_n} goes to infinity, it follows that when n is large enough, there are arbitrarily long segments A_n of ∂D_{β_n} outside any fixed disk. Since $f_{\beta_n}(\partial D_{\beta_n})$ is bounded away from zero and infinity, it follows that for all $z \in A_n$ the argument of $\beta_n z$ stays in a wedge about the imaginary axis. That is, given any $L > 0$ there exist $R > 0$ and arcs A_n of ∂D_{β_n} outside Δ_R whose Euclidean diameter is greater than L and such that one of the following two inequalities

$$|\arg(\beta_n z) - \pi/2| < \pi/4 \quad \text{or} \quad |\arg(\beta_n z) + \pi/2| < \pi/4 \tag{2}$$

holds for all $z \in A_n$. This implies, however, by taking L large enough, that as z varies continuously along A_n we can make $\arg e^{\beta_n z}$ vary from 0 to 2π any number of times. On the other hand, as z varies along A_n, it follows from inequalities (2) that the variation of $\arg(z/2 + \alpha_{\beta_n} z^2)$ remains bounded. Therefore, taking n large enough we can make the image $f_{\beta_n}(A_n)$, which is a sub-arc of $h_n(\mathbb{T}_{1/2})$, wind around the origin any number of times. This contradicts the fact that $h_n \to h$ uniformly as $n \to \infty$ on the compact set $\mathbb{T}_{1/2} \subset \Delta$ proving the claim.

Fix β and let x and y be any two points on ∂D_β. Denote by I and I' the two Jordan arcs they determine on ∂D_β and label them so that $f_\beta(I)$ is shorter than $f_\beta(I')$. Let L be the straight segment joining x and y. Since ∂D_β is a quasi-circle, the quantity

$$Q(\beta) = Q(I, L) = \mathrm{Diam}(I)/|L|$$

is bounded for all pairs (x, y) on ∂D_β. It will suffice to show that there is an upper bound on $Q(I, L)$ for all $\beta \in \Lambda$.

By Lemma 2.3, we have

$$|f_\beta(I)| \preccurlyeq |f_\beta(x) - f_\beta(y)|. \tag{3}$$

From (3) and the definitions of Diam and length, we have

$$|f_\beta(I)| \preccurlyeq |f_\beta(x) - f_\beta(y)| \preccurlyeq \text{ Diam}(f_\beta(L)) \preccurlyeq |f_\beta(L)|. \tag{4}$$

Let q be a point on the closed segment L such that $\max_{z\in L} |f'_\beta(z)|$ is achieved so that Q2

$$|f_\beta(L) \le |f'_\beta(q)||L|. \tag{5}$$

Now fix $R \ge 2$ and consider the annulus

$$A_R = \{z \mid 2\,\text{Diam}(I)/3R \le |z - x| \le 3\,\text{Diam}(I)/4R\}$$

centered at the endpoint x of I. Let $\hat{I}$ be one of the closed components of $I \cap A_R$ that connects the two boundary components of A. It follows that $|\hat{I}| \ge \text{Diam}(I)/12R$.

Let p be a point on $|\hat{I}|$ such that $\min_{z\in|\hat{I}|} |f'_\beta(z)|$ is achieved so that

$$|f_\beta(\hat{I})| \ge |f'_\beta(p)||\hat{I}|. \tag{6}$$

Combining these relations we have

$$\frac{|f'_\beta(q)|}{|f'_\beta(p)|} \ge \frac{|f_\beta(L)|}{|f_\beta(\hat{I})|}\frac{|\hat{I}|}{\text{Diam}(I)}\frac{\text{Diam}(I)}{|L|} \ge \frac{1}{12R}\frac{|f_\beta(L)|}{|f_\beta(\hat{I})|}Q(I, L). \tag{7}$$

Note that by (4), we always have

$$|f_\beta(\hat{I})| \preccurlyeq |f_\beta(L)|.$$

Putting this into (7) we have

$$Q(I, L) \preccurlyeq \frac{|f'_\beta(q)|}{|f'_\beta(p)|}. \tag{8}$$

In the first part of this proof we proved that $\overline{D}_\beta$ is contained in some compact set E of the complex plane for every $\beta \in \Lambda$. From that it follows that p and q belong to a compact set of the complex plane and hence the ratio $e^{\beta(p-q)}$ is bounded away from both zero and infinity. Therefore, from the formula $f'_\beta(z) = \alpha\beta(1 - z)(c_\beta - z)e^{\beta z}$ we see that the size of the ratio $|f'_\beta(q)|/|f'_\beta(p)|$ depends on how close the critical points are to p.

We claim that if neither critical point is close to p, the ratio $|f'_\beta(q)|/|f'_\beta(p)|$ is bounded. To see this, suppose that

$$|p - 1| \ge \text{Diam}(I)/6R \quad \text{and} \quad |p - c_\beta| \ge \text{Diam}(I)/6R. \tag{9}$$

Since $p \in \hat{I}$, we have

$$2\text{Diam}(I)/3R \le |p - x| \le 3\text{Diam}(I)/4R. \tag{10}$$

From this and $|L| \le |I|$ we get

$$|q - p| \le |q - x| + |x - p| \le |L| + |x - p| \preccurlyeq |I|. \tag{11}$$

Combining (9) and (11) we have

$$|q - 1| \le |p - 1| + |q - p| \preccurlyeq |p - 1|. \tag{12}$$

Replacing 1 by c_β in the relations above we obtain

$$|q - c_\beta| \le |p - c_\beta| + |q - p| \preccurlyeq |p - c_\beta|. \tag{13}$$

It follows that if the quasi-conformal constants K_β are unbounded, the constant $Q(\beta)$, and hence the ratio $|f'_\beta(q)|/f'_\beta(p)|$, can be made arbitrarily large by taking an appropriate $\beta \in \Lambda$. This, together with (12) and (13), implies that for any choice of R, one of the inequalities in (9) does not hold for this $\beta \in \Lambda$. In other words, for any $R > 0$, we can find $\beta \in \Lambda$ such that there is a critical point of f_β within $\mathrm{Diam}(I)/6R$ of p. This critical point lies in the annulus

$$B_R = \{z \mid \mathrm{Diam}(I)/2R < |z - x| < \mathrm{Diam}(I)/R\}.$$

Because R was arbitrary in the above argument, we can take β such that there are also critical points of f_β in the annuli $B_{R/2}$ and $B_{R/4}$. These three annuli are disjoint however, so that f_β must have at least three critical points. Since it only has two, we conclude that the K_β are bounded. □

PROPOSITION 2.1. *Let $\beta_n \to \beta_0$. Then $\partial D_{\beta_n} \to \partial D_{\beta_0}$ and $\overline{D_{\beta_n}} \to \overline{D_{\beta_0}}$ with respect to the Hausdorff metric.*

Proof. By Lemma 2.4, there is a $1 < K < \infty$ such that, for every β in a neighborhood of β_0, h_β can be extended to a K-quasi-conformal homeomorphism of the whole plane. By abuse of notation denote the extension of h_{β_n} again by h_{β_n}. Passing to a subsequence we may assume that h_{β_n} converges to a quasi-conformal homeomorphism of the plane that we denote by h. Since the maps are holomorphic in a neighborhood of the origin and $\beta_n \to \beta_0$, $h'_{\beta_n}(0) \to h'_{\beta_0}(0)$ as $n \to \infty$ and there is an L, $1 < L < \infty$, such that for all n, $1/L < h'_{\beta_n}(0) < L$. Each h_{β_n} and h_{β_0} is holomorphic on Δ and extends to a homeomorphism of $\overline{\Delta}$. It follows that $h|_\Delta$ is also holomorphic and conjugates f_{β_0} to the linear map $L_{1/2}$. This implies that $h|_{\overline{\Delta}} = h_{\beta_0}|_{\overline{\Delta}}$. Since $h_{\beta_n} \to h$ uniformly in any compact set of the complex plane, it follows that

$$h_{\beta_n}(\partial\Delta) \to h_{\beta_0}(\partial\Delta) \quad \text{and} \quad h_{\beta_n}(\overline{\Delta}) \to h_{\beta_0}(\overline{\Delta})$$

with respect to the Hausdorff metric. The lemma follows. □

To complete the proof of Theorem 2.1 we turn our attention now to neighborhoods of the boundary points of $\Sigma_{1/2}$. It turns out to be more convenient to consider the family of functions $l_\beta(\xi) = (\xi/2 + \alpha\xi^2/\beta)e^\xi$ linearly conjugate to $f_\beta(z)$ by the map $\xi = \beta z$. Set $l_\infty(\xi) = \xi e^\xi/2$; then $l_\beta \to l_\infty$ as $\beta \to \infty$.

Denote by U_β and U_∞ the maximal linearization domains of $l_\beta(\xi)$ and $l_\infty(\xi)$ centered at the origin. Then we have the following result.

LEMMA 2.5. *For any $M > 2$, consider the family*

$$\{l_\beta \mid |\beta| \ge M\} \cup \{l_\infty\}.$$

Then there is a constant $K > 1$, depending only on M, such that for all functions in the family ∂U_β is a K-quasi-circle.

Proof. Using the linear conjugation we see that ∂D_β and ∂U_β are quasi-circles with the same constant and both contain the same number of critical points. The argument of Lemma 2.2 applied to l_∞ shows that U_∞ is also a quasi-circle. Since the family is compact, the argument in the proof of Lemma 2.4 can be applied to obtain the uniform constant of quasi-conformality. □

As an immediate corollary we have the following result.

COROLLARY 2.1. *There is a constant $K > 1$ such that for all $\beta \in \Sigma_{1/2}$ with $|\beta| \geq M$, ∂D_β is a K-quasi-circle containing at least one of the critical points. Moreover for $|\beta|$ large, it contains only one, the critical point c_β.*

Proof. The first statement follows directly from Lemma 2.5. For the second, by an argument similar to the first half of the proof of Lemma 2.4, it follows that for all $|\beta| \geq M$, $\overline{U}_\beta$ is contained in some compact set E'. Suppose $|\beta|$ is so large that it does not belong to E'. Then, since the critical points of l_β are β and βc_β, ∂U_β can only contain the critical point βc_β. □

Remark 2.2. The forward orbit of the critical point β may, however, land inside D_β; for example if β is large and negative.

Next set $f_0(z) = z/2 - z^2/4$ and note that $\alpha_\beta \to -1/4$ as $\beta \to 0$; therefore $f_\beta \to f_0$ uniformly on any compact set of the complex plane. It follows that for any $m < 1$ the family

$$\{f_\beta \mid \beta \leq m\} \cup \{z/2 - z^2/4\}$$

is a compact family. Moreover the boundary of the maximal linearization domain containing the origin of the function $z/2 - z^2/4$ is a quasi-circle. We have the following result.

COROLLARY 2.2. *There is a constant $K > 1$ such that for all $\beta \in \Sigma_{1/2}$ with $|\beta| < m$, ∂D_β is a K-quasi-circle containing at least one of the critical points. Moreover, for $|\beta|$ small, it contains only one, the critical point* 1.

Proof. Applying the proof of Lemma 2.4 to this family we obtain uniformity of the quasi-conformal constant.

Let D_0 denote the maximal domain containing the origin on which f_0 is conjugate to a linear map; ∂D_0 must contain the unique critical point of f_0. Because $f_\beta \to f_0$ uniformly on compact sets, there is a compact set $E \subset \mathbb{C}$ such that, when β is small enough, there are two open topological disks $0 \in U_\beta \subset V_\beta \subset E$ such that $f_\beta : U_\beta \to V_\beta$ is a polynomial-like map of degree two and therefore that f_β is quasi-conformally conjugate to the quadratic polynomial f_0. For such β, there is only one critical point on ∂D_β and this point lies inside E. When $|\beta|$ is small enough, $|c_\beta| \approx |2/\beta|$ and is outside E. It follows that ∂D_β contains only the critical point 1 of f_β. □

Remark 2.3. Again, while the second critical point does not lie inside D_β, for some small values of β its forward orbit may fall into D_β; for example if β is small and real.

Remark 2.4. Another view on Corollary 2.2 suggested by the referee is the following: for β near 0, the restriction of f_β to a large disk centered at the origin is a quadratic-like map hybrid equivalent to $z \mapsto z/2 + z^2$. Moreover, the dilatation of the conjugacy between the two tends to 1 as $\beta \to 0$. It is also worth noting that there is a similar hybrid equivalence between g_β and $z \mapsto e^{2\pi i\theta} z + z^2$ which proves the Main Theorem when β is close to 0 or -2.

Proof of Theorem 2.1. Note that the corollaries imply the uniformity of the quasi-conformal constant in neighborhoods of the boundary points 0 and ∞ of $\Sigma_{1/2}$. The proof of Theorem 2.1 is completed by noting that the maps near ∞ and 0 are respectively conformally conjugate to the maps near -1 and -2 by the map $z \to z/c_\beta$. Therefore there is uniformity of the quasi-conformal constant and analogous behavior of the critical points on the boundary of D_β in these neighborhoods as well.

3. *The parameter space $\Sigma_{1/2}$*

Let $\gamma \subset S^2 \setminus \{0, -1, -2, \infty\}$ be the set which consists of all the values β for which ∂D_β passes through both critical points of f_β.

THEOREM 3.1. (Structure theorem for $\Sigma_{1/2}$) *The set γ is a simple closed curve which separates $\{-2, \infty\}$ and $\{0, -1\}$, such that for every $\beta \in \Sigma_{1/2}$, if β lies in the component of $S^2 \setminus \gamma$ which contains $\{-2, \infty\}$, ∂D_β passes through the critical point c_β but not the critical point 1, and if β lies in the other component, ∂D_β passes through the critical point 1 but not the critical point c_β. Moreover γ is invariant under the map $\beta \to -(\beta + 2)/(\beta + 1)$.*

A direct calculation shows the following result.

LEMMA 3.1. *$c_\beta = 1$ if and only if $\beta = -1 + i$ or $-1 - i$.*

To find points on the set γ, we consider any continuous curve $\eta : (0, 1) \to \Sigma_{1/2} - \{-1 + i, -1 - i, \}$ such that $\lim_{t\to 0} \eta(t) = 0$ and $\lim_{t\to 1} \eta(t) = \infty$. Let

$$t_0 = \sup\{t \mid 0 < t < 1,\ \partial D_{\eta(t)} \text{ passes through } 1\}$$

and set $\beta_0 = \eta(t_0)$. By definition, $c_{\beta_0} \neq 1$.

LEMMA 3.2. *∂D_{β_0} passes through both c_{β_0} and 1.*

Proof. By Corollaries 2.1 and 2.2, there is a compact set $E \subset \mathbb{C}$ such that the point $\beta_0 \in E$ for any curve η. Therefore as $t \to t_0$, $\eta(t) \to \beta_0$. By Proposition 2.1, $d_H(\partial D_{\eta(t)}, \partial D_{\beta_0}) \to 0$ as $t \to t_0$ where $d_H(A, B)$ denotes the Hausdorff distance between sets A and B. Now by the definition of t_0, there is a sequence $t_k \to t_0^-$ such that $\partial D_{\eta(t_k)}$ passes through 1 for every $k \geq 1$ and thus $1 \in \partial D_{\beta_0}$. Similarly, there is a sequence $t_k \to t_0^+$ such that $\partial D_{\eta(t_k)}$ passes through c_β for every $k \geq 1$ and thus $c_{\beta_0} \in \partial D_{\beta_0}$ also. □

LEMMA 3.3. *For each $\beta \in \gamma$, there are exactly two components of $f_\beta^{-1}(f_\beta(D_\beta))$ each of which is attached to ∂D_β at one of the two critical points c_β and 1. Moreover, one of them is bounded and the other one is unbounded. In particular, both components are attached to 1 if $c_\beta = 1$.*

Proof. Let v_1 and v_c be the critical values $f(1)$ and $f(c_\beta)$ respectively. For $i = 1, c$, draw paths σ_i from v_i to the origin. For each $i = 1, c$, there are two components of $f_\beta^{-1}(\sigma_i)$ with endpoint at i. One connects i to the origin and the other either connects it to the (unique) other pre-image of the origin or is an asymptotic path extending to infinity. In the first case, $f_\beta^{-1}(\sigma_i)$ is contained in the unique bounded component U_0 of $f_\beta^{-1}(f_\beta(D_\beta))$, and in the second, it is contained in an unbounded component U_∞ of $f_\beta^{-1}(f_\beta(D_\beta))$ that, in turn, is contained in the asymptotic tract of the origin. Both these components lie outside D_β.

To see that the unbounded component U_∞ is also unique, recall that there are only two asymptotic values, zero and infinity. Each has an asymptotic tract and these are separated by the two infinite rays $R_\beta^\pm = \{z \mid \arg(\beta z) = \pm\pi/2\}$ whose arguments differ by π. These are therefore the only infinite rays $r(t)$ such that $\lim_{t\to 1} f_\beta(r(t)) \neq 0, \infty$, that is, the Julia rays.

If there were an unbounded component $V_\infty \neq U_\infty$, then both V_∞ and U_∞ would lie in the asymptotic tract of zero. Since they are different components of $f_\beta^{-1}(f_\beta(D_\beta))$, $V_\infty \cap U_\infty = \emptyset$. The boundary of each would have to be asymptotic in one direction to some ray $r_U(t)$, respectively, $r_V(t)$, different from either of the rays $R_\beta^\pm$. Since neither $r_U(t)$ nor $r_V(t)$ can belong to any component of $f_\beta^{-1}(f_\beta(D_\beta))$ it must be one of $R_\beta^\pm$, giving us a contradiction. Note that this argument also shows that the infinite ends of the boundary of U_∞ are asymptotic respectively to the rays $R_\beta^\pm$. □

In the proof of Lemma 3.3, we saw that the boundary of the unbounded component U is asymptotic to both of the rays $R_\beta^\pm$. This implies that the Julia set of f_β is *thin* at infinity. The forward orbits of both the critical points 1 and c_β are attracted to the origin since they both lie on ∂D_β. Using a standard pull-back argument (for instance, see [**13,** proof of Theorem 3.2.9]), it is straightforward to prove the following.

LEMMA 3.4. *For each $\beta \in \gamma$, the Julia set of f_β has zero Lebesgue measure.*

We now set up a parametrization of the set γ. Recall that for each $\beta \in \gamma$, $h_\beta : \Delta \to D_\beta$ is the univalent map such that $h_\beta(0) = 0$, $h'_\beta(0) > 0$ and $h_\beta^{-1} \circ f_\beta \circ h_\beta(z) = z/2$. Since ∂D_β is a quasi-circle, it follows that h_β can be homeomorphically extended to $\partial\Delta$.

Define A_β to be the angle between $h_\beta^{-1}(1)$ and $h_\beta^{-1}(c_\beta)$ measured counterclockwise. Then $0 \le A_\beta \le 2\pi$. Define $\chi(\beta) = 1$ if the bounded component of $f_\beta^{-1}(f_\beta(D_\beta))$ is attached to 1; define $\chi(\beta) = -1$ otherwise. Identify the pair (0, 1) with the pair $(2\pi, -1)$, and the pair $(0, -1)$ with the pair $(2\pi, 1)$. Under this identification, to each $\beta \in \gamma$, we can assign a unique pair $\mathbf{I}_\beta = (A_\beta, \chi(\beta))$. From Remark 2.1 and the fact that c_β depends continuously on β we have the following result.

PROPOSITION 3.1. *The map $\beta \to \mathbf{I}_\beta$ is continuous on the set γ.*

The next lemma says that the value $\beta \in \gamma$ is uniquely determined by the pair $\mathbf{I}_\beta = (A_\beta, \chi(\beta))$.

LEMMA 3.5. *Let $\beta_1, \beta_2 \in \gamma$. If $\mathbf{I}_{\beta_1} = \mathbf{I}_{\beta_2}$, then $f_{\beta_1} = f_{\beta_2}$ and therefore, $\beta_1 = \beta_2$.*

The idea of the proof is to show that if $\mathbf{I}_{\beta_1} = \mathbf{I}_{\beta_2}$ then f_{β_1} is conformally conjugate to f_{β_2}. Note that since both critical points are attracted to the origin there is only one grand orbit of components of the Fatou set.

Proof. Let us give a description of the combinatorics of f_{β_1}; those for f_{β_2} will be the same. For readability we omit the subscript. The description we give of the grand orbit of $D = D_\beta$ works for either $\beta = \beta_1$ or β_2. Let U denote the unbounded component and V the bounded component of $f_\beta^{-1}(f_\beta(D))$ outside D. Assume that U is attached to c_β and V is attached to 1. The same argument can be applied in the other case.

Since the map h_β can be continuously extended to a homeomorphism between $\overline{\Delta}$ and $\overline{D}$, we can define a continuous family of curves λ_r, $0 \le r \le 1$, by

$$\lambda_r(t) = h_\beta(re^{it}), \quad t \in \mathbb{R}.$$

Define $t_0 \in [0, 2\pi)$ by $\lambda_1(t_0) = c_\beta$.

Next we lift the curves λ_r, $1/2 < r \le 1$, using a normalized inverse branch of f_β taking D to U to get a continuous family of curves Λ_r, $1/2 < r \le 1$,

$$\Lambda_r(t) = f_\beta^{-1}(\lambda_r(t)), \quad t \in \mathbb{R}.$$

From the continuity of $\Lambda_r(t)$ with respect to r it follows that

$$\Lambda_{1/2} = \{\Lambda_{1/2}(t) \mid t \in \mathbb{R}\} = \partial D \cup \partial U \cup \partial V.$$

We normalize so that $\Lambda_{1/2}(f_\beta(1)) = 1$; this determines the normalization for the curves when $r > 1/2$.

The curves $\Lambda_r = \{\Lambda_r(t) \mid t \in \mathbb{R}\}$ for $1/2 < r < 1$ lie outside $\overline{(D \cup U \cup V)}$ and are infinite curves asymptotic at one end to R_β^+ and asymptotic at the other to R_β^-. The map f_β from Λ_r onto λ_r is infinite-to-one.

It follows that $\Lambda_1 = f_\beta^{-1}(\partial D)$ is a curve with the same asymptotic and covering properties. It thus separates $f_\beta^{-1}(D)$ from its complement. That is, both $f_\beta^{-1}(D)$ and its complement in $\mathbb{C}$ are simply connected. Note that $f_\beta^{-1}(D)$ contains $D \cup U \cup V$.

To keep track of the pre-images of D, U, and V we need an addressing scheme similar to the one described for the model for quadratics in [**14**]. Here, the coverings are infinite-to-one. Let $y_0 = \Lambda_1(t_0)$ where $t_0 = \arg h_{\beta_1}^{-1}(c_{\beta_1}) \in [0, 2\pi)$. The other pre-images are naturally labeled by $y_n = \Lambda(t_0 + 2\pi n)$.

Denote the complement of $\overline{f_\beta^{-1}(D)}$ by Y. In Y, label by U_0 the component of $f_\beta^{-1}(U)$ attached to Λ_1 at y_0. Then label the components attached at y_n by U_n.

There is a branch of $f_\beta^{-1}(\Lambda_1)$ between each pair U_i and U_{i+1}; label it $\Lambda_{1,i}$; it extends to infinity in both directions and the map from $\Lambda_{1,i}$ to Λ_1 is one-to-one. It is the boundary of a simply connected component of the complement of $f_\beta^{-2}(D)$ that we label Y_i. Set $y_{i,0} = f_\beta^{-1}(y_0)$ and label the other pre-images accordingly.

In this way, increasing the number of subscripts at each stage, we label each of the components of $f_\beta^{-k}(D)$ and each of the components of its complement for all $k \ge 2$.

We now use subscripts and superscripts to differentiate between objects associated to β_1 and β_2. For instance, D_1 and D_2 are the maximal linearization domains and Λ_r^1 and Λ_r^2 are used to denote the curve family Λ_r for f_{β_1} and f_{β_2}, respectively.

Let $H : D_1 \to D_2$ be the univalent map defined by $f_{\beta_1} H = H f_{\beta_2}$ such that $H(1) = 1$, and $H(c_{\beta_1}) = c_{\beta_2}$. Let $\phi_0 : \mathbb{C} \to \mathbb{C}$ be a quasi-conformal extension of H such that $\phi_0(\infty) = \infty$. We will define a sequence of quasi-conformal maps $\phi_n : \mathbb{C} \to \mathbb{C}$ inductively using the dynamics.

First let us define ϕ_1 and show how the condition $\mathbf{I}_{\beta_1} = \mathbf{I}_{\beta_2}$ is used. Define $\phi_1 = \phi_0 = H$ on D_1. Using the addressing scheme to choose the appropriate inverse branch of f_{β_2}, we define $\phi_1 : U_1 \to U_2$, $V_1 \to V_2$ by $\phi_1 = f_{\beta_2}^{-1} \circ \phi_0 \circ f_{\beta_1}$. For a point in

$$f_{\beta_1}^{-1}(D_1) \setminus \overline{(D_1 \cup U_1 \cup V_1)}$$

define

$$\phi_1 = f_{\beta_2}^{-1} \circ H \circ f_{\beta_1},$$

where the inverse is chosen so that if $z = \Lambda_r^1(t)$ then $\phi_1(z) = \Lambda_r^2(t)$.

We now have a map $\phi_1 : f_{\beta_1}^{-1}(D_1) \to f_{\beta_2}^{-1}(D_2)$. Since $\mathbf{I}_{\beta_1} = \mathbf{I}_{\beta_2}$, ϕ_1 is continuous at both the critical points 1 and c_{β_1} and hence holomorphic on $f_{\beta_1}^{-1}(D_1)$.

To extend ϕ_1 to a quasi-conformal homeomorphism of $\mathbb{C}$, define ϕ_1 on $\mathbb{C} - f_{\beta_1}^{-1}(D_1)$ by $\phi_1 = f_{\beta_2}^{-1} \circ \phi_0 \circ f_{\beta_1}$. This is well defined because $\mathbb{C} - f_{\beta_1}^{-1}(D_1)$ is simply connected and there is no critical value of f_{β_2} outside D_2.

Now let us assume that for every $1 \le k \le n$, we have a quasi-conformal homeomorphism $\phi_k : \mathbb{C} \to \mathbb{C}$ defined so that $\phi_k : f_{\beta_1}^{-k}(D_1) \to f_{\beta_2}^{-k}(D_2)$ is a holomorphic isomorphism such that for all $z \in f_{\beta_1}^{-k}(D_1)$,

$$f_{\beta_1}(z) = \phi_{k-1}^{-1} \circ f_{\beta_2} \circ \phi_k(z).$$

Define ϕ_{n+1} as follows. Let W be a component of $f_{\beta_1}^{-n-1}(D_1) - f_{\beta_1}^{-n}(D_1)$ and let Λ be a boundary component of W which is also a boundary component of $f^{-n}(D_1)$. Define ϕ_{n+1} on W by $\phi_{n+1} = f_{\beta_2}^{-1} \circ \phi_n \circ f_{\beta_1}$, where the inverse branch of f_{β_2} is chosen respecting the addressing scheme so that on Λ, $\phi_{n+1} = \phi_n$. Note that ϕ_{n+1} is well defined on W because W is simply connected and $\phi_n(f_{\beta_1}(W))$ does not contain any critical values of f_{β_2}. Now we can define $\phi_{n+1} : f_{\beta_1}^{-n-1}(D_1) \to f_{\beta_2}^{-n-1}(D_{\beta_2})$ to be a holomorphic isomorphism such that $\phi_{n+1} = \phi_n$ on $f_{\beta_1}^{-n}(D_1)$ and on $f_{\beta_1}^{-n-1}(D_1)$,

$$f_{\beta_1}(z) = \phi_n^{-1} \circ f_{\beta_2} \circ \phi_{n+1}(z). \tag{14}$$

It then follows that the boundary of some component Y of $\mathbb{C} - f_{\beta_1}^{-n-1}(D_1)$ is mapped by ϕ_{n+1} to the boundary of some component Y' of $\mathbb{C} - f_{\beta_2}^{-n-1}(D_{\beta_2})$ with the same address. Note that the component Y is mapped by f_{β_1} one-to-one and onto some component Y_i of $\mathbb{C} - f_{\beta_1}^{-n}(D_1)$ and similarly, the component Y' is mapped by f_{β_2} one-to-one and onto some component Y_i' of $\mathbb{C} - f_{\beta_2}^{-n}(D_{\beta_2})$. By equation (14), it follows that $\phi_n(\partial Y_i) = \partial Y_i'$ and therefore $\phi_{n+1}(\partial Y) = \partial Y'$. Now we define $\phi_{n+1} : Y \to Y'$ by setting $\phi_{n+1} = f_{\beta_2}^{-1} \circ \phi_n \circ f_{\beta_1}$. In this way we extend ϕ_{n+1} to all the components of $\mathbb{C} - f_{\beta_1}^{-n-1}(D_1)$ and obtain a quasi-conformal homeomorphism $\phi_{n+1} : \mathbb{C} \to \mathbb{C}$.

By induction, we have a sequence of quasi-conformal homeomorphisms $\{\phi_n\}$ of the complex plane such that each ϕ_n is conformal on $f_{\beta_1}^{-n}(D_1)$ and its Beltrami coefficient satisfies

$$\|\mu_{\phi_n}\|_\infty \le \|\mu_{\phi_1}\|_\infty < 1.$$

Taking a convergent subsequence of $\{\phi_n\}$, we get a pair of limit quasi-conformal homeomorphisms of the sphere, ϕ and ψ, which fix 0, 1, and ∞ and satisfy the functional relation $f_{\beta_1}(z) = \phi^{-1} \circ f_{\beta_2} \circ \psi(z)$. It follows from the above construction that $\phi = \psi$ on the grand orbit of D_1. Since both critical points are attracted to the origin, by Remark 2.1 this grand orbit is the full Fatou set of f_{β_1}. Since the Fatou set of f_{β_1} is dense on the complex plane, $\phi = \psi$ everywhere. Since ϕ is conformal on $\bigcup_{0\le k<\infty} f_{\beta_1}^{-k}(D_1)$, which by Lemma 3.4 has full measure, it is conformal everywhere and must be the identity, completing the proof. □

In the next lemma we show that $\mathbf{I}_\beta$ is surjective.

LEMMA 3.6. *For each pair* (θ, χ) *where* $0 \le \theta \le 2\pi$ *and* $\chi = 1$ *or* -1, *there is a unique* $\beta \in \gamma$ *such that* $\mathbf{I}_\beta = (\theta, \chi)$.

Proof. Recall that when $\beta = -1 + i$ or $-1 - i$, $c_\beta = 1$. In both cases the two components of $f_\beta^{-1}(f_\beta(D_\beta))$, which are on the outside of D_β, are attached to ∂D_β at 1; the configurations are complex conjugates of one another. These cases realize the combinatorial pairs, $(0, +1)$ which is identified with $(2\pi, -1)$ and $(0, -1)$ which is identified with $(2\pi, 1)$.

Suppose now that $0 < \theta < 2\pi$. Choose some curve η as defined for Lemma 3.2 and let $\beta_0 = \eta(t_0)$. Under conjugation by $z \mapsto z/c_\beta$, the sign of $\chi(\beta)$ will reverse and $A(\beta)$ will become $2\pi - A(\beta)$. We therefore restrict our consideration to the assumption that $\chi = \chi(\beta_0)$. We want to construct a function f_β such that $\mathbf{I}_\beta = (\theta, \chi)$.

For $t > 0$, set $\mathbb{D}_t = \{z \mid |z| < t\}$. Take r small enough that $\mathbb{D}_r$ is contained in $f_{\beta_0}(D_{\beta_0})$. Take any two points $x_1, x_2 \in \partial\mathbb{D}_r$ such that the counterclockwise angle from x_1 ro x_2 is equal to θ. Define a quasi-conformal homeomorphism $g : D_{\beta_0} \setminus \mathbb{D}_r \to f_{\beta_0}(D_{\beta_0}) \setminus \mathbb{D}_{r/2}$ such that

$$g|_{\partial D_{\beta_0}} = f_{\beta_0}|_{\partial D_{\beta_0}}, \quad g^2(\partial D_{\beta_0}) = \partial\mathbb{D}_r, \quad g|_{\partial\mathbb{D}_r(z)} = z/2$$

and

$$g^2(1) = x_1, \quad g^2(c_{\beta_0}) = x_2.$$

Such a g obviously exists. Define

$$F(z) = \begin{cases} f_{\beta_0}(z) & \text{for } z \notin D_{\beta_0}, \\ z/2 & \text{for } z \in \mathbb{D}_r, \\ g(z) & \text{for } z \in D_{\beta_0} \setminus \mathbb{D}_r. \end{cases} \tag{15}$$

We next define an F-invariant complex structure on the Riemann sphere that we identify with the Beltrami differential μ, $\|\mu\|_\infty < 1$. Denote the standard structure by μ_0 and define

$$\mu(z) = \begin{cases} \mu_0(z) & \text{for } z \in \mathbb{D}_r, \\ (g^2)^*\mu_0(z) & \text{for } D_{\beta_0} \setminus \mathbb{D}_r, \\ \mu(F^n(z))\overline{F_z^n}/F_z^n & \text{for } z \in F^{-n}(D_{\beta_0}) \text{ with } n \ge 1 \text{ and} \\ 0 & \text{otherwise.} \end{cases} \tag{16}$$

Now let ω be the quasi-conformal map which solves the Beltrami equation with coefficient μ and which fixes 0, 1, and ∞. Then $G = \omega \circ F \circ \omega^{-1}$ is an entire function.

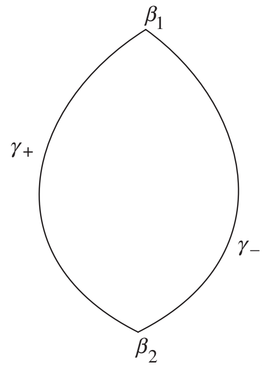

FIGURE 1. The curve $\gamma \subset \Sigma_{1/2}$.

Since ω and its inverse are Hölder continuous at infinity, it follows that G is of finite order. Since f_{β_0} has an asymptotic value, G is transcendental. From the construction of G, it follows that G has an asymptotic value at zero, has two zeros and two critical points, and has an attracting fixed point at the origin with multiplier $1/2$; G therefore belongs to $\Sigma_{1/2}$. Moreover, both critical points lie in the boundary of the maximal linearization domain of G centered at the origin. It follows that there is a $\beta \in \gamma$ such that $G = f_\beta$. Note that the angle between 1 and c_β is equal to the angle between $f_\beta^2(1)$ and $f_\beta^2(c_\beta)$. Since the restriction of ω to $\mathbb{D}_r$ is the linearization map of f_β, it follows that the angle between $f_\beta^2(1)$ and $f_\beta^2(c_\beta)$ is equal to the angle between x_1 and x_2, which is θ. This implies that the map f_β realizes the pair (θ, χ) and by Lemma 3.5, β is unique. $\square$

For each $0 < \xi < 2\pi$, by Lemmas 3.5 and 3.6, there is a unique value denoted by $\beta_+(\xi) \in \gamma$ such that $\mathbf{I}_{\beta_+} = (\xi, +1)$ and a unique value denoted by $\beta_-(\xi) \in \gamma$ such that $\mathbf{I}_{\beta_-} = (\xi, -1)$.

LEMMA 3.7. *The maps* $\beta_+, \beta_- : (0, 2\pi) \to S^2 \setminus \{-1, -2, 0, \infty\}$ *are continuous.*

Proof. We only prove the continuity of β_+. The same argument proves the continuity of β_-.

Assume β_+ is not continuous at some $0 < \xi < 2\pi$. Then there is a sequence $\xi_n \to \xi$ and some $\delta > 0$ such that $|\beta_+(\xi_n) - \beta_+(\xi)| > \delta$. By Corollaries 2.1 and 2.2 we see that in a small neighborhood of each singularity of $\Sigma_{1/2}$, ∂D_β contains exactly one critical point and therefore that the sequence $\{\beta_+(\xi_n)\}$ is contained in some compact set $K \subset S^2 \setminus \{-1, -2, 0, \infty\}$. Passing to a convergent subsequence, we may assume that $\beta_+(\xi_n) \to \beta$ for some β.

An argument similar to that of Lemma 3.2 proves, however, that ∂D_β passes through both 1 and c_β so that $\beta \in \gamma$. By Proposition 3.1, $\mathbf{I}_\beta = (\xi, +1)$ and by Lemma 3.5, $\beta_+(\xi) = \beta$. This contradiction completes the proof. $\square$

We now have all the ingredients to prove the Structure theorem for $\Sigma_{1/2}$.

Proof of Theorem 3.1. It is not difficult to see that

$$\lim_{\xi\to 0}\beta_+(\xi)=\lim_{\xi\to 2\pi}\beta_-(\xi)=\beta_1,$$
$$\lim_{\xi\to 2\pi}\beta_+(\xi)=\lim_{\xi\to 0}\beta_-(\xi)=\beta_2$$

and

$$\{\beta_1,\,\beta_2\}=\{-1+i,\,-1-i\}.$$

In addition, by Lemma 3.5, both β_+ and β_- are injective. It follows that (see Figure 1)

$$\gamma=\beta_+([0,\,2\pi])\cup\beta_-([0,\,2\pi])=\gamma_1\cup\gamma_2$$

is a simple closed curve . In fact, when β varies along one of the curves of γ_1 or γ_2, the component of $f_\beta^{-1}(f_\beta(D_\beta))$ attached to 1 is bounded and when β varies along the other one, the component is unbounded.

Set $\sigma:\beta\to-(\beta+2)/(\beta+1)$. The map $\xi=z/c_\beta$ conjugates f_β to $f_{\sigma(\beta)}$ so that γ is invariant under σ. In addition, any continuous curve in $\Sigma_{1/2}$ joining zero to infinity must intersect γ by Lemma 3.2 so that γ separates zero and infinity.

Let Ω_{int}, Ω_{ext} denote the bounded and unbounded components of $\Sigma_{1/2}-\gamma$. It follows that zero is a puncture of Ω_{int} and infinity is a puncture of Ω_{ext}. Since $\sigma(0)=-2$, it follows that for β in a small neighborhood of -2, ∂D_β passes through only c_β. The curve γ thus must separate 0 and -2 and therefore -2 is a puncture of Ω_{ext}. Similarly, since $\sigma(-1)=\infty$, γ separates -1 and infinity, -1 is a puncture of Ω_{int}. Since γ is invariant under σ, $\sigma(\Omega_{\text{int}})=\Omega_{\text{ext}}$ and $\sigma(\Omega_{\text{ext}})=\Omega_{\text{int}}$. □

4. *The surgery map* **S**

In this section we will define a surgery map $\mathbf{S}:\Omega_{\text{int}}\to\Sigma_\lambda$ which can then be continuously extended to $\overline{\Omega_{\text{int}}}$. The main idea is based on a construction from [**21**], which allows one to construct a Siegel disk from an attracting fixed point.

The idea behind the construction is to replace the contraction map on D_β by a rotation. This is done by constructing a model map. First the map is constructed on the boundary of D_β and its images and pre-images; then it is extended from these curves to the whole plane as a quasi-conformal map. The surgery map is then defined using the Ahlfors-Bers Measurable Riemann mapping to assign a map in Σ_λ to the model.

We begin by recalling some basic facts about real-analytic curves. A curve η is called *real-analytic* if, for each $x\in\eta$, there is a domain D with $x\in D$ and a univalent map h defined on D such that $h(D\cap\eta)$ is a segment of $\mathbb{R}$ (or equivalently a circle). We need the following generalized version of the Schwarz reflection principle [**1**].

LEMMA 4.1. *Let U be a domain such that $\eta\subset\partial U$ is an open and real-analytic curve segment. Suppose f is a holomorphic function defined on U such that f can be continuously extended to η and $f(\eta)$ is also a real-analytic curve segment. Then f can be holomorphically continued to a larger domain which contains η in its interior.*

We now use $\beta\in\overline{\Omega_{\text{int}}}$ to construct a real analytic circle homeomorphism. For $\beta\in\Omega_{\text{int}}$, let U_β, V_β denote the unbounded components of $\widehat{\mathbf{C}}-\partial D_\beta$ and $\widehat{\mathbf{C}}-f_\beta(\partial D_\beta)$, respectively.

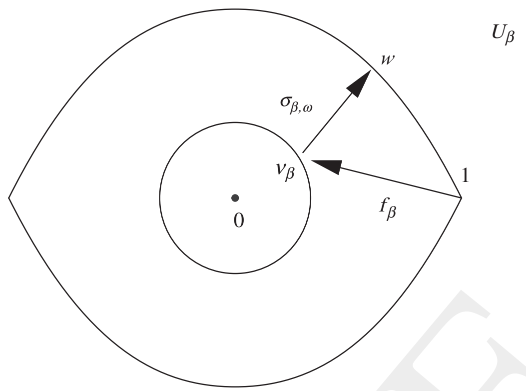

FIGURE 2. The topological circle mapping $\sigma_{\beta,w} \circ f_\beta : \partial D_\beta \to \partial D_\beta$.

The first step is to construct a homeomorphism from ∂U_β to itself whose rotation number is the fixed θ. Let $v_\beta = f_\beta(1) \in \partial V_\beta$. By the Riemann Mapping theorem, for each $w \in \partial U_\beta$, there is a unique conformal isomorphism $\sigma_{\beta,w} : V_\beta \to U_\beta$ such that $\sigma_{\beta,w}(v_\beta) = w$ and $\sigma_{\beta,w}(\infty) = \infty$. Note that as w varies on ∂U_β, the restricted maps $\{(\sigma_{\beta,w} \circ f_\beta)|_{\partial U_\beta}\}$ form a continuous and monotone family of topological circle homeomorphisms. By [**11,** Proposition 11.1.9], it follows that there is a unique w, say $w_\beta \in \partial U_\beta$, such that the rotation number of $(\sigma_{\beta,w_\beta} \circ f_\beta)|_{\partial U_\beta}$ is the θ we fixed in §1. To simplify the notation, we denote σ_{β,w_β} by σ_β.

We now define a circle homeomorphism by conjugating the map we just constructed. Let $\psi_\beta : \widehat{\mathbf{C}} - \overline{\Delta} \to U_\beta$ be the Riemann map such that $\psi_\beta(\infty) = \infty$ and $\psi_\beta(1) = 1$. By Theorem 2.1, $\partial U_\beta = \partial D_\beta$ is a quasi-circle. The curve $\partial V_\beta = f_\beta(\partial D_\beta)$ is *real-analytic* since it is the h_β-image of the circle $\{z \mid |z| = 1/2\}$, where, as usual, $h_\beta : \Delta \to \partial D_\beta$ is the univalent map that conjugates f_β to the linear map $z \mapsto z/2$. The map

$$s_\beta = \psi_\beta^{-1} \circ \sigma_\beta \circ f_\beta \circ \psi_\beta : \partial\Delta \to \partial\Delta$$

is the desired critical circle homeomorphism with rotation number θ. The following lemma shows it is real analytic.

LEMMA 4.2. *There is an open annular neighborhood A of $\partial\Delta$ such that for every $\beta \in \overline{\Omega_{\text{int}}}$, the circle homeomorphism $s_\beta : \partial\Delta \to \partial\Delta$ can be analytically extended to A. For $\beta \in \Omega_{\text{int}}$, s_β has one double critical point at* 1. *For $\beta \in \gamma \subset \partial\Omega_{\text{int}}$, if $c_\beta \neq 1$, s_β has two double critical points at* 1 *and $\psi_\beta^{-1}(c_\beta)$; otherwise, s_β has a critical point at* 1 *of local degree five.*

Proof. We will prove only the first assertion of the lemma. The remaining assertions will follow from this proof.

Note that by setting $f_0(z) = z/2 - z^2/4$ and $f_{-1}(z) = ze^{-z}/2$, $\overline{\Omega_{\text{int}}}$ is homeomorphic to the closed unit disk and is therefore compact. Thus it is sufficient to prove that for every $\beta_0 \in \overline{\Omega_{\text{int}}}$, there is an open neighborhood A of $\partial\Delta$ and an open neighborhood U of β_0 in $\overline{\Omega_{\text{int}}}$ such that for every $\beta \in U$, s_β can be analytically extended to A. We shall prove this assuming that $\beta_0 \in \Omega_{\rm int} \cup \{0, -1\}$ so that ∂D_β contains only the critical point 1 of f_{β_0}. The case that $\beta_0 \in \partial\Omega_{\rm int} \setminus \{0, -1\}$ can be proved in a similar way.

First take a small half neighborhood N'_{β_0} of 1 which is attached to the unit circle from the outside of the unit disk. Note that if N'_{β_0} is small enough, the boundary segment of N'_{β_0}, which lies on the unit circle, is mapped by $f_{\beta_0} \circ \psi_{\beta_0}$ to a real-analytic curve segment on ∂V_{β_0}. Applying Lemma 4.1, $f_{\beta_0} \circ \psi_{\beta_0}$ can be holomorphically extended to an open neighborhood N_{β_0} of 1 such that $f_{\beta_0} \circ \psi_{\beta_0}$ has local degree three at 1.

Let $W_{\beta_0} = f_{\beta_0} \circ \psi_{\beta_0}(N_{\beta_0})$. We may take N_{β_0} small enough so that the following holomorphic continuation is valid. Let $W'_{\beta_0} = V_{\beta_0} \cap W_{\beta_0}$. Note that the boundary segment of W'_{β_0} which lies on ∂V_{β_0} is real-analytic and is mapped by $\psi_{\beta_0}^{-1} \circ \sigma_{\beta_0}$ to a Euclidean arc segment. By Lemma 4.1 again, $\psi_{\beta_0}^{-1} \circ \sigma_{\beta_0}$ can be holomorphically continued to W_{β_0} and the continuation maps W_{β_0} homeomorphically onto some neighborhood of $s_{\beta_0}(1)$. It follows that s_{β_0} can be analytically extended to the open neighborhood N_{β_0} of 1 and 1 is a double critical point of s_{β_0}. Since the maps f_β, ψ_β, and σ_β are open and continuous in β, it follows that one can, using the same method, obtain a neighborhood N_β for each β in a small enough neighborhood U of β_0, such that the intersection of all these neighborhoods contains a neighborhood N of 1. That is, for every $\beta \in U \subset \overline{\Omega_{\rm int}}$ there is a common open neighborhood N of 1 such that s_β can be analytically extended to N.

Now for every $z \in \partial\Delta \setminus N$, s_{β_0} is holomorphic in a half neighborhood B'_{β_0} of z exterior to the unit circle. We can take B'_{β_0} small enough so that s_{β_0} maps B'_{β_0} homeomorphically to a half neighborhood of $s_{\beta_0}(z)$. By Lemma 4.1 one can construct an open neighborhood B_{β_0} of z such that s_{β_0} can be analytically and homeomorphically extended to B_{β_0}. Since $\partial\Delta \setminus N$ is compact, there exist finitely many points $z_i \in \partial\Delta \setminus N$, $1 \le i \le n$, such that

$$\partial\Delta \subset N \cup \bigcup_{i=1}^{n} B^i_{\beta_0},$$

where $B^i_{\beta_0}$ is the corresponding neighborhood of z_i. Again since the maps f_β, ψ_β, σ_β, and thus s_β, are open and continuous in β, the corresponding neighborhoods B^i_β each contain an open neighborhood B^i in $B^i_{\beta_0}$ for every β in a small enough neighborhood U of β_0. That is, there exist open neighborhoods B^i of z_i, $1 \le i \le n$, such that

$$\partial\Delta \subset N \cup \bigcup_{i=1}^{n} B^i,$$

and moreover, for every $\beta \in U$, s_β can be analytically extended to B_i, $1 \le i \le n$. Now set

$$A = N \cup \bigcup_{i=1}^{n} B^i.$$

It follows that A is an open neighborhood of $\partial\Delta$ and that s_β can be analytically extended to A for every $\beta \in U$.

If $\beta_0 \in \partial\Omega_{\rm int} \setminus \{0, -1\}$, neighborhoods of both critical points need to be considered. The argument is then essentially the same. This completes the proof of the Lemma. □

We now need the following theorem due to Herman and Swiatek ([H], [Sw]).

HERMAN–SWIATEK THEOREM. *Let $s : \partial\Delta \to \partial\Delta$ be a real-analytic critical circle homeomorphism of rotation number θ. Then s is quasi-symmetrically conjugate to the rigid rotation R_θ if and only if θ is of bounded type. Moreover, the quasi-symmetric constant of the conjugacy depends only on θ and the size of the annular neighborhood of $\partial\Delta$ over which s extends analytically.*

From the Herman–Swiatek theorem, for each $\beta \in \Omega_{\text{int}}$, the circle homeomorphism s_β defined in Lemma 4.2 is quasi-symmetrically conjugate to the rigid rotation R_θ. Let A be the open annular neighborhood of $\partial\Delta$ given in Lemma 4.2. Then every map in the family $\{s_\beta \mid \beta \in \overline{\Omega_{\text{int}}}\}$ can be analytically extended to A. Applying the Herman–Swiatek theorem, we have the following result.

LEMMA 4.3. *There exists a constant K, $1 < K < \infty$, such that for any $\beta \in \overline{\Omega_{\text{int}}}$, there is a quasi-symmetric homeomorphism p_β satisfying:*

(1) $p_\beta(1) = 1$;
(2) $s_\beta = p_\beta \circ R_\theta \circ p_\beta^{-1}$;
(3) *the quasi-symmetric constant of p_β is bounded by K.*

In order to construct the model map, we will need to consider quasi-conformal extensions of quasi-symmetric homeomorphisms of the circles $\partial\Delta$ and $\partial\Delta_{1/2}$. Such extensions can be defined using either the Beurling–Ahlfors or Douady–Earle extensions. For our purposes it will be necessary to normalize the extensions so that they fix the origin.

Let g be a quasi-symmetric homeomorphism of $\partial\Delta$. Using the covering map of the upper half plane $e^{2\pi i z} : \mathbb{H} \to \Delta \setminus \{0\}$, g can be lifted to a quasi-symmetric homeomorphism $G : \mathbb{R} \to \mathbb{R}$, invariant under the translation $x \mapsto x + 1$. Let $\widetilde{G} : \mathbb{H} \to \mathbb{H}$ denote the Douady-Earle extension of g. Since it is also translation invariant, it can be pushed down to a homeomorphism $\widetilde{g} : \Delta \setminus \{0\} \to \Delta \setminus \{0\}$ and extended to Δ setting $\widetilde{g}(0) = 0$. This gives a quasi-conformal extension of g fixing the origin. To extend a quasi-symmetric map of $\partial\Delta_{1/2}$, we use the covering map $e^{2\pi i z}/2 : \mathbb{H} \to \Delta_{1/2} \setminus \{0\}$. Below we will refer to the map $\widetilde{g}$ as the *normalized extension* of g.

As before we denote by h_β the univalent map with $h_\beta(0) = 0$ and $h'_\beta(0) > 0$ that conjugates the action of $L_{1/2}$ on Δ to the action of f_β on $\overline{D}_\beta$ and we use $\psi_\beta : \widehat{\mathbb{C}} - \overline{\Delta} \to U_\beta$ to denote the Riemann map such that $\psi_\beta(1) = 1$ and $\psi_\beta(\infty) = \infty$.

Now set $\phi_\beta = \sigma_\beta^{-1}\psi_\beta$. Then $\phi_\beta : \widehat{\mathbf{C}} - \overline{\Delta} \to V_\beta$ is the Riemann map such that $\phi_\beta(\infty) = \infty$ and $\phi_\beta(v_\beta = \psi^{-1}(w_\beta)$. Q3

LEMMA 4.4. *There is a positive constant M such that for every $\beta \in \overline{\Omega_{\text{int}}}$, the maps:*

(1) $\psi_\beta^{-1} \circ h_\beta : \partial\Delta \to \partial\Delta$; *and*
(2) $L_{1/2} \circ \phi_\beta^{-1} \circ h_\beta : \partial\Delta_{1/2} \to \partial\Delta_{1/2}$;

are all M-quasi-symmetric homeomorphisms.

Proof. By Theorem 2.1, there is a $K \geq 1$, independent of $\beta \in \overline{\Omega_{\text{int}}}$, such that the curves ∂D_β are K-quasi-circles. Since $\overline{\Omega_{\text{int}}}$ is compact, there is an $M \geq 1$ such that the maps h_β, ϕ_β, ψ_β and p_β can be extended to M-quasi-conformal homeomorphisms of the plane which fix the origin. This implies the lemma. □

Each of these quasi-symmetric homeomorphisms has a normalized quasi-conformal extension as does the map p_β. We denote them as follows:

$$\Psi_\beta = \widetilde{\psi_\beta^{-1} \circ h_\beta} : \overline{\Delta} \to \overline{\Delta},$$

$$\Phi_\beta = L_{1/2} \circ \widetilde{\phi_\beta^{-1} \circ h_\beta} : \overline{\Delta}_{1/2} \to \overline{\Delta}_{1/2}$$

and

$$P_\beta = \widetilde{p_\beta} : \overline{\Delta} \to \overline{\Delta}.$$

From Lemmas 4.3 and 4.4 it follows that the complex dilatation of these maps is uniformly bounded. That is, the following result is true.

LEMMA 4.5. *There is a constant* $0 < k < 1$ *such that for every* $\beta \in \Omega_{\text{int}} \cup \gamma$,

$$|\mu_{\Psi_\beta}(z)| < k, \quad |\mu_{\Phi_\beta}(z)| < k \quad \textit{and} \quad |\mu_{P_\beta}(z)| < k$$

hold for almost every point $z \in \mathbb{C}$.

Remark 4.1. These quasi-conformal maps are just what we need to construct the model maps F_β in which the contraction on D_β is replaced by a rotation. The subtlety here is due to the fact that we have to make the model F_β depend continuously on β. The construction would be much simpler if one had only to construct a single model map. For instance, the reader may refer to [**21**] to see the construction of the model map for $e^{2\pi i\theta}\sin(z)$.

Define $\widehat{\sigma}_\beta(z) : \mathbb{C} \to \mathbb{C}$ to be the normalized quasi-conformal extension of σ_β by setting

$$\widehat{\sigma}_\beta(z) = \begin{cases} \sigma_\beta(z) & \text{for } z \in V_\beta, \\ h_\beta \circ \Psi_\beta^{-1} \circ L_{1/2}^{-1} \circ \Phi_\beta \circ h_\beta^{-1}(z) & \text{otherwise.} \end{cases} \tag{17}$$

Thus $\widehat{\sigma}_\beta : V_\beta \to U_\beta$ and $\widehat{\sigma}_\beta : \overline{f_\beta(D_\beta)} \to \overline{D_\beta}$.

Set $R_\beta = P_\beta^{-1} \circ \Psi_\beta \circ h_\beta^{-1}$. Finally, we define the model map $F_\beta : \mathbb{C} \to \mathbb{C}$ by

$$F_\beta(z) = \begin{cases} \widehat{\sigma}_\beta \circ f_\beta(z) & \text{for } z \in U_\beta, \\ R_\beta^{-1} \circ R_\theta \circ R_\beta(z) & \text{otherwise.} \end{cases} \tag{18}$$

By Lemma 4.5 and the construction of F_β the quasi-conformal maps F_β are uniformly quasi-conformal. That is, given by the following result. Q4

LEMMA 4.6. *There is a constant* $0 < k < 1$ *such that for every* $\beta \in \Omega_{\text{int}} \cup \gamma$,

$$\sup_{z \in \mathbb{C}} |\mu_{F_\beta}(z)| \le k.$$

The support of μ_{F_β} *is contained in* $\bigcup_{k \ge 0} F_\beta^{-k}(D_\beta)$.

To define the surgery map we want to construct a map in Σ_λ from the model F_β. To do this we define a complex structure that we identify with the Beltrami differential μ_β on the Riemann sphere that is compatible with the dynamics as follows: for $z \in \mathbb{C}$, let $m \ge 0$ be the least integer such that $F_\beta^m(z) \in D_\beta$. If m is finite define $\mu_\beta(z)$ to be the pull back of $\mu_{R_\beta}(F_\beta^m(z))$ by F_β^m. Otherwise, set $\mu_\beta(z) = 0$. In this way we get a F_β-invariant complex structure μ_β on the whole Riemann sphere satisfying $\|\mu_\beta\|_\infty \le k < 1$. Let ω_β be the quasi-conformal homeomorphism of the Riemann sphere solving the Beltrami equation with coefficient μ_β fixing 0, 1 and ∞. Then $T_\beta(z) = \omega_\beta \circ F_\beta \circ \omega_\beta^{-1}(z)$ is an entire function which has a Siegel disk of rotation number θ. By construction, the boundary of the Siegel disk is a quasi-circle passing through the critical point 1.

LEMMA 4.7. $T_\beta \in \Sigma_\lambda$.

Proof. We first claim that F_β has exactly two zeros which in turn implies that T_β has exactly two zeros. From the construction, the origin is fixed and is the only zero in the complement of U_β, $\bar{D}_\beta$. In U_β, f_β has exactly one zero and since $\widehat{\sigma}_\beta(0) = 0$, this is a zero of F_β. Since $\widehat{\sigma}_\beta$ is a homeomorphism, this proves the claim.

The homeomorphism ω_β preserves the critical structure of F_β so that T_β has exactly two critical points, $\omega_\beta(1)$ and $\omega_\beta(c_\beta)$, whose orders correspond to those of 1 and c_β and these points coincide precisely when $c_\beta = 1$. Because ω_β fixes 1, it is a critical point of T_β.

We claim that the origin is an asymptotic value for T_β. Let $\eta(t)$ be an asymptotic path for f_β so that $\lim_{t\to 1} \eta(t) = \infty$ and $\lim_{t\to 1} f_\beta(\eta(t)) = 0$. We may assume without loss of generality that $f_\beta(\eta(t))$ is not in V_β so that $\widehat{\sigma}_\beta \circ f_\beta(\eta(t))$ is not in U_β. It follows that $\lim_{t\to 1} F_\beta(\eta(t)) = 0$ and that $\lim_{t\to 1} T_\beta(\eta(t)) = 0$ proving the claim.

Since ω_β is a quasi-conformal homeomorphism of the Riemann sphere, both it and its inverse are Hölder continuous at infinity. Therefore, because f_β is an entire function of finite order, so is T_β.

By construction T_β has a Siegel disk of rotation number θ centered at the origin and $T'_\beta(1) = 0$. It must therefore be that $T_\beta \in \Sigma_\lambda$. □

Recall that we denote the map in Σ_λ corresponding to β by g_β. We have therefore shown that $T_\beta = g_{\beta'}$ for some $\beta' \in \Sigma$. We thus define the surgery map

$$\mathbf{S} : \overline{\Omega_{\text{int}}} \to \Sigma_\lambda$$

as follows: to each $\beta \in \Omega_{\text{int}} \cup \gamma$ set Q5

$$\mathbf{S}(\beta) = T_\beta = g_{\beta'}$$

and for the two punctures $\{0, -1\}$ of Ω_{int} set

$$\mathbf{S}(0) = 0 \quad \text{and} \quad \mathbf{S}(-1) = -1.$$

In the next section, we will prove that $\mathbf{S}$ is continuous on $\overline{\Omega_{\text{int}}}$. To simplify notation, we will identify the map $\mathbf{S}(\beta)$ with the corresponding parameter $\beta' \in \Sigma_\lambda$.

5. *The continuity of the surgery map* S

The proof of the continuity of the surgery map is based on a similar proof in [**20,** §12].First though, we need a lemma about quasi-conformal conjugacy classes in Σ_λ. The proof holds just as well for any Σ_t, $|t| < 1$. Q6

LEMMA 5.1. *The quasi-conformal conjugacy class Q of every g_β in Σ_λ is an open set or a point. In particular, for $\beta \in \gamma$, the quasi-conformal conjugacy class of $g_{\mathbf{S}(\beta)}$ is a point.*

Proof. Assume first that the critical points of g_β are distinct and that $g_{\beta'} \neq g_\beta$ belongs to Q. Then there is a quasi-conformal homeomorphism of the complex plane ϕ satisfying $\phi^{-1} \circ g_\beta \circ \phi = g_{\beta'}$. Let μ_ϕ be the Beltrami differential of ϕ corresponding to the complex structure on $\mathbb{C}$ invariant with respect to g_β. Then, using the 'Bers μ-trick' (see for example [**6**] or [**20,** Theorem 7.1]), the structures corresponding to $t\mu$ for $|t| < 1/\|\mu\|_\infty$ are all invariant with respect to g_β. If we denote the solutions to the Beltrami equations for $t\mu$ by ϕ_t, then the maps $\phi_t^{-1} \circ g_\beta \circ \phi_t$ are all holomorphic. Arguing as in the proof of Lemma 4.7 we deduce they belong to Σ_λ. Let $g_{\beta(t)} = \phi_t^{-1} \circ g_\beta \circ \phi_t$. Since $t \mapsto t\mu$ is holomorphic, the same is true for $t \mapsto \phi_t$ by the analytic dependence on parameters of solutions to Beltrami equations [**3**]. Thus $t \to g_{\beta(t)}$ is holomorphic. This then implies that $t \mapsto c_{\beta(t)}$ and hence $t \mapsto \beta(t)$ is holomorphic for $|t| < 1/\|\mu\|_\infty$. It follows that the quasi-conformal class of $g_{\mathbf{S}(\beta)}$ is either an open set or a single point.

If $\beta \in \gamma$, the boundary of the Siegel disk of $g_{\mathbf{S}(\beta)}$ contains two critical points but in any neighborhood of β there are points β' for which the boundary of the Siegel disk of $g_{\mathbf{S}(\beta')}$ contains only one critical point so that $g_{\mathbf{S}(\beta)}$ and $g_{\mathbf{S}(\beta')}$ are not even topologically conjugate. The quasi-conformal class Q is therefore a single point. $\square$

Remark 5.1. The conjugacy classes depend on the orbit structure of the critical point which does not lie on the boundary of the Siegel disk.

THEOREM 5.1. *The surgery map* $\mathbf{S} : \overline{\Omega_{\text{int}}} \to \Sigma_\lambda$ *defined in the last section is continuous.*

Proof. We show first that if $\beta_\infty \in \overline{\Omega_{\text{int}}} - \{0, -1\}$, $\mathbf{S}$ is continuous at β_∞. It suffices to show that $\mathbf{S}(\beta_n) \to \mathbf{S}(\beta_\infty)$ if $\beta_n \to \beta_\infty$.

By construction, it follows that F_β depends continuously on β and therefore that $F_{\beta_n} \to F_{\beta_\infty}$ uniformly on compact subsets of the complex plane. Using the same notation as in the the previous section, we have

$$\mathbf{S}(\beta_n) = \omega_{\beta_n} \circ F_n \circ \omega_{\beta_n}^{-1} \quad \text{and} \quad \mathbf{S}(\beta_\infty) = \omega_{\beta_\infty} \circ F_{\beta_\infty} \circ \omega_{\beta_\infty}^{-1}.$$

By Lemma 4.6, for all n, $\|\mu_{\beta_n}\|_\infty \le k < 1$ so that, passing to a convergent subsequence, we can find a quasi-conformal map ω_∞ such that $\omega_{\beta_n} \to \omega_\infty$ and

$$\mathbf{S}(\beta_n) \to G = \omega_\infty \circ F_{\beta_\infty} \circ \omega_\infty^{-1}.$$

As before, $G \in \Sigma_\lambda$ and by definition, G is quasi-conformally conjugate to $\mathbf{S}(\beta_\infty)$. If the quasi-conformal class of $S(\beta_\infty)$ is a point we are done. If it is not, we have to prove $S(\beta_\infty) = G$.

Now assume $S(\beta_\infty)$ is not quasi-conformally rigid. Let N be a neighborhood of G in Σ_λ containing $\mathbf{S}(\beta_n)$ for large n. By Lemma 5.1 it follows that $\mathbf{S}(\beta_n)$ is quasi-conformally conjugate to G and hence to $\mathbf{S}(\beta_\infty)$. It also follows that F_{β_n} and F_{β_∞} are quasi-conformally conjugate for large n.

The theorem will follow if we can prove that $\omega_{\beta_\infty} = \omega_\infty$ so that $\mathbf{S}(\beta_\infty) = G$. This will follow by standard quasi-conformal theory (see for example [**12**]) if we can show that $\mu_{\beta_n} \to \mu_{\beta_\infty}$ with respect to the spherical measure.

Let area(E) denote the Lebesgue area in the spherical metric of a measurable set E in the sphere. For $\epsilon > 0$, define

$$Q_n^\epsilon = \{z \in \mathbb{C} \mid |\mu_{\beta_n}(z) - \mu_{\beta_\infty}(z)| > \epsilon\}.$$

We claim that

$$Q_n^\epsilon \subset \bigcup_{k\geq 0} F_{\beta_n}^{-k}(D_{\beta_n}) \cup \bigcup_{k\geq 0} F_{\beta_\infty}^{-k}(D_{\beta_\infty}). \tag{19}$$

To see this, note that if $z \notin \bigcup_{k\geq 0} F_{\beta_n}^{-k}(D_{\beta_n}) \cup \bigcup_{k\geq 0} F_{\beta_\infty}^{-k}(D_{\beta_\infty})$, then $\mu_{\beta_n}(z) = \mu_{\beta_\infty}(z) = 0$ and hence $z \notin Q_n^\epsilon$.

To prove the theorem, it is sufficient to prove that for any $\epsilon > 0$ and $\delta > 0$, there is an N large enough such that for all $n \geq N$, one has the following inequality

$$\text{area}(Q_n^\epsilon) \leq \delta.$$

To this end, fix $\epsilon > 0$ and $\delta > 0$. Since D_{β_∞} is F_{β_∞}-invariant, it follows that for M large enough,

$$\text{area}\left(\bigcup_{k>M} F_{\beta_\infty}^{-k}(D_{\beta_\infty}) - \bigcup_{0\leq k\leq M} F_{\beta_\infty}^{-k}(D_{\beta_\infty})\right) \tag{20}$$

can be made as small as desired. From this, the area distortion theorem for quasi-conformal mappings (see for example, [**12,** Theorem 5.2]) and the fact that F_{β_n} and F_{β_∞} are quasi-conformally conjugate by maps with uniformly bounded dilatation, it follows that there is an L so large that

$$\text{area}\left(\bigcup_{k>L} F_{\beta_\infty}^{-k}(D_{\beta_\infty}) - \bigcup_{0\leq k\leq L} F_{\beta_\infty}^{-k}(D_{\beta_\infty})\right) < \delta/5 \tag{21}$$

and

$$\text{area}\left(\bigcup_{k>L} F_{\beta_n}^{-k}(D_{\beta_n}) - \bigcup_{0\leq k\leq L} F_{\beta_n}^{-k}(D_{\beta_n})\right) \leq \delta/5. \tag{22}$$

In fact, for such an L, since $D_{\beta_n} \to D_{\beta_\infty}$ in the Hausdorff topology by Proposition 2.1, and since $F_{\beta_n} \to F_{\beta_\infty}$, there is an open topological disk B and an $N_1 > 0$ such that for all $n > N_1$, $\overline{B} \subset D_{\beta_n} \cap D_{\beta_\infty}$,

$$\text{area}\left(\bigcup_{0\leq k\leq L} F_{\beta_\infty}^{-k}(D_{\beta_\infty}) - \bigcup_{0\leq k\leq L} F_{\beta_\infty}^{-k}(B)\right) \leq \delta/5 \tag{23}$$

and

$$\text{area}\left(\bigcup_{0\leq k\leq L} F_{\beta_n}^{-k}(D_{\beta_n}) - \bigcup_{0\leq k\leq L} F_{\beta_\infty}^{-k}(B)\right) \leq \delta/5. \tag{24}$$

Since $\overline{B} \subset D_{\beta_n} \cap D_{\beta_\infty}$, there is an open topological disk D such that $\overline{B} \subset D \subset \overline{D} \subset D_{\beta_n} \cap D_{\beta_\infty}$ for all n large enough. Note that in (18), the map R_{β_n} is defined on D for all large enough n and, moreover, $R_{\beta_n} \to R_{\beta_\infty}$ uniformly on D. This implies that

$$R_{\beta_n} \circ F_{\beta_n}^L \to R_{\beta_\infty} \circ F_{\beta_\infty}^L$$

uniformly on $\bigcup_{0\leq k\leq L} F_{\beta_\infty}^{-k}(B)$. Since the dilatations μ_{β_n} of the maps $R_{\beta_n} \circ F_{\beta_n}^L$ are uniformly bounded, they converge in the $L^1(\bigcup_{0\leq k\leq L} F_{\beta_\infty}^{-k}(B)$ norm to the dilatation μ_{β_∞} Q7
of $R_{\beta_\infty} \circ F_{\beta_\infty}^L$. In particular, there is an N_2 such that for all $n > N_2$, we have

$$\text{area}\left(Q_n^\epsilon \cap \bigcup_{0\leq k\leq L} F_{\beta_\infty}^{-k}(B)\right) < \delta/5. \tag{25}$$

Let $N = \max\{N_1, N_2\}$. From equations (20)–(25), we derive that for all $n > N$,

$$\text{area}(Q_n^\epsilon) \leq \delta.$$

This implies that $\mu_{\beta_n} \to \mu_{\beta_\infty}$ with respect to spherical measure. By Lemma 4.6, there is a uniform bound k on all the $\|\mu_{\beta_n}\|_\infty$. Passing to a convergent subsequence, we conclude $\omega_{\beta_n} \to \omega_{\beta_\infty}$ uniformly on compact sets in the plane. This implies that $\omega_{\beta_\infty} = \omega_\infty$ and thus $\mathbf{S}(\beta_\infty) = G$ which is what was to be proved. Thus $\mathbf{S}$ is continuous at the points in $\overline{\Omega_{\text{int}}} - \{0, -1\}$.

Now let us show that $\mathbf{S}$ is continuous at the punctures 0 and -1. We need only to show that $\lim_{\beta\to 0} \mathbf{S}(\beta) = 0$ and $\lim_{\beta\to -1} \mathbf{S}(\beta) = -1$.

First let us prove that $\lim_{\beta\to 0} \mathbf{S}(\beta) = 0$. Let z_β be the non-zero solution of $f_\beta(z_\beta) = 0$; it is therefore also a solution of $F_\beta(z_\beta) = 0$. As $\beta \to 0$, $z_\beta \to 2$. By Lemma 4.6, $\omega_\beta(z_\beta)$ stays bounded away from zero and infinity. As $\beta \to 0$, $c_\beta \to \infty$. Again by Lemma 4.6, $\omega_\beta(c_\beta) \to \infty$. In other words, as $\beta \to 0$, the zero of $g_{\mathbf{S}(\beta)}$ distinct from the origin stays bounded away from the origin and infinity, and the critical point of $g_{\mathbf{S}(\beta)}$ distinct from 1 approaches infinity. From the formula for c_β, it follows that $\mathbf{S}(\beta) \to 0$ as $\beta \to 0$.

A similar argument proves that $\lim_{\beta\to -1} \mathbf{S}(\beta) = -1$. In fact, as $\beta \to -1$, $z_\beta \to \infty$, and $c_\beta \to \infty$, or, in other words, as $\beta \to -1$, the zero of $g_{\mathbf{S}(\beta)}$ distinct from the origin, and the critical point of $g_{\mathbf{S}(\beta)}$ distinct from 1, both approach infinity. From the formula for z_β, it follows that $\lim_{\beta\to -1} \mathbf{S}(\beta) = -1$. □

6. *The proof of the Main theorem*

Recall that γ is the union of two Jordan arcs, γ_+ and γ_-, which connect $\beta_1 = -1 + i$ and $\beta_2 = -1 - i$, such that when β varies along one of them, the component of $f_\beta^{-1}(f_\beta(D_\beta))$ which is attached to ∂D_β at 1 is bounded, and when β varies along the other one, this component is unbounded.

For $\beta \in \gamma$, denote the the Siegel disk of $\mathbf{S}(\beta)$ by $\Delta_{\mathbf{S}(\beta)}$; it is a quasi-circle passing through both of the critical points 1 and $c_{\mathbf{S}(\beta)}$. Let $h_{\mathbf{S}(\beta)} : \Delta \to \Delta_{\mathbf{S}(\beta)}$ be the holomorphic conjugation map such that $h_{\mathbf{S}(\beta)}(1) = 1$. Define the angle from 1 to $c_{\mathbf{S}(\beta)}$ to be the angle from $h^{-1}_{\mathbf{S}(\beta)}(1)$ to $h^{-1}_{\mathbf{s}(\beta)}(c_{\mathbf{S}(\beta)})$ measured counterclockwise; denote it by $A_{\mathbf{S}(\beta)}$. By the construction of the surgery map $\mathbf{S}$ and Lemma 3.3, it follows that there is exactly one component of $g^{-1}_{\mathbf{S}(\beta)}(\Delta_{\mathbf{S}(\beta)})$ attached to $\partial\Delta_{\mathbf{S}(\beta)}$ at each of the critical points, 1 and $c_{\mathbf{S}(\beta)}$. Denote the component which is attached at 1 by U_β. Since $\mathbf{S}$ is continuous, it follows that $A_{\mathbf{S}(\beta)}$ depends continuously on β. Therefore, as β varies along one of the curves $\gamma_\pm$, $A_{\mathbf{S}(\beta)}$ varies continuously from 0 to 2π and U_β is bounded, and as β varies along the other one, $A_{\mathbf{S}(\beta)}$ varies continuously from 0 to 2π and U_β is unbounded. As a direct consequence, we have the following result.

COROLLARY 6.1. $\mathbf{S}(\gamma_+) \cap \mathbf{S}(\gamma_-) = \{\mathbf{S}(\beta_1), \mathbf{S}(\beta_2)\}$.

LEMMA 6.1. *For β, $\beta' \in \gamma_\pm$, if $A_{\mathbf{S}(\beta)} = A_{\mathbf{S}(\beta')}$, then $\mathbf{S}(\beta) = \mathbf{S}(\beta')$.*

Proof. Since β, β' belong to the same arc $\gamma_\pm$, both U_β and $U_{\beta'}$ are bounded or both are unbounded. This together with the condition $A_{\mathbf{S}(\beta)} = A_{\mathbf{S}(\beta')}$ imply that $g_{\mathbf{S}(\beta)}$ and $g_{\mathbf{S}(\beta')}$ have the same combinatorial information.

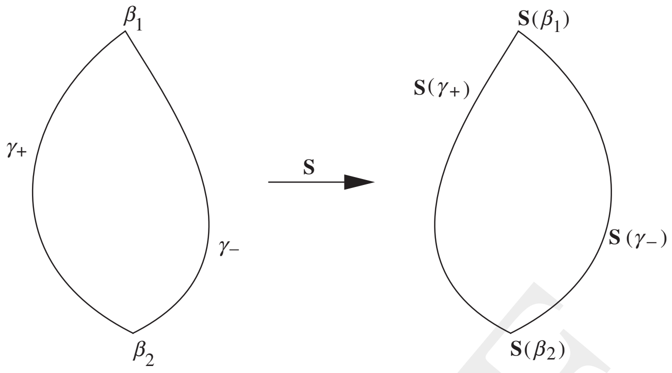

FIGURE 3. The map $\mathbf{S} : \gamma \to \mathbf{S}(\gamma)$.

Since $A_{\mathbf{S}(\beta)} = A_{\mathbf{S}(\beta')}$, there is a univalent map $h : \Delta_{\mathbf{S}(\beta)} \to \Delta_{\mathbf{S}(\beta')}$ such that $h(1) = 1$, $h(c_{\mathbf{S}(\beta)}) = h(c_{\mathbf{S}(\beta')})$ and $g_{\mathbf{S}(\beta)} = h^{-1} \circ g_{\mathbf{S}(\beta')} \circ h$. Q8

Take $\phi_0 : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ to be a quasi-conformal homeomorphism such that $\phi_0|_{\Delta_{\mathbf{S}(\beta)}} = h$ and $\phi_0(\infty) = \infty$. Since $\partial \Delta_{\mathbf{S}(\beta)}$ is a quasi-circle, this is always possible. Now let us use induction to define a sequence $\{\phi_n\}$ of quasi-conformal homeomorphisms of the sphere. We need a scheme to assign addresses to the components of $g_{\mathbf{S}(\beta)}^{-k}(\Delta_{\mathbf{S}(\beta)})$ for each positive integer k. This may be done in essentially the same manner indicated in the proof of Lemma 3.5.

Given this symbolic description of the components we assume that now ϕ_n is defined and define ϕ_{n+1}. First define $\phi_{n+1} = \phi_n$ on $g_{\mathbf{s}(\beta)}^{-n}(\Delta_{\mathbf{S}(\beta)})$. For each component W of $g_{\mathbf{s}(\beta)}^{-n-1}(\Delta_{\mathbf{S}(\beta)})$ which is not a component of $g_{\mathbf{s}(\beta)}^{-n}(\Delta_{\mathbf{S}(\beta)})$, find the corresponding component W' of $g_{\mathbf{s}(\beta')}^{-n-1}(\Delta_{\mathbf{S}(\beta')})$ that has the same address as U. Define $\phi_{n+1} : W \to W'$ by $\phi_{n+1}(z) = g_{\mathbf{S}(\beta')}^{-1} \circ \phi_n \circ g_{\mathbf{S}(\beta)}(z)$. Now let Y be a component of $\mathbb{C} - g_{\mathbf{s}(\beta)}^{-n-1}(\Delta_{\mathbf{S}(\beta)})$. It is not difficult to see that W is simply connected and unbounded. Let Y' be the corresponding component of $\mathbb{C} - g_{\mathbf{S}(\beta')}^{-n-1}(\Delta_{\mathbf{S}(\beta')})$ and define

$$\phi_{n+1} : Y \to Y' \quad \text{by } \phi_{n+1}(z) = g_{\mathbf{S}(\beta')}^{-1} \circ \phi_n \circ g_{\mathbf{S}(\beta)}(z).$$

This inductive process defines a sequence $\{\phi_n\}$ of quasi-conformal homeomorphisms of the sphere. From the construction, it follows that for each $n \geq 1$, we have:

(1) ϕ_n is holomorphic on $g_{\mathbf{s}(\beta)}^{-n}(\Delta_{\mathbf{S}(\beta)})$;

(2) $\phi_{n+1} = \phi_n$ on $g_{\mathbf{S}(\beta)}^{-n}(\Delta_{\mathbf{S}(\beta)})$;

(3) $g_{\mathbf{S}(\beta)} = \phi_n^{-1} \circ g_{\mathbf{S}(\beta')} \circ \phi_{n+1}$;

(4) $\|\mu_{\phi_{n+1}}\|_\infty = \|\mu_{\phi_n}\|_\infty$; and

(5) ϕ_n fixes 0, 1 and ∞.

From property (4), it follows that there is a constant $k < 1$ such that $\|\mu_{\phi_n}\|_\infty \leq k$ for all n. Passing to convergent subsequences, we get two quasi-conformal homeomorphisms of the sphere, ϕ and ψ, fixing 0, 1 and ∞, such that the supports of μ_ϕ and μ_ψ are contained in the grand orbit of the Siegel disk $\Delta_{g_{\mathbf{S}(\beta)}}$. Moreover, $\phi = \psi$ on this grand orbit. By Remark 2.1, since both β and β' lie on $\gamma_\pm$, both critical points are attracted to the origin and the complement of this grand orbit does not contain any other Fatou components and so is the Julia set. Thus $\phi = \psi$ on a dense set of the complex plane and therefore everywhere. It follows that $g_{\mathbf{S}(\beta)}$ and $g_{\mathbf{S}(\beta')}$ are quasi-conformally conjugate to each other. By the second assertion of Lemma 5.1, we get $g_{\mathbf{S}(\beta)} = g_{\mathbf{S}(\beta')}$. $\square$

LEMMA 6.2. *The sets* $\mathbf{S}(\gamma_+) \subset \Sigma_\lambda$ *and* $\mathbf{S}(\gamma_-) \subset \Sigma_\lambda$ *are simple Jordan arcs.*

Proof. We show $\mathbf{S}(\gamma_+)$ is a simple Jordan arc. The same argument applies to $\mathbf{S}(\gamma_-)$. By Lemma 6.1, we have a map $\chi : [0, 2\pi] \to \mathbf{S}(\gamma_+)$ defined by assigning to each $\alpha \in [0, 2\pi]$ that $\mathbf{S}(\beta) \in \mathbf{S}(\gamma_+)$ such that $A_{\mathbf{S}(\beta)} = \alpha$. Obviously the map χ is injective and surjective. Now let us show that it is continuous. Let $\alpha_n \to \alpha$ be a sequence such that $\chi(\alpha_n) = \mathbf{S}(\beta_n) \to \mathbf{S}(\beta')$ and $\chi(\alpha) = \mathbf{S}(\beta)$. Now $A_{\mathbf{S}(\beta')} = \lim_{n\to\infty} A_{\mathbf{S}(\beta_n)} = \lim_{n\to\infty} \alpha_n = \alpha$ and $A_{\mathbf{S}(\beta)} = \alpha$. Lemma 6.1 implies $\mathbf{S}(\beta') = \mathbf{S}(\beta)$ so that χ is continuous at α. This means that $\chi : [0, 2\pi] \to \mathbf{S}(\gamma_+)$ is a homeomorphism and the curves are simple as claimed. $\square$

LEMMA 6.3. $\mathbf{S}(\gamma)$ *is a simple closed curve in* Σ_λ, *consisting of all maps* f *in* Σ_λ *such that the boundary of the Siegel disk of* f *is a quasi-circle passing through both critical points. Moreover, the topological degree of the map* $\mathbf{S} : \gamma \to \mathbf{S}(\gamma)$ *is either* 1 *or* -1.

Proof. It follows from Corollary 6.1 and 6.2 that $\mathbf{S}(\gamma)$ is a simple closed curve in Σ_λ.

Now suppose $\beta \in \Sigma_\lambda$ is such that $\partial\Delta_{g_\beta}$ is a quasi-circle passing through both 1 and c_β. Then there is some $\beta' \in \gamma$ such that the angle between the critical points of g_β is the same as the angle between the critical points of $\mathbf{S}(\beta')$ and such that the components U_β and $U_{\mathbf{S}(\beta')}$ are either both bounded or are both unbounded. Then, arguing as in the proof of Lemma 6.1 we deduce that $\mathbf{S}(\beta')$ and g_β are quasi-conformally conjugate to each other, and by the second assertion of Lemma 5.1, we get $\mathbf{S}(\beta') = g_\beta$. This implies that $\beta \in \mathbf{S}(\gamma)$.

To see the topological degree is 1 or -1, note that by Lemma 6.2 each $\gamma_\pm$ is simple and on the endpoints

$$\mathbf{S}^{-1}(\mathbf{S}(\beta_1)) = \{\beta_1\} \quad \text{and} \quad \mathbf{S}^{-1}(\mathbf{S}(\beta_2)) = \{\beta_2\}. \qquad \square$$

Let $\Gamma = \mathbf{S}(\gamma)$. Recall that the linear conjugation $z \mapsto z/c_\beta$ induces a map $\sigma : \Sigma_\lambda \to \Sigma_\lambda$: $\beta \to -(\beta+2)/(\beta+1)$.

LEMMA 6.4. Σ_λ *is symmetric about* Γ *under the map* σ.

Proof. Since the map $\sigma : \beta \to -(\beta+1)/(\beta+1)$ is induced by the linear conjugation $z \mapsto z/c_\beta$, it follows that Γ is invariant under the map σ and moreover, $\sigma : \Gamma \to \Gamma$ is a homeomorphism.

Let us denote the bounded component of $\Sigma_\lambda - \Gamma$ by Θ_{int} and the unbounded one by Θ_{ext}. By the first assertion of Lemma 6.3, it follows that $\mathbf{S}(\overline{\Omega_{\text{int}}})$ is contained either in Θ_{int} or in Θ_{ext}. This is because otherwise there would be a point $\beta \in \Omega_{\text{int}}$ such that $\mathbf{S}(\beta) \in \Gamma$; that is, the boundary of the Siegel disk of $\mathbf{S}(\beta)$ would contain both the critical points. But this is impossible from the construction of the surgery map. Since by Lemma 6.3, the topological degree of $\mathbf{S} : \gamma \to \Gamma$ is either 1 or -1, it follows that $\overline{\Omega_{\text{int}}}$ is mapped either onto $\overline{\Theta_{\text{int}}}$ or onto $\overline{\Theta_{\text{ext}}}$. In fact, if this were not true, we could make a homeomorphic change of coordinates and reduce the situation to that of a continuous map from the closed unit disk to itself restricting to the identity on the boundary. If the map were not surjective, we would get a deformation retract of the closed unit disk to its boundary; this is impossible. Note that $\overline{\Omega_{\text{int}}}$ is compact and $\mathbf{S}$ is continuous on $\overline{\Omega_{\text{int}}}$. It follows that $\mathbf{S}(\overline{\Omega_{\text{int}}})$ is bounded. We thus have $\mathbf{S}(\overline{\Omega_{\text{int}}}) = \overline{\Theta_{\text{int}}}$. Since $\mathbf{S}(\{0, 1\}) = \{0, 1\}$, it follows that $\{0, -1\} \subset \Theta_{\text{int}}$.

Because σ maps the set $\{0, -1\}$ to the set $\{-2, \infty\}$, we see that $\sigma(\Theta_{\text{int}}) = \Theta_{\text{ext}}$ and $\sigma(\Theta_{\text{ext}}) = \Theta_{\text{int}}$. □

We now have all the ingredients to prove the main theorem. We recall the statement.

MAIN THEOREM. *Let θ be a bounded type irrational number. Then for any $\beta \in \hat{\mathbb{C}} \setminus \{0, -1, -2, \infty\}$, the boundary of the invariant Siegel disk of the entire map,*

$$f_\beta(z) = e^{2\pi i\theta}\left(z - \frac{\beta + 2}{\beta + 1} z^2\right) e^{\beta z},$$

is a quasi-circle passing through one or both the critical points of $f_\beta(z)$.

Proof. For $\beta \in \Theta_{\text{int}}$, the theorem is implied by the surjectivity of the surgery map $\mathbf{S}: \Omega_{\text{int}} \to \Theta_{\text{int}}$. For $\beta \in \Theta_{\text{ext}}$, by Lemma 6.4, there is a $\beta' \in \Theta_{\text{int}}$ such that g_β and $g_{\beta'}$ are linearly conjugate to each other. □

The following theorem summarizes our results and is the structure theorem for Σ_λ.

THEOREM 6.1. (Structure theorem of Σ_λ) *There is a simple closed curve $\Gamma \subset \Sigma_\lambda$ dividing it into two twice punctured disks such that for $\beta \in \Gamma$ the boundary of the Siegel disk passes through both critical points; for β in the bounded component of $\Sigma_\lambda - \Gamma$, punctured at the points* $\{0, -1\}$*, the boundary of the Siegel disk contains the critical point* 1 *but not the critical point c_β; and for β in the unbounded component of $\Sigma_\lambda - \Gamma$, punctured at the points* $\{-2, \infty\}$*, the boundary of the Siegel disk contains the critical point c_β but not the critical point* 1*. Moreover, Γ is invariant under the map $\beta \to -(\beta + 2)/(\beta + 1)$.*

Acknowledgements. The authors would like to thank the referee of a previous version of this paper. His or her many comments were very helpful in improving it. The second author is supported partially by National Basic Research Program of China (973 Program) 2007CB814800.

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Please check word order and hyphenation in 'the one complex dimensional parameter space'. (NB one-dimensional would be hyphenated.)

Q2 (page 7)
There are frequent paragraph breaks between equations (4) and (10). Please check.

Q3 (page 19)
Closing bracket missing here. Please check.

Q4 (page 20)
Check wording and punctuation in 'That is, given by the following result.'

Q5 (page 21)
The paragraph break after 'as follows:' has been removed. Please check.

Q6 (page 21)
The paragraph break after the first line of section 5 has been removed. Please check.

Q7 (page 23)
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Q8 (page 25)
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