ENUMERATING PALINDROMES IN RANK TWO FREE GROUPS

JANE GILMAN AND LINDA KEEN

ABSTRACT. Conjugacy classes of primitive words in the free group of rank two can indexed by the rationals. A representative for each conjugacy class is given by a recursive enumeration scheme and the representative, usually called a Farey word, is denoted by $W_{p/q}$ where p and q are relatively prime integers. Primitive pairs, that is, pairs of words that generate the free group, correspond to pairs of words $(W_{p/q}, W_{r/s})$ with $|ps - rq| = 1$. Here we give a new enumerative scheme for the conjugacy class representatives of primitive words in the free group of rank two. We denote these words by $E_{p/q}$ and prove that $E_{p/q}$ is a palindrome if pq is even and the product of two palindromes if pq is odd. We prove that the pairs $(E_{p/q}, E_{r/s})$ again generate the group when $|ps - rq| = 1$.

1. INTRODUCTION

It is well known that up to conjugacy primitive generators in a rank two free group can be indexed by the rationals and that pairs of primitives that generate the group can be obtained by the number theory of the Farey sequences. The words are denoted by $W_{p/q}$ where p and q are always taken to be relatively prime integers and a primitive pair will be of the form $(W_{p/q}, W_{r/s})$ obeying the Farey neighbor condition $|ps - rq| = 1$.

It is also well known that up to conjugacy a primitive word can always be written as either a palindrome or a product of palindromes and that certain pairs of palindromes will generate the group.

In this paper we give a new enumerative scheme for conjugacy classes of primitive words, still indexed by the rationals (Theorem 3.1). We denote the words representing each conjugacy class by $E_{p/q}$. In addition to proving that the enumeration scheme gives a unique representative for each conjugacy class containing a primitive, we prove that the words in this scheme are either palindromes or products of palindromes and thus give a new proof of that result. Further we show that pairs of

these words $(E_{p/q}, E_{r/s})$ are primitive pairs that generate the group if $|pq - rs| = 1$ (Theorem 3.2).

Pairs of primitive generators arise in the discreteness algorithm for $PSL(2, \mathbb{R})$ representations of two generator groups [4, 5]. This enumerative scheme will be useful in extending discreteness criteria to $PSL(2, \mathbb{C})$ representations where the hyperbolic geometry of palindromes plays an important role [3].

2. Preliminaries

The main object here is a two generator free group which we denote by $G = \langle A, B \rangle$. A word $W = W(A, B) \in G$ is, of course, an expression of the form $A^{n_1} B^{m_1} A^{n_2} \cdots B^{n_r}$ for some set of $2r$ integers $n_1, ..., n_r, m_1, ..., n_r$. The expression $W(C, D)$ denotes the word W with A and B replaced by C and D. The expressions $W(B, A)$, $W(A^{-1}, B), W(A^{-1}, B^{-1})$ and $W(A, B^{-1})$ have the obvious meaning.

We write $\bar{X}$ as well as X^{-1} for the inverse of a word X.

Definition 1. *A word $W = W(A, B) \in G$ is* primitive *if there is another word $V = V(A, B) \in G$ such that W and V generate G. V is called a* primitive associate *of W and the unordered pair W and V is called a* pair of primitive associates *or a* primitive pair *for short.*

Definition 2. *A word $W = W(A, B) \in G$ is a* palindrome *if it reads the same forward and backwards.*

In [1] we found connections between a number of different forms of primitive words and pairs of primitive words in a two generator free group. We recall some results.

Theorem 2.1. *([1, 7]) $W = W(A, B)$ in $G = \langle A, B \rangle$ is primitive if and only if, up to cyclic reduction and inverse, it has the form*

$$(1) \qquad A^\epsilon B^{n_1} A^\epsilon B^{n_2} \ldots A^\epsilon B^{n_p}$$

or the form

$$(2) \qquad B^\epsilon A^{n_1} B^\epsilon A^{n_2} \ldots B^\epsilon A^{n_p}$$

where $\epsilon = \pm 1$ and $n_j \geq 1, j < p, |n_j - n_{j+1}| \leq 1$ where j is taken $\mod p$, and, if $\sum_{i=1}^{p} n_i = q$ then p and q are relatively prime, $(p, q) = 1$.

We denote this primitive $W(A, B)$ by $W_{p/q}$. Two primitive words $W_{p/q}$ and $W_{r/s}$ are associates if and only if neighbor property $|ps - qr| = 1$ holds.

The exponents in (1) and (2) are termed the *primitive exponents* of the word.

In what follow when we use r/s to denote a rational, we assume that r and s are integers, $s \neq 0$ and $(r,s) = 1$.

To state the next theorem, we need the concept of Farey addition for fractions.

Definition 3. *If $\frac{p}{q}, \frac{r}{s} \in \mathbb{Q}$, the* Farey sum *is*

$$\frac{p}{q} \oplus \frac{r}{s} = \frac{p+r}{q+s}$$

Two fractions are called Farey neighbors *if $|ps - qr = 1|$.*

If $\frac{p}{q} < \frac{r}{s}$ then it is a simple computation to see that

$$\frac{p}{q} < \frac{p}{q} \oplus \frac{r}{s} < \frac{r}{s}$$

and that both pairs of fractions

$$(\frac{p}{q}, \frac{p}{q} \oplus \frac{r}{s}) \text{ and } (\frac{p}{q} \oplus \frac{r}{s}, \frac{r}{s})$$

are Farey neighbors if $(p/q, r/s)$ are.

One creates the Farey diagram in the upper half-plane by marking each fraction by a point on the real line and joining each pair of Farey neighbors by semi-circle orthogonal to the real line. The point here is that because of the properties above none of the semi-circles intersect in the upper half plane. This gives a tessellation of the hyperbolic plane where the semi-circles (which are hyperbolic geodesics) joining a pair of neighbors and its Farey sum form a hyperbolic triangle.

Fix any point ζ on the positive imaginary axis. Given a fraction, $\frac{p}{q}$, there is a hyperbolic geodesic γ from ζ to $\frac{p}{q}$ that intersects a minimal number of these triangles.

Definition 4. *We determine a* Farey sequence *for $\frac{p}{q}$ by choosing the new endpoint of each new common edge in the sequence of triangles traversed by γ.*

Definition 5. *The* Farey level *or the* level *of p/q, $Lev(p/q)$ is the number of triangles traversed by γ*

Given p/q, we can find the smallest and largest rationals m/n and r/s that are neighbors of p/q. These also have the property that they are the only neighbors with lower Farey level. That is, $m/n < p/q < r/s$ and $Lev(p/q) < Lev(m/n)$, $Lev(p/q) < Lev(r/s)$, and if u/v is any other neighbor $Lev(p/q) > Lev(u/v)$.

Definition 6. *We call the smallest and the largest neighbors of the rational p/q the* distinguished neighbors *of p/q.*

Note that we can tell whether a distinguished neighbor c/d is smaller (respectively larger) than p/q by whether $cq - pd = 1$ (respectively $cq - pd = -1$). We also refer to the distinguished neighbors of p/q as being the unique *parents of p/q*.

Farey sequences are related to continued fraction expansions of fractions (see for example, [6]). In particular, write

$$\frac{p}{q} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots + \frac{1}{a_k}}}} = [a_0, \ldots, a_k]$$

where $a_j > 0$, $j = 1 \ldots k$ and set $\frac{p_n}{q_n} = [a_0, \ldots, a_n]$.

The approximating fractions can be computed recursively from the continued fraction for p/q as follows:

$$p_0 = a_0, q_0 = 1 \text{ and } p_1 = a_0 a_1 + 1, q_1 = a_1$$

$$p_j = a_j p_{j-1} + p_{j-2} , q_j = a_j q_{j-1} + q_{j-2} \; j = 2, \ldots, k.$$

The Farey level can be expressed in terms of the continued fraction $p/q = [a_0, \ldots a_k]$ by the formula

$$Lev(p/q) = \sum_{j=0}^{k} a_j.$$

The distinguished neighbors of p/q have continued fractions

$$[a_0, \ldots, a_{k-1}] \text{ and } [a_0, \ldots, a_{k-1}, a_k - 1].$$

The Farey sequence contains the approximating fractions as a subsequence. The points of the Farey sequence between $\frac{p_j}{q_j}$ and $\frac{p_{j+1}}{q_{j+1}}$ have continued fraction expansions

$$[a_0, a_1, \ldots a_j + 1], [a_0, a_1, \ldots, a_j + 2], \ldots, [a_0, a_1, \ldots a_j + a_{j+1} - 1].$$

The next theorem shows how distinguished neighbors of words are related to the pairs of primitive associates in theorem 2.1.

Theorem 2.2. *([1, 7, 10]) The primitive words $W_{p/q}$ in $G = \langle A, B \rangle$ can be enumerated inductively using Farey sequences of fractions as follows: set*

$$W_{0/1} = A, \quad W_{1/0} = B$$

Given $\frac{p}{q}$, let $\frac{m}{n}$ and $\frac{r}{s}$ be its distinguished neighbors labeled so that

$$\frac{m}{n} < \frac{p}{q} < \frac{r}{s}.$$

Then

$$W_{\frac{p}{q}} = W_{\frac{m}{n} \oplus \frac{r}{s}} = W_{r/s} \cdot W_{m/n}.$$

A pair $W_{p/q}, W_{r/s}$ is a pair of primitive associates if and only if $\frac{p}{q}, \frac{r}{s}$ are neighbors.

We note that the product of the words $W_{m/n} \cdot W_{r/s}$ is conjugate to the product $W_{r/s} \cdot W_{m/n}$ regardless of which rational is larger, but that in order for the scheme to work we need to be consistent. In this iteration scheme we always choose the product where $ms - rn = 1$. We emphasize that $W_{p/q}$ always denotes the word obtained using this enumeration scheme.

We also note that the $W_{p/q}$ words can be expanded using their continued fraction exponents instead of their primitive exponents. Stated most simply, for generators A and B, if we have the primitive pair of words (X, Y), then either

(3) $$(X, Y) \mapsto (XY^{a_k}, Y) \text{ or } (X, Y) \mapsto (Y, XY^{a_k})$$

The following exhibits the primitive words with the continued fraction expansion exponents in its most precise form.

Theorem 2.3. *[1] If $[a_0, \ldots, a_k]$ is the continued fraction expansion of p/q, the primitive word $W_{p/q}$ can be written in a form that exhibits the continued fraction. As above let the continued fraction approximants be $p_j/q_j = [a_0, \ldots, a_j]$. They are alternately larger and smaller than p/q.*
Set

$$W_{0/1} = A, \ W_{1/0} = B \text{ and } W_{1/1} = BA.$$

If $p_{k-2}/q_{k-2} > p/q$ set

$$W_{p/q} = W_{p_{k-2}/q_{k-2}}(W_{p_{k-1}/q_{k-1}})^{a_k}$$

and set

$$W_{p/q} = (W_{p_{k-1}/q_{k-1}})^{a_k} W_{p_{k-2}/q_{k-2}}$$

otherwise.

In [9] Piggott characterizes conjugacy classes of primitive elements as those which contain cyclically reduced words of odd length and pairs of palindromes that generate the group. He also shows that each cyclically reduced primitive element is either a palindrome or a product of palindromes. We obtain new proofs of these results using our $E_{p/q}$ enumeration scheme. Note that our condition on the parity of pq translates into a condition on the parity of $p + q$.

In the case when pq is odd, the two palindromes in the product can be chosen in various ways. We will make a particular choice in our enumeration scheme.

3. Enumerating primitives: palindromes and products

We can now establish the E enumeration scheme: set

$$E_{0/1} = P_{0/1} = A \text{ and } E_{1/0} = P_{1/0} = B$$

These are trivially palindromes. At the next level we have $Q_{1/1} = BA$. To give the induction scheme: we assume we have $m/n < p/q < r/s$, $p/q = (m+r)/(n+s)$ and $m/n, r/s$ are the distinguished neighbors of p/q.

Case 1 pq - odd: Here the order of multiplication is

$$E_{p/q} = E_{r/s} E_{m/n}.$$

Case 2 pq - even: Here the order is reversed

$$E_{p/q} = E_{m/n} E_{r/s}.$$

Theorem 3.1. *The primitive elements of a two generator free group can be enumerated recursively as $E_{p/q}$. Each $E_{p/q}$ is a cyclic permutation of $W_{p/q}$. The enumeration scheme uses Farey sequences so that when pq is even, $E_{p/q}$ is a palindrome, denoted by $P_{p/q}$, and when pq is odd, $E_{p/q}$ is a product of palindromes denoted by $Q_{p/q}$.*

- *For pq even, there is a unique palindrome conjugate to $W_{p/q}$. We denote it by $P_{p/q}$*
- *For pq odd, set $Q_{p/q} = P_{m/n} P_{r/s}$ where m/n and r/s are the distinguished neighbors of p/q with m/n the smaller one.*

Remark 3.1. *Note that although there are several ways the conjugacy class of $W_{p/q}$ can be factored as products of palindromes, in this theorem we specifically choose the one that makes the enumeration scheme work.*

Not only are the words in this enumeration scheme primitive, we have

Theorem 3.2. *Let $\{W_{p/q}\}$ or $\{E_{p/q}\}$ denote the words in the two enumeration schemes. Then if $(p/q, p'/q')$ are neighbors the pair $W_{p/q}, W_{p'/q'}$ and the pair $E_{p/q}, E_{p'/q'}$ are both pairs of primitive associates.*

Before we give the proofs we note that Theorem 2.2 gives one way to choose a specific representative for each conjugacy class of primitive words. Theorem 3.1 gives another way. Our notation distinguishes between these representatives: We write $W_{p/q}$ for the primitive form from theorem 2.1, and $E_{p/q}$ (e.g. $P_{p/q}$ or $Q_{p/q}$ as pq is even or odd) for the primitive form from theorem 3.1.

In this new enumeration scheme, we will see that words labeled with fractions that are Farey neighbors again give rise to primitive pairs.

Note that cyclic permutations are obtained by conjugation but we cannot necessarily simultaneously conjugate both elements of a primitive pair in the form of theorem 2.1 to get to the corresponding primitive pair in theorem 3.1.

Proof. (**proof of Theorem 3.1**) It is a simple observation that in every Farey triangle with vertices $m/n, p/q, r/s$ one of the vertices is *odd* and the other two are *even*. To see this simply use the fact that $mq - np, ps - rq, ms - nr$ are all congruent to 1 modulo 2. (This also gives the equivalence of parity cases for pq and the $p + q$ used by other authors.) In a triangle where pq is even it may be the smaller distinguished neighbor is even and the larger odd or vice-versa and we take this into account in discussing the enumeration scheme. We note that in general if X and Y are palindromes, then so is $(XY)^t X$ for any positive integer t. The proof proceeds by induction on the level.

The idea behind the proof is that each rational has a pair of parents and each parent in turn has two parents so there are at most four *grandparents* to consider. The parents and grandparents may not all be distinct. The cases considered below correspond to the possible ordering of the grandparents as rational numbers and also the possible orders of their levels.

In case 1) by induction we will get the product of distinguished neighbor palindromes.

In case 2) we will need to show that we get palindromes.

The set up shows that we have palindromes for level 0, ($\{0/1, 1/0\}$) and the correct product for level 1, $\{1/1\}$.

Assume the scheme works for all rationals with level less than N and assume $Lev(p/q) = N$. Since $m/n, r/s$ are distinguished neighbors of p/q both their levels are less than N.

Let $p/q = [a_0, a_1, \ldots, a_k]$, $\sum_0^k a_j = N$. and assume we are in case 2 where pq is even.

If

$$Lev(r/s) > Lev(m/n),$$

we have

(4) $\quad m/n = [a_0, a_1, \ldots, a_{k-1}]$ and $r/s = [a_0, a_1, \ldots, a_{k-1}, a_k - 1]$.

Then m/n is the smaller distinguished neighbor of r/s and the larger distinguished neighbor is

(5) $$w/z = [a_0, a_1, \ldots, a_{k-1} - 2].$$

The smaller distinguished neighbor of m/n is

(6) $$u/v = [a_0, \ldots, a_{k-2}, a_{k-1} - 1]$$

and the larger distinguished neighbor is

(7) $$x/y = [a_0, \ldots, a_{k-2}].$$

If rs is odd we have, by the induction hypothesis

$$E_{r/s} = Q_{r/s} P_{w/z} P_{m/n}$$

and

$$E_{p/q} = P_{p/q} = P_{m/n} Q_{r/s} = P_{m/n}(P_{w/z} P_{m/n})$$

which is a palindrome.

If mn is odd we have by the induction hypothesis

$$E_{m/n} = Q_{m/n} = P_{x/y} P_{u/v}$$

and by equations (4), (5), (6))and (7)

$$P_{r/s} = Q_{m/n}^{(a_k - 1)} P_{x/y} = (P_{x/y} P_{u/v})^{(a_k - 1)} P_{x/y}$$

so that

$$P_{p/q} = Q_{m/n} P_{r/s} = (P_{x/y} P_{u/v})^{a_k} P_{x/y}$$

is a palindrome.

If

$$Lev(r/s) < Lev(m/n),$$

we have

(8) $m/n = [a_0, a_1, \ldots, a_{k-1}, a_k - 1]$ and $r/s = [a_0, a_1, \ldots, a_{k-1}]$.

Then r/s is the larger distinguished neighbor of m/n and the smaller distinguished neighbor is

(9) $$w/z = [a_0, a_1, \ldots, a_{k-1} - 2].$$

The larger distinguished neighbor of r/s is

(10) $$x/y = [a_0, \ldots, a_{k-2}, a_{k-1} - 1]$$

and the smaller distinguished neighbor is

(11) $$u/v = [a_0, \ldots, a_{k-2}].$$

If mn is odd we have, by the induction hypothesis

$$E_{m/n} = Q_{m/n} = P_{r/s} P_{w/z}$$

and

$$E_{p/q} = P_{p/q} = Q_{m/n} P_{r/s} = P_{r/s}(P_{w/z} P_{r/s})$$

which is a palindrome.

If rs is odd we have, by the induction hypothesis

$$E_{r/s} = Q_{r/s} = P_{x/y}P_{u/v}$$

and by equations (8), (9), (10))and (11)

$$P_{m/n} = P_{u/v}Q_{r/s}^{(a_k-1)} = P_{u/v}(P_{x/y}P_{u/v})^{(a_k-1)}$$

so that

$$P_{p/q} = P_{m/n}Q_{r/s} = P_{u/v}(P_{x/y}P_{u/v})^{a_k}$$

is a palindrome.

We have yet to establish that the $E_{p/q}$ words are primitive but this follows from the proof of Theorem 3.2. $\qquad\square$

3.1. **Proof of Theorem3.2.**

Proof. The proof is by induction on the maximum level of p/q and p'/q' and is the same for either set of words. We write the proof for the E form since it is a well known fact for the W form of the words even though the proof given here is new.

At level 1, the theorem is clearly true: (A, B), (A, BA) and (BA, B) are all primitive pairs.

Assume now that the theorem holds for any pair both of whose levels are less than N.

(1) Let $(p/q, p'/q')$ be a pair of neighbors with $Lev(p/q) = N$.

(2) Let $m/n, r/s$ be the distinguished neighbors of p/q and assume $m/n < p/q < r/s$.

   (a) Then $m/n, r/s$ are neighbors and both are have level less than N so that by the induction hypothesis $(E_{m/n}, E_{r/s})$ is a pair of primitive associates.

   (b) It follows that all of the pairs

   $$(E_{m/n}, E_{m/n}E_{r/s}),\ (E_{m/n}, E_{r/s}E_{m/n}),$$

   $$(E_{m/n}E_{r/s}, E_{r/s})\ \text{and}\ (E_{r/s}E_{m/n}, E_{r/s})$$

   are pairs of primitive associates since we can retrieve the original pair $(E_{m/n}, E_{r/s})$ from any of them.

   (c) Since $E_{p/q} = E_{r/s}E_{m/n}$ or $E_{p/q} = E_{m/n}E_{r/s}$ we have proved the theorem if p'/q' is one of the distinguished neighbors.

(3) If p'/q' is not one of the distinguished neighbors, then either $p'/q' = (tp+m)/(tq+n)$ for some $t > 0$ or $p'/q' = (tp+r)/(tq+s)$ for some $t > 0$.

(a) Assume for definiteness $p'/q' = (tp + m)/(tp + n)$; the argument is the same in the other case.

(b) Note that the pairs $p/q, (jp + m)/(jp + n)$ are neighbors for all $j = 1, \ldots t$.

(c) We have already shown $E_{m/n}, E_{p/q}$ is a pair of primitive associates. The argument above applied to this pair shows that $E_{m/n}, E_{(p+m)/(q+m)}$ is also a pair of primitive associates. Applying the argument t times proves the theorem for the pair $E_{p'/q'}, E_{p/q}$.

$\square$

An immediate corollary is

Corollary 3.1. *The scheme of Theorem 3.1 also gives a scheme for enumerating primitive palindromes alone and a scheme for enumerating primitives that are palindromic products alone.*

We thank Vidur Malik whose thesis [8] suggested that we look at palindromes.

References

1. Gilman, Jane and Keen, Linda, Word sequences and intersection numbers. Complex manifolds and hyperbolic geometry (Guanajuato, 2001), 231–249, Contemp. Math., 311, Amer. Math. Soc., Providence, RI, 2002
2. Gilman, Jane and Keen, Linda, *Palindromes and Generators of Free Groups of Rank Two*, manuscript to be submitted to the proceeding of the conference on Teichmuller theorey held in honor of W.H. Harvey in Anogia, Greece.
3. Gilman, Jane and Keen, Linda, *Discreteness and Palindromes in Kleinian groups*, in preparation.
4. Gilman, Jane, *Informative Words and Discreteness*, Contemp. Math **421**, (2007) 147-155. Cont Math.
5. Gilman, Jane, *Algorithms, Complexity and Discreteness Criteria in $PSL(2, \mathbb{C})$*, Journal D'Analyse Mathematique, Vol 73, (1997), 91-114.
6. Hardy, G.H. and Wright, E. M., *An introduction to the theory of numbers* Oxford, Clarendon Press 1938.
7. Keen, Linda and Series, Caroline, *Pleating Coordinates for the Maskit Embedding of Teichmüller space for a punctured torus*, Topology, Vol. **32** #4, (1993),719-749.
8. Malik, Vidur, *Curves Generated on Surfaces by the Gilman-Maskit Algorithm*, Ph D thesis, Rutgers University, Newark, NJ (2007).
9. Piggott, Adam, *Palindromic primitives and palindromic bases in the free group of rank two* J. Algebra 304 (2006), no. 1, 359–366.
10. Series, Caroline *The Geometry of Markoff Numbers*, Math. Intelligencer **7** #3,(1985), 20-29.
11. Wright, David J., *Searching for the cusp*, Spaces of Kleinian Groups, LMS Lecture Notes **329**, Cambridge U. Press (2004), 1-36.

DEPARTMENT OF MATHEMATICS, RUTGERS UNIVERSITY, NEWARK, NJ 07079
E-mail address: gilman@rutgers.edu

DEPARTMENT OF MATHEMATICS, CUNY, LEHMAN COLLEGE AND GRADUATE CENTER, BRONX, 10468
E-mail address: linda.keen@lehman.cuny.edu