Review Sheet

(1) Graphical analysis - orbit analysis - phase portraits

Sample exercise: Use graphical analysis to describe the fate of all orbits of the function $x^2 - 1$. Draw the phase portrait.

Sample exercise: Use graphical analysis to describe the fate of all orbits of the function $x^3 - x$. Draw the phase portrait.

(2) Theorems from calculus: Intermediate value theorem, Mean value theorem. Memorize the statements - see how they are used to prove theorems about the existence of fixed points.

(3) Fixed and periodic points - know the definitions of a fixed point, periodic point. What it means to be an attracting, repelling and neutral periodic point. What are the conditions on the derivative of a fixed point? How does this condition work for a periodic cycle?

Sample exercise: Are there attracting fixed points for the function x^2-1 and if so what are they? Are there repelling fixed points and if so what are they? Are there attracting periodic cycles and if so what are they?

(4) Bifurcations: know the definitions of saddle node and period doubling bifurcations.

Sample exercise: Draw three graphs that show the saddle-node bifurcation in the family $E_\lambda(x) = e^x + \lambda$.

Sample exercise: Draw three graphs that show the period doubling bifurcation in the family $Q_c = x^2 + c$.

Sample exercise: In the logistic family $F_\lambda = \lambda x(1 - x)$ - for what values of λ is there an attracting fixed point at $x = 0$? Describe the bifurcation when $\lambda = 3$

(5) The Quadratic family: Draw the graphs of $Q_{-2} = x^2 - 2$ and Q^2_{-2} on the same set of axes. Prove the theorem

Theorem. 1. *The function Q_{-2} has at least 2^n periodic points of period n*

(6) Know the definition of the Cantor middle thirds set and the ternary expansions of the points in the set.

Sample exercise: Find the ternary expression for 1/4

Sample exercise: Consider the tent map $T(x) = 3x$ if $x \leq 1/2$ and $T(x) = 3x-3$ if $x \geq 1/2$. Suppose x has ternary expansion $x = .a_1a_2a_3\ldots$. What is the expansion of $T(x)$ - there are two cases.

(7) Study the diagrams on pages 84 - 85. For each point c on the horizontal axis, the show periodic cycle that the critical point is attracted to is plotted above it. Note how the cycles bifurcate - first there is period doubling, then it seems to be chaotic - later you see period three cycles and they undergo period doubling.

The graphs on page 90 - 91 show the period doubling phenomena for Q_c for various values of c.

(8) The sequence space Σ with the shift map σ describes a dynamical system. It a model for various dynamical systems.

It is a metric space. Know the definition of metric space.

The shift is a continuous function from Σ to itself. Know the definition of a continuous function and how you apply it to σ.

Know the definition of a conjugacy between two metric spaces.

Know the nested intervals theorem:

Theorem. 2. *If I_n is a sequence of nested closed intervals in the real line, $I_{n+1} \subset I_n$, and if their lengths go to zero and n goes to infinity, then there is a unique point in their intersection $\cap_1^\infty I_n$.*

Sample exercises:

(a) Find all points in Σ whose distance from $000\ldots$ is exactly $1/3$.
(b) What are the periodic points under the shift? Which points are not periodic?
(c) Find the homeomorphism between the sequence space Σ and the Cantor middle thirds space Λ. Check all the homeomorphism properties

(9) List the properties of a chaotic dynamical system. Know the definition of a dense subset of a given set. Know the statement of the theorem that density is preserved by a continuous map. Know the theorem

Theorem. 3. *If two dynamical systems are conjugate, they are either both chaotic or neither is.*

Sample questions:

(a) Define a point in Σ that has a dense orbit.
(b) Decide whether the sequence

$$T = \{s_0 s_1 s_2 \ldots \mid \text{no two consecutive } s_j = 0\}$$

is dense in Σ.

(c) Prove that the set of endpoints of the removed intervals in the Cantor middle thirds set is a dense subset of the Cantor set.
(d) Graph the doubling function given by $D(x) = 2x$ if $0 \leq x < 1/2$ and $D(x) = 2x - 1$ if $1/2 < x < 1$. Prove that it acts chaotically on $[0, 1)$.