Homework Assignment of Complex Analysis II from Prof. Hu

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The following exercise problems are related to

(i) Bloch's Theorem, Montel-Caratheodory Theorem, Great Picard Theorem and techniques of their proofs;

(ii) the final examination of Complex Analysis I in Fall 2009.

1. (a) Let f be an analytic function on the disk $D(a,r)$ centered at a and of radius r such that $|f'(z) - f'(z)| < |f'(a)|$ for all z in $D(a,r) \setminus \{a\}$. Show that f is one-to-one on $D(a,r)$.

   (b) If f is analytic in a neighborhood of a and $f'(a) \neq 0$, then f is injective on a neighborhood of a. (Hint: By using the first part, one can develop a proof much simpler than the one by using Rouche's Theorem.)

2. (a) Let f be an analytic function defined on a simply connected domain Ω. If $f(z) \neq 0$ for all $z \in \Omega$, then for any positive integer n, there exists an analytic function g defined on Ω such that $f(z) = (g(z))^n$ for all $z \in \Omega$.

   (b) If f is analytic in a neighborhood of a and $f'(a) = 0$, then f is not injective on a neighborhood of a. (Hint: By using the first part, one can develop a proof by using Open Mapping Theorem instead of Rouche's Theorem.)

3. (a) Every entire function f mapping the unit circle into itself has to be of the form $f(z) = az^n$ for some positive integer n and some constant a with $|a| = 1$.

   (b) Every meromorphic function defined on the complex plane C that maps the unit circle into itself has to be of the form $f(z) = az^n$ for non-zero integer n and some constant a with $|a| = 1$.

4. Let f be analytic on the unit disk $D = \{z : |z| < 1\}$ and suppose that $f(0) = 0$, $f'(0) = 1$ and $|f(z)| \leq M$ for all $z \in D$. Show that $M \geq 1$ and $f(D)$ contains the open disk centered at the origin and of radius $\frac{1}{6M}$.

5. Apply Schottky's Theorem to prove Montel-Caratheodory Theorem.

6. Apply Montel-Caratheodory Theorem to prove Great Picard Theorem.

7. Show that every entire one-to-one function f has to be of the form $f(z) = az + b$ for two constants a and b with $a \neq 0$.

8. Show that if f is an entire function that is not a polynomial then f assumes every complex number except one infinitely many times.

9. Show that if f is a meromorphic function on the complex plane C such that $\hat{C} \setminus f(C)$ contains at least three points then f is a constant function.