Math 70300

Homework 7

Due: within 72 hours

1. Let $u$ be harmonic in a region $G$ and suppose that the closed disc $\overline{D(a,R)}$ is contained in $G$. Show that

$$u(a) = \frac{1}{\pi R^2} \int_{\overline{D(a,R)}} u(x, y) \, dx \, dy.$$  

*Hint:* Use polar coordinates.

For every $r$ between 0 and $R$ we have by the mean value property:

$$u(a) = \frac{1}{2\pi} \int_0^{2\pi} u(a + re^{i\theta}) \, d\theta.$$  

We integrate this equality multiplied by $r$, since $dx \, dy = r \, dr \, d\theta$ in polar coordinates. This gives

$$\int_0^R u(a) r \, dr = \frac{1}{2\pi} \int_0^R \int_0^{2\pi} u(a + re^{i\theta}) \, d\theta \, r \, dr.$$  

Using polar coordinates centered at $a$, we cover the whole disc $\overline{D(a,R)}$ when $0 \leq r \leq R$ and $0 \leq \theta \leq 2\pi$. We remark again that $dx \, dy = r \, dr \, d\theta$. So the right-hand side becomes

$$\frac{1}{2\pi} \int_{\overline{D(a,R)}} u(x, y) \, dx \, dy. \quad (1)$$  

The left-hand side is the integral of a linear function

$$\int_0^R u(a) r \, dr = u(a) \int_0^R r \, dr = u(a) \frac{R^2}{2}. \quad (2)$$  

We just combine (1) and (2) to get the result.

2. Prove Hadamard’s three circles theorem: Let $f(z)$ be holomorphic in an open set containing the annulus

$$r_1 \leq |z| \leq r_2, \quad 0 < r_1 < r_2.$$  

Show that with the notation $M(r) = \sup_{|z|=r} |f(z)|$

$$M(r) \leq M(r_1) \frac{\log r_2 - \log r}{\log r_2 - \log r_1} M(r_2) \frac{\log r - \log r_1}{\log r_2 - \log r_1}.$$  

We want to apply the maximum modulus principle to a function of the form $z^\alpha f(z)$. The problem is that for non integer $\alpha$ the function $z^\alpha$ is not holomorphic in a disc or annulus containing 0, as it requires to define first a branch of $\log(z)$ and then define
\[ z^\alpha = e^{\alpha \log(z)} \]. And we know that there is no annulus centered at 0 on which the logarithmic function is defined as a holomorphic function. However, the problem is superficial. The maximum modulus principle can be applied in any small neighborhood off 0, where \( z^\alpha f(z) \) can be defined. Moreover, the modulus of \( z^\alpha \) is independent of the branch of \( \log(z) \) we will use, as \( |z^\alpha| = |z|^\alpha \). To understand why the maximum modulus of \( z^\alpha f(z) \) is achieved on the boundary of the annulus for any branch of \( z^\alpha \) is suffices to assume that it is achieved inside at a point \( z_0 \). Then is a small neighborhood of \( z_0 \) we violate the standard maximum modulus principle. We get

\[ r^\alpha M(r) \leq \max(r_1^\alpha M(r_1), r_2^\alpha M(r_2)). \] (3)

The standard trick is to choose \( \alpha \) so that the two expressions on the right become equal. This gives

\[ r_1^\alpha M(r_1) = r_2^\alpha M(r_2) \iff \alpha \log r_1 + \log M(r_1) = \alpha \log r_2 + \log M(r_2) \]

\[ \iff \alpha = \frac{\log M(r_1) - \log M(r_2)}{\log r_2 - \log r_1}. \]

We take logarithms in (3) and substitute our choice of \( \alpha \) to get

\[ \alpha \log r + \log M(r) \leq \alpha \log r_1 + \log M(r_1) \iff \log M(r) \leq \frac{\log M(r_1) - \log M(r_2)}{\log r_2 - \log r_1} (\log r_1 - \log r) + \log M(r_1) \]

\[ \iff \log M(r) \leq \left( \frac{\log r_1 - \log r}{\log r_2 - \log r_1} + 1 \right) \log M(r_1) + \frac{\log r - \log r_1}{\log r_2 - \log r_1} \log M(r_2). \]

We only now need to add the fractions to get the result in logarithmic form:

\[ \frac{\log r_1 - \log r}{\log r_2 - \log r_1} + 1 = \frac{\log r_1 - \log r + \log r_2 - \log r_1}{\log r_2 - \log r_1} = \frac{\log r_2 - \log r}{\log r_2 - \log r_1}. \]

3. Fix \( R > 0 \). Show that, if \( n \) is large enough, then

\[ P_n(z) = 1 + z + \frac{z^2}{2!} + \cdots + \frac{z^n}{n!} \]

has no zeroes in \( \{ z : |z| \leq R \} \).

We apply Rouché’s theorem to \( f(z) = e^z \) and \( g(z) = P_n(z) - e^z \). We fix \( R \). We consider on \( |z| = R \) the modulus of \( f \) and \( g \).

\[ |f(z)| = |e^z| = e^{\Re(z)} \geq e^{-R}, \quad \text{since} \quad -R \leq \Re(z) \leq R. \]

Moreover, using the Taylor series of \( e^z \), we get

\[ |g(z)| = |P_n(z) - e^z| = \left| \sum_{k=n+1}^{\infty} \frac{z^k}{k!} \right| \leq \sum_{k=n+1}^{\infty} \frac{|z|^k}{k!} = \sum_{k=n+1}^{\infty} \frac{R^k}{k!} \rightarrow 0, \quad n \rightarrow \infty \]

as this is the remainder in the (convergent) Taylor series of \( e^R \). In particular, for \( n \) sufficiently large, \( |g(z)| \leq e^{-R} \). So we can apply Rouché’s theorem to get that for \( |z| \leq R \) the functions \( e^z \) and \( f(z) + g(z) = P_n(z) \) have the same number of zeros. Since \( e^z \) never has zeros, the result follows.
4. Let $|f(z)| \leq 1$ for $|z| < 1$ be a non-constant analytic function. Prove that

(i) If $f(0) > 0$, then
\[ \frac{|f(0) - |z||}{1 - |f(0)z|} \leq |f(z)| \leq \frac{|f(0) + |z||}{1 + |f(0)z|}. \]

*Hint:* Apply Schwarz lemma to an appropriate composition of functions. Where is the circle of radius $r$ mapped by the standard linear fractional transformations of the unit disc (assume they have real coefficients)?

Consider $T : \mathbb{D} \to \mathbb{D}$ given by
\[ T(w) = \frac{w - f(0)}{1 - f(0)w} \]

the linear fractional transformation of $\mathbb{D}$ such that $T(f(0)) = 0$. We consider $T \circ f : \mathbb{D} \to \mathbb{D}$ which has $(T \circ f)(0) = 0$. We apply the Schwarz lemma to it:
\[ |T(f(z))| \leq |z|, \quad |z| < 1. \]

We have
\[ T^{-1}(z) = \frac{z + f(0)}{1 + f(0)z}. \]

We fix $|z| = r$. For real $f(0)$ we determine the image of the circle $|z| = r$ by $T^{-1}$. Since $T^{-1}$ is a linear fractional transformation, the image has to be a circle (inside the unit disc). Since $T^{-1}$ has real coefficients, it maps the real line to the real line. Moreover,
\[ T^{-1}(0) = f(0), \quad T^{-1}(r) = \frac{r + f(0)}{1 + f(0)r}, \quad T^{-1}(-r) = \frac{-r + f(0)}{1 - rf(0)}. \]

Since $T^{-1}(0) = f(0)$ is real and $T^{-1}(\infty) = 1/f(0)$ is real, the images have to be symmetric with respect to the image circle. Points symmetric with respect to a circle lie on the ray from the center. This proves that the center of the image circle is real. (Notice the center of $T^{-1}(\{z, |z| = r\})$ is not $T^{-1}(0)$).

We remark that $1 - rf(0) > 0$. Assume first that $f(0) > r$. The point closest to the origin for the image circle is $T^{-1}(-r)$ and the point further is $T^{-1}(r)$ Check:
\[ T^{-1}(r) = \frac{r + f(0)}{1 + f(0)r} > T^{-1}(-r) = \frac{-r + f(0)}{1 - rf(0)}. \]
\[ \Leftrightarrow r + f(0) - rf(0)^2 > -r + f(0) - r^2f(0) + rf(0)^2 \Leftrightarrow 2r > 2rf(0)^2, \]
i.e. $f(0) < 1$, which is true. Now by $|T(f(z))| \leq |z|$ we see that $f(z)$ belongs to this image circle. So we get the inequalities
\[ \frac{f(0) - r}{1 - rf(0)} \leq |f(z)| \leq \frac{r + f(0)}{1 + f(0)r}. \]
which is exactly what we want to prove.

In the case \( f(0) < r \) we only need to prove the right-hand inequality. In this case \( T^{-1}(-r) < 0 \) and we see that

\[
T^{-1}(r) = \frac{r + f(0)}{1 + f(0)r} > |T^{-1}(-r)| = \frac{r - f(0)}{1 - rf(0)}
\]

\[\Leftrightarrow r + f(0) - r^2f(0) - rf(0)^2 > r - f(0) + r^2f(0) - rf(0)^2 \Leftrightarrow 2f(0) > 2r^2f(0),\]

which holds as \( f(0) > 0 \) and \( r < 1 \). So the point on the circle further away from 0 is \( T^{-1}(r) \). This gives the desired inequality.

(ii) Show that the inequality is true in general, without the assumption \( f(0) > 0 \), by using an appropriate rotation.

Define \( g(z) = f(z)e^{-i \arg f(0)} \). Then \( g \) maps the unit disc to the unit disc and \( g(0) = |f(0)| > 0 \). Moreover, \( |g(z)| = |f(z)| \) and this suffices to prove the result. If \( f(0) = 0 \), the result is just the Schwarz lemma for the right-hand inequality and obvious for the left-hand inequality.

5. (a) Show that \( w = \tan(\pi z/4) \) maps the infinite strip \( -1 < \Re(z) < 1 \) onto the unit disk.

We consider the map

\[ w = g(z) = e^{i\pi z/2}. \]

We explain why \( g \) maps the strip \( |\Re(z)| \leq 1 \) to the right hand-plane \( \Re(w) \geq 0 \). It maps the line \( \Re(z) = 1 \) to the ray \( \{ix, x \geq 0\} \), as, if we set \( z = 1 + iy \), we get

\[ e^{i\pi(1+iy)/2} = e^{i\pi/2}e^{-\pi y/2} = ie^{-\pi y/2}. \]

Similarly, if \( z = -1 + iy \) we have \( e^{i\pi z/2} = -ie^{-\pi y/2} \). So \( g \) maps \( \Re(z) = -1 \) to \( \{ix, x \leq 0\} \). Moreover, \( g(0) = 1 \). The fact that the strip has width 2 and we multiply by \( i\pi/2 \), makes the function \( g \) one-to-one as \( e^z \) has period \( 2\pi i \). Now we need to compose with the standard map from the right-half plane \( \Re(w) \geq 0 \) to the unit disc given by

\[ T(w) = \frac{w - 1}{i(w + 1)} \implies T(e^{i\pi z/2}) = \frac{e^{i\pi z/2} - 1}{i(e^{i\pi z/2} + 1)} = \frac{e^{i\pi z/4} - e^{-i\pi z/4}}{i(e^{i\pi z/4} + e^{-i\pi z/4})} = \tan(\pi z/4). \]

Remark: The \( i \) in the mapping \( T \) just gives the correct rotation (=automorphism of \( \mathbb{D} \)) to get exactly \( \tan(\pi z/4) \). We also see that indeed on the imaginary axis \( ix \) the numerator has the same modulus as the denominator: \( |T(ix)| = |(ix - 1)/(ix + 1)| = 1 \). Moreover, \( T(1) = 0 \). This explains why \( T \) maps the right-half plane to the unit disc.

(b) Let \( f(z) \) be a holomorphic function on \( |z| < 1 \) with \( |\Re f(z)| < 1 \) and \( f(0) = 0 \). Show that

\[
|\Re(f(z))| \leq \frac{4}{\pi} \arctan |z|, \quad |\Im(f(z))| \leq \frac{2}{\pi} \log \frac{1 + |z|}{1 - |z|}.
\]
Hint: Use Exercise 5 in Homework 6.

We use \( \phi(z) = z \) and \( \psi(z) = S^{-1}(z) \), where \( S(z) = T \circ g(z) = \tan(\pi z/4) \). We get that \( f(D(0, r)) \subset S^{-1}(D(0, r)) = g^{-1}(T^{-1}(D(0, r))) \) for \( 0 < r < 1 \). Finding the preimage of the circle \( C(0, r) \) under \( S \) will give bounds for the \( \Re f(z) \) and \( \Im f(z) \). We first determine the image of the circle \( C(0, r) \) under \( T^{-1} \). We solve to get \( T^{-1}(z) = (iz + 1)/(1 - iz) \). It maps circles into circles or lines. Since the only point mapped to infinity is \(-i \notin C(0, r), T^{-1}(C(0, r)) \) is a circle inside \( T^{-1}(\mathbb{D}) = \{ z, \Re(z) > 0 \} \). We get

\[
T^{-1}(ir) = \frac{-r + 1}{1 + r}, \quad T^{-1}(-ir) = \frac{r + 1}{1 - r}, \quad T^{-1}(0) = 1, \quad T^{-1}(\infty) = -1.
\]

As in problem 4, we get that the points 0 and \( \infty \) are mapped by \( T^{-1} \) to points symmetric with respect to \( T^{-1}(C(0, r)) \) and their line contains the center \( C \) of this circle. So the center is on the real axis. It is located at the midpoint of \( T^{-1}(ir) \) and \( T^{-1}(-ir) \) and the radius \( R \) is half the distance between them. We calculate:

\[
C = \frac{1}{2} \left( \frac{(1 - r)^2 + (r + 1)^2}{1 - r^2} \right) = \frac{1 + r^2}{1 - r^2}, \quad R = \frac{1 - (1 - r)^2 + (r + 1)^2}{2} = \frac{2r}{1 - r^2}.
\]

We also have

\[
g^{-1}(w) = \frac{2}{i\pi} \log w, \quad \Re g^{-1}(w) = \frac{2}{\pi} \arg w, \quad \Im g^{-1}(w) = -\frac{2}{\pi} \log |w|.
\]

The point of maximum modulus on the circle \( T^{-1}(C(0, r)) \) is the right-most point \( T^{-1}(-ir) = (r + 1)/(1 - r) \). This gives

\[
|\Im f(z)| \leq \frac{2}{\pi} \log \frac{r + 1}{1 - r}.
\]

which gives the one required inequality with \( |z| = r \). For the other, the situation is a bit more complicated because we need to give bounds for the argument on circle \( T^{-1}(C(0, r)) \). We draw the tangent line from the origin to the circle and use trigonometry. If the maximum argument (=angle) in Figure 5 is \( a \) then

\[
\sin a = \frac{2r/(1 - r^2)}{(1 + r^2)/(1 - r^2)} = \frac{2r}{1 + r^2} \Rightarrow \cos a = \frac{1 - r^2}{1 + r^2} \Rightarrow \tan a = \frac{2r}{1 - r^2},
\]

(solve \( \cos^2 a + \sin^2 a = 1 \), for instance). This gives the inequality

\[
|\Re f(z)| \leq \frac{2}{\pi} \arctan \frac{2r}{1 - r^2} = \frac{2}{\pi} \arctan \frac{2|z|}{1 - |z|^2}.
\]

This is not the inequality in the form given, but the discrepancy can be explained using trigonometric identities. We have

\[
\tan(a) = \frac{2 \tan(a/2)}{1 - \tan^2(a/2)} \Rightarrow \tan(a/2) = r \Rightarrow a = 2 \arctan r = 2 \arctan |z|.
\]
6. Let $f(z)$ be holomorphic in $|z| < R$ with Taylor expansion $f(z) = \sum a_n z^n$ and set

$$I_2(r) = \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta, \quad 0 \leq r < R.$$ 

Show that

(a) $I_2(r) = \sum_{n=0}^{\infty} |a_n|^2 r^{2n}$.

We substitute the Taylor series of $f(z)$ to get

$$I_2(r) = \frac{1}{2\pi} \int_0^{2\pi} \sum_{n=0}^{\infty} a_n r^n e^{in\theta} \sum_{m=0}^{\infty} \bar{a}_m r^m e^{-im\theta} d\theta = \frac{1}{2\pi} \sum_{n,m=0}^{\infty} a_n \bar{a}_m r^{n+m} \int_0^{2\pi} e^{i(n-m)\theta} d\theta,$$

by uniform convergence of the Taylor series inside the radius of convergence. Since

$$\int_0^{2\pi} e^{ij\theta} d\theta = \left\{ \begin{array}{ll} 2\pi, & j = 0, \\ 0, & j \neq 0, \end{array} \right.$$ 

we get

$$I_2(r) = \frac{1}{2\pi} \sum_{n=0}^{\infty} |a_n|^2 r^{2n} 2\pi.$$

(b) $I_2(r)$ is increasing.
This follows from (a), since each term increases for larger \( r \).

(c) \(|f(0)|^2 \leq I_2(r) \leq M(r)^2\), with \( M(r) = \sup_{|z|=r} |f(z)| \).

We have \( a_0 = f(0) \), so that \(|f(0)|^2 \leq \sum_{n=0}^{\infty} |a_n|^2 r^{2n}\), as \( |a_0|^2 \) is the first term in the series of positive terms. For the second inequality we resort to the definition of \( I_2(r)\): since \(|f(re^{i\theta})| \leq M(r)\), we get

\[
I_2(r) = \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta \leq \frac{1}{2\pi} \int_0^{2\pi} M(r)^2 d\theta = M(r)^2.
\]

(d) \( \log I_2(r) \) is a convex function of \( \log r \), when \( f \) is not identically zero. This means

\[
\log I_2(r) \leq \frac{\log r_2 - \log r}{\log r_2 - \log r_1} \log I_2(r_1) + \frac{\log r - \log r_1}{\log r_2 - \log r_1} \log I_2(r_2).
\]

**Hint:** Set \( u = \log r \), \( J(u) = I_2(e^u) \), show that \( d^2 \log J(u)/du^2 = (JJ'' - J^2)/J^2 \) and use the Cauchy-Schwarz inequality.

With this definition of \( J(u)\) and the quotient rule we get

\[
(\log J(u))' = \frac{J'(u)}{J(u)}, \quad (\log J(u))'' = \frac{J''(u)J(u) - J'(u)J'(u)}{J^2(u)}.
\]

Now using (a) and \( r = e^u \) we get

\[
J(u) = \sum_{n=0}^{\infty} |a_n|^2 e^{2nu}, \quad J'(u) = \sum_{n=0}^{\infty} |a_n|^2 2ne^{2nu}, \quad J''(u) = \sum_{n=0}^{\infty} |a_n|^2 (2n)^2 e^{2nu}.
\]

Now we apply Cauchy-Schwarz inequality for sequences in the form

\[
\left| \sum_n c_n d_n \right|^2 \leq \sum_n |c_n|^2 \sum_n |d_n|^2.
\]

We take \( c_n = a_n e^{nu} \) and \( d_n = 2n\bar{a}_n e^{nu} \) to get

\[
J'(u)^2 \leq \sum_{n=0}^{\infty} |a_n|^2 e^{2nu} \sum_{n=0}^{\infty} |a_n|^2 (2n)^2 e^{2nu} = J(u)J''(u).
\]

This implies that \((\log J(u))'' \geq 0\), i.e. \( \log J(u) \) is a convex function of \( u \). This means that the graph is less than any secant segment on its graph. We fix \( u_1 = \log r_1 \) and \( u_2 = \log r_2 \). The segment between them is \( au_1 + (1-a)u_2 \). To get \( u \) we choose \( a = (u_2 - u)/(u_2 - u_1) \) i.e.

\[
u = \frac{u}{u_2 - u_1} u_1 + \frac{u - u_1}{u_2 - u_1} u_2.
\]

We join the points \((u_1, \log J(u_1))\) and \((u_2, \log J(u_2))\) on the graph of \( \log J(u) \) with a line segment of slope

\[
m = \frac{\log J(u_2) - \log J(u_1)}{u_2 - u_1} = \frac{\log I_2(r_2) - \log I_2(r_1)}{\log r_2 - \log r_1}
\]

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with equation
\[
y = m(u - u_1) + \log J(u_1) = \frac{\log I_2(r_2) - \log I_2(r_1)}{\log r_2 - \log r_1} (u - \log r_1) + \log I_2(r_1).
\]

At a given \( u = \log r \) this line segment is higher in the plane than the point on the graph \((u, \log J(u))\):
\[
\log J(u) = \log I_2(r) \leq \frac{\log I_2(r_2) - \log I_2(r_1)}{\log r_2 - \log r_1} (\log r - \log r_1) + \log I_2(r_1)
\]
\[
= \left(1 + \frac{\log r_1 - \log r}{\log r_2 - \log r_1}\right) \log I_2(r_1) + \frac{\log r - \log r_1}{\log r_2 - \log r_1} \log I_2(r_2).
\]

Now we add the fraction and 1 in parentheses to get the required inequality.

7. Let \( U(\xi) \) be piecewise continuous and bounded for all real \( \xi \). Show that
\[
P_U(z) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y}{(x - \xi)^2 + y^2} U(\xi) d\xi
\]
is a harmonic function in the upper half plane with boundary values \( U(\xi) \) at points of continuity. This is the Poisson integral for the half-plane.

This is really an exercise in changing variables in the integral using
\[
T: \mathbb{H} \to \mathbb{D}, \quad w = T(z) = \frac{z - i}{z + i}, \quad T'(z) = \frac{2i}{(z + i)^2}.
\]
We set \( T(\xi) = e^{it} \). We define \( V(t) = V(e^{it}) = U(\xi), \) i.e. we set \( V \circ T = U \) and we consider the circle to be parametrized with \( t \in [0, 2\pi] \). Since \( i e^{it} dt = T'(\xi) d\xi \), we get that
\[
dt = \frac{T'(\xi) d\xi}{i T(\xi)} = \frac{2i/(\xi + i)^2}{i(\xi - i)/(\xi + 1)} d\xi = \frac{2d\xi}{\xi^1 + 1}.
\]

We know that the Poisson integral \( P_V(w) \) is a harmonic function in the unit disc with boundary values \( V(e^{it}) \) at points of continuity. Since the composition of a harmonic function with a holomorphic function is still harmonic, we need to show that \( P_V(w) \) can be written as \( P_U(z) \) in the \( z, \xi \) variables. The boundary behavior is obvious since \( T(\mathbb{R}) \) is the unit circle. We have with \( w = re^{i\theta} = T(z) \)
\[
P_V(w) = \frac{1}{2\pi} \int_0^{2\pi} \Re \left( \frac{1 + re^{i(\theta - t)}}{1 - re^{i(\theta - t)}} \right) V(e^{it}) dt = \frac{1}{2\pi} \int_0^{2\pi} \Re \left( \frac{1 + we^{-it}}{1 - we^{-it}} \right) V(e^{it}) dt
\]
\[
= \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - |w|^2}{|1 - we^{-it}|^2} V(e^{it}) dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1 - |T(z)|^2}{|1 - T(z)|^2} U(\xi) \frac{2}{\xi^2 + 1} d\xi
\]
\[
= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1 - |(z - i)/(z + i)|^2}{|1 - T(\xi)(z - i)/(z + i)|^2} U(\xi) \frac{2}{\xi^2 + 1} d\xi = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{|z + i|^2 - |z - i|^2}{|z + i - T(\xi)(z - i)|^2} U(\xi) \frac{d\xi}{\xi^2 + 1}
\]
We calculate the numerator and denominator in the fraction with \( z = x + iy \):

\[
|z + i|^2 - |z - i|^2 = x^2 + (y + i)^2 - (x^2 + (y - 1)^2) = 4y,
\]

\[
|(z+i)-(z-i)(\xi+i)/(\xi-i)|^2 = \frac{4}{\xi^2 + 1}((x-\xi)^2 + y^2).
\]

This gives finally

\[
P_{\nu}(T(z)) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{4y}{4((x-\xi)^2 + y^2)} U(\xi) d\xi,
\]

which is exactly the given formula.

8. Let

\[
P_r(t) = \Re \left( \frac{1 + z}{1 - z} \right), \quad z = re^{it}
\]

be the Poisson kernel for the unit disc \(|z| < 1\). Let \( U(\theta) \) be a continuous function of the interval \([0, \pi]\) with \( U(0) = U(\pi) = 0 \). Show that the function

\[
u(re^{i\theta}) = \frac{1}{2\pi} \int_{0}^{\pi} \{P_r(t - \theta) - P_r(t + \theta)\} U(t) dt
\]

is harmonic in the half-disc

\[
\{re^{i\theta}, 0 \leq r < 1, 0 \leq \theta \leq \pi\}
\]

and has the following limiting behavior on the boundary:

\[
\lim_{z \to e^{i\theta_0}} u(z) = U(\theta_0), \quad 0 < \theta_0 < \pi
\]

\[
u(x) = 0, \quad -1 < x < 1.
\]

We define the following function as an extension of \( U \) from \([0, \pi]\) to the interval \([-\pi, \pi]\):

\[
U(\theta) = \begin{cases} 
U(\theta), & \theta \in [0, \pi] \\
-U(-\theta), & \theta \in [-\pi, 0]
\end{cases}
\]

We recall that the Poisson kernel is an even function of the angle. We consider \( P_{\nu}(z) \), the convolution integral of \( U \) with the Poisson kernel \( (z = re^{i\theta}) \):

\[
P_{\nu}(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(\theta - t)U(t) dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(\theta-t)(-U(-t))dt + \frac{1}{2\pi} \int_{0}^{\pi} P_r(\theta-t)U(t)dt
\]

\[
= -\frac{1}{2\pi} \int_{0}^{\pi} P_r(\theta+s)U(s)ds + \frac{1}{2\pi} \int_{0}^{\pi} P_r(\theta-t)U(t)dt = \frac{1}{2\pi} \int_{0}^{2\pi} \{P_r(\theta - t) - P_r(\theta + t)\} U(t)dt.
\]
This explains our choice for extending $U$ as an odd function: we get the formula given. Moreover, we know that $P_U(z)$ is harmonic on the whole unit disc, so, in particular, it is harmonic in the upper half of it. We examine the limiting behavior. Since $U(\theta)$ is continuous, its extension is continuous on the whole circle, so Schwarz’ theorem gives

$$\lim_{r \to 1} P_U(re^{it}) = U(e^{it}) = U(e^{it}), \quad t \in [0, \pi].$$

It remains to show the behavior on the segment $(-1, 1)$. These points are interior to the unit disc. For $0 < x < 1$ we have $\theta = 0$, which gives

$$u(x) = \frac{1}{2\pi} \int_0^\pi \{P_x(t - 0) - P_x(t + 0)\} U(t)dt = \frac{1}{2\pi} \int_0^\pi 0U(t)dt = 0.$$

For $-1 < x < 0$, we have $\theta = \pi$. Since $P_r(\pi + t) = P_r(-\pi - t)$ we get

$$2\pi u(x) = \int_0^\pi \{P_{-x}(\pi - t) - P_{-x}(\pi + t)\} U(t)dt =$$

$$\int_0^\pi \left\{ \Re \left( \frac{1 + re^{i(\pi-t)}}{1 - re^{i(\pi-t)}} \right) - \Re \left( \frac{1 + re^{i(-\pi-t)}}{1 - re^{i(-\pi-t)}} \right) \right\} U(t)dt = \int_0^\pi \Re \left( \frac{1 - re^{-it}}{1 + re^{-it}} - \frac{1 - re^{-it}}{1 + re^{-it}} \right) U(t)dt = 0$$

since $e^{i\pi} = -1 = e^{-i\pi}$.