Math 70300

Homework 6

Due: December 5

1. Let $f(z)$ be a holomorphic function in the disc $|z| < R_1$ and set

$$M(r) = \sup_{|z|=r} |f(z)|, \quad A(r) = \sup_{|z|=r} \Re(f(z)), \quad 0 \leq r < R_1.$$

(a) Show that $M(r)$ is monotonic and, in fact, strictly increasing, unless f is a constant.

This is nothing more than the maximum modulus principle. If $r < s < R_1$, we apply the maximum modulus principle on $|z| \leq s$. This implies that for $|z| \leq s$ $|f(z)| \leq M(s)$, in particular this holds for $|z| = r$ and gives by taking supremum $M(r) \leq M(s)$. Equality for $|z| = r$ implies equality in the maximum modulus principle, so f is constant.

(b) Show that $A(r)$ is monotonic and, in fact, strictly increasing, unless f is constant.

We introduce $g(z) = e^{f(z)}$ so that $|g(z)| = e^{\Re f(z)}$. This gives with the obvious notation $M(g, r) = \sup_{|z|=r} |g(z)|$

$$M(g, r) = e^{A(r)}, r < R_1.$$

Since g is holomorphic $M(g, r)$ is increasing and by (a) strictly increasing, unless g is a constant. Since e^x is also strictly increasing, $A(r)$ is strictly increasing, unless $g(z)$ is a constant. But in this case $g(z) = e^{f(z)} = k$ and this gives $e^{\Re f(z)} = |k| \implies \Re f(z) = \log |k|$ is constant. But we know that analytic functions with constant real part are also constant.

2. Assume $f(z)$ is a holomorphic function on $|z| \leq 1$ with $|f(z)| \leq 1$. Show that

$$\frac{|f'(z)|}{1 - |f(z)|^2} \leq \frac{1}{1 - |z|^2}.$$

When does equality hold for a point z_0 inside $|z| < 1$?

Fix $z \in \mathbb{D}$. We set

$$\phi_a(w) = \frac{w - a}{1 - \bar{a}w}, \quad |w| < 1, \quad |a| < 1.$$

We remark that

$$\phi_a'(w) = \frac{1 - |a|^2}{(1 - \bar{a}w)^2}, \quad \phi_a'(0) = 1 - |a|^2, \quad \phi_a'(a) = \frac{1}{1 - |a|^2}.$$

We consider the function

$$g = \phi_{f(z)} \circ f \circ \phi_{-z} : \mathbb{D} \to \mathbb{D},$$

$$g(0) = \phi_{f(z)}(f(\phi_{-z}(0))) = \phi_{f(z)}(f(z)) = 0.$$

By Schwarz lemma $|g'(0| \leq 1$, with equality only if g is a rotation, i.e. $g(w) = e^{i\theta}w$, $\theta \in [0, 2\pi]$. We calculate with the chain rule.

$$|\phi'_{f(z)}(f(\phi_{-z}(0)))f'(\phi_{-z}(0))\phi'_{-z}(0)| \leq 1 \Leftrightarrow |\phi'_{f(z)}(f(z))||f'(z)||\phi'_{-z}(0)| \leq 1$$

$$\Leftrightarrow \left|\frac{1}{1-|f(z)|^2}\right| |f'(z)||1-|z|^2| \leq 1 \Leftrightarrow \frac{|f'(z)|}{1-|f(z)|^2} \leq \frac{1}{1-|z|^2}.$$

If equality holds then $g(w) = e^{i\theta}w$. We get (note that $\phi_a^{-1} = \phi_{-a}$)

$$\phi_{f(z_0)} \circ f \circ \phi_{-z_0} = e^{i\theta} \text{ Id } \Leftrightarrow f(\phi_{-z_0}(w)) = \phi_{-f(z_0)}(e^{i\theta}w) \Leftrightarrow f(z) = \phi_{-f(z_0)}(e^{i\theta}\phi_{z_0}(z)).$$

3. Let $\mathbb{D} = \{z : |z| < 1\}$. Suppose that $f : \mathbb{D} \to \mathbb{D}$ is analytic, $f(1/3) = 0$ and $f'(1/3) = 0$. Show that $|f(0)| \leq 1/9$.

   Let

   $$\phi(z) = \phi_{-1/3}(z) = \frac{z + 1/3}{1 + z/3}$$

   be the linear fractional transformation with $\phi : \mathbb{D} \to \mathbb{D}$ and $\phi(-1/3) = 0$, $\phi(0) = 1/3$. Then $f \circ \phi : \mathbb{D} \to \mathbb{D}$ and $f(\phi(0)) = f(1/3) = 0$, so we can apply Schwarz lemma. The result is $|f(\phi(z))| \leq |z|$, $|z| < 1$. Applying this for $z = -1/3$ we get

   $$|f(\phi(-1/3))| \leq 1/3 \Leftrightarrow |f(0)| \leq 1/3.$$

   This is short of what we want to prove, since we have not used $f'(1/3) = 0$. To exploit the information on the derivative, we consider the function

   $$g(z) = \frac{f(\phi(z))}{z}.$$

   Since $(f \circ \phi)(0) = f(1/3) = 0$, $g(z)$ has in fact a removable singularity at 0. Since $(f \circ \phi)'(0) = f'(\phi(0))\phi'(0) = f'(1/3)\phi'(0) = 0$, we get the value $g(0) = 0$. We apply Schwarz lemma to get $|g(z)| \leq |z|$. We set $z = -1/3$ to get

   $$|g(-1/3)| = \left|\frac{f(\phi(-1/3))}{1/3}\right| = \left|\frac{f(0)}{1/3}\right| \leq 1/3 \Longrightarrow |f(0)| \leq \frac{1}{9}.$$

4. Prove that if $f(z) : \mathbb{H} \to \mathbb{H}$ is an analytic function from the upper-half plane to itself, then

   $$\frac{|f(z) - f(z_0|}{|f(z) - \overline{f(z_0)}|} \leq \frac{|z - z_0|}{|z - \overline{z_0}|}, \quad z, z_0 \in \mathbb{H}$$

and

$$\frac{|f'(z)|}{\Im f(z)} \leq \frac{1}{\Im z}, \quad z \in \mathbb{H}.$$

When does equality hold?

We map the upper-half plane conformally to the unit disc $\mathbb{D}$ with appropriate holomorphic transformations. More, precisely we define

$$\phi(z) = \frac{z - z_0}{z - \bar{z}_0}, \quad z \in \mathbb{H}, \quad \psi(w) = \frac{w - f(z_0)}{w - \overline{f(z_0)}}, \quad w \in \mathbb{H}.$$

Since for $z \in \mathbb{R}$, $|z - z_0| = |z - \bar{z}_0|$, we have $\phi(\mathbb{R}) = \{z, |z| = 1\}$. Notice that we mapped z_0 to the center of the disc and its symmetric point with respect to the real axis, i.e. $\bar{z}_0$ to the point symmetric with respect to the circle, i.e. ∞. This shows that $\phi : \mathbb{H} \to \mathbb{D}$. Similar considerations apply for ψ and, in particular, $\psi(f(z_0)) = 0$. We consider the function

$$g = \psi \circ f \circ \phi^{-1} : \mathbb{D} \to \mathbb{D},$$
$$g(0) = \psi(f(\phi^{-1}(0))) = \psi(f(z_0)) = 0,$$

so that we can apply Schwarz lemma. This gives $|g(\zeta)| \leq |\zeta|$. We set $\phi(z) = \zeta \Leftrightarrow z = \phi^{-1}(\zeta)$. This gives

$$|\psi(f(\phi^{-1}(\zeta)))| \leq |\zeta| \Leftrightarrow |\psi(f(z))| \leq |\phi(z)| \Leftrightarrow \left| \frac{f(z) - f(z_0)}{f(z) - \overline{f(z_0)}} \right| \leq \left| \frac{z - z_0}{z - \bar{z}_0} \right|, \quad z, z_0 \in \mathbb{H}.$$

For the second inequality we use $|g'(0)| \leq 1$. First we notice that

$$\phi'(z) = \frac{z_0 - \bar{z}_0}{(z - \bar{z}_0)^2} \Longrightarrow \phi'(z_0) = \frac{1}{2i\Im z_0}.$$

Similarly

$$\psi'(w) = \frac{f(z_0) - \overline{f(z_0)}}{(w - \overline{f(z_0)})^2} \Longrightarrow \psi'(f(z_0)) = \frac{1}{2i\Im f(z_0)}.$$

By the chain rule we get

$$|\psi'(f(\phi^{-1}(0)))||f'(\phi^{-1}(0))||(\phi^{-1})'(0)| \leq 1 \Leftrightarrow |\psi'(f(z_0))||f'(z_0)|\frac{1}{|\phi'(z_0)|} \leq 1,$$

since $(\phi^{-1})'(z) = 1/\phi'(\phi^{-1}(z))$. As a result

$$\frac{|f'(z_0)|}{|2i\Im f(z_0)|} \leq \frac{1}{|2i\Im z_0|}$$

and this clearly implies the inequality for an arbitrary point z_0 in $\mathbb{H}$. If equality holds in either of the inequalities, $g(\zeta) = c\zeta$ with $|c| = 1$. This gives

$$g(\zeta) = \psi(f(\phi^{-1}(\zeta))) = c\zeta \Leftrightarrow \psi(f(z)) = c\phi(z) \Leftrightarrow f(z) = \psi^{-1}(c\phi(z)).$$

5. Suppose $z = \phi(\zeta)$ and $w = \psi(\zeta)$ are one-to-one analytic maps from the unit disc $D(0,1)$ onto the regions G_1 and G_2. Set $\phi(0) = z_0$ and $\psi(0) = w_0$. Let $0 < r < 1$ and $\Omega_1(r) = \phi(D(0,r))$, $\Omega_2(r) = \psi(D(0,r))$. Assume $f : G_1 \to G_2$ be a holomorphic map with $f(z_0) = w_0$. Show that

$$f(\Omega_1(r)) \subset \Omega_2(r).$$

We have that $g = \psi^{-1} \circ f \circ \phi : \mathbb{D} \to \mathbb{D}$. Moreover, $g(0) = \psi^{-1}(f(\phi(0))) = \psi^{-1}(f(z_0)) = \psi^{-1}(w_0) = 0$. This means we can apply Schwarz lemma to get $|g(\zeta)| \leq |\zeta|$. Let $z \in \Omega_1(r) \Leftrightarrow z = \phi(\zeta), |\zeta| < r$. Then

$$|\psi^{-1}(f(z))| = |\psi^{-1}(f(\phi(\zeta)))| \leq |\zeta| < r$$

This gives that $f(z) \in \psi(D(0,r)) = \Omega_2(r)$.

6. Show that if an entire function f maps the real axis into itself and the imaginary axis into itself, then f is an odd function, i.e., $f(-z) = -f(z)$ for any z.

Give two proofs, which are really different.

First proof: Use of Taylor series. We have in the whole complex plane

$$f(z) = \sum_{n=0}^{\infty} a_n z^n.$$

We plug $z = x \in \mathbb{R}$ and force $f(x)$ to be real:

$$f(x) = \overline{f(x)} \Leftrightarrow \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \overline{a_n} x^n, \quad x \in \mathbb{R}.$$

By uniqueness of the Taylor expansion we get

$$a_n = \overline{a_n}, \quad n = 1, 2, \ldots \Longrightarrow a_n \in \mathbb{R}.$$

Now we plug $z = ix$, $z \in \mathbb{R}$ and force $f(ix)$ to be purely imaginary.

$$f(ix) = -\overline{f(ix)} \Leftrightarrow \sum_{n=0}^{\infty} a_n i^n x^n = -\sum_{n=0}^{\infty} \overline{a_n i^n x^n} = -\sum_{n=0}^{\infty} a_n (-i)^n x^n.$$

By uniqueness of the Taylor series we get

$$a_n i^n = -a_n(-i)^n \Leftrightarrow a_n = -(-1)^n a_n \Longrightarrow a_{2k} = 0, k = 1, 2, \ldots.$$

Since the Taylor series has only odd terms, the function $f(z)$ is odd.

Second proof: We use the Schwarz reflection principle twice. Since f is real on the real axis, symmetric points with respect to the real line are mapped to symmetric points with respect to the real axis. This gives

$$f(z) = \overline{f(\bar{z})}, \quad z \in \mathbb{C}.$$

Since f maps the imaginary axis to the imaginary axis, it maps points symmetric with respect to the imaginary axis to points symmetric with respect to the imaginary axis. We notice that the symmetric point of z has the same imaginary part and opposite real part, i.e. $z^* = -\bar{z}$. Similarly $w^* = -\bar{w}$. This gives

$$f(z) = -\overline{f(z^*)} = -\overline{f(-\bar{z})}.$$

Combined the two equations for $f(z)$ we get

$$f(z) = -\overline{(f(-\bar{z})} = -\overline{f(\overline{-z})} = -f(-z), \quad z \in \mathbb{C}.$$

7. (a) Consider two rectangles $R = [0,a] \times [0,b]$ and $R' = [0,a'] \times [0,b']$. Suppose $f : R \to R'$ is a homeomorphism which is holomorphic in the interior of R and maps a-sides to a'-sides and b-sides to b'-sides (i.e., $f([0,a] \times \{0\}) = [0,a'] \times \{0\}$, $f([0,a] \times \{b\}) = [0,a'] \times \{b'\}$, $f(\{0\} \times [0,b]) = \{0\} \times [0,b']$, and $f(\{a\} \times [0,b]) = \{a'\} \times [0,b']$). Show that $a/b = a'/b'$.

We use the Schwarz reflection principle applied first along each side of R. Since f maps $\{0\} \times [0,b]$ to $\{0\} \times [0,b']$ we reflect the rectangle R along the imaginary axis and R' similarly. The reflections are given by $z^* = -\bar{z}$, $w^* = -\overline{w}$. The analytic continuation of f is given by $f(z^*) = (f(z))^* \Leftrightarrow f(-\bar{z}) = -\overline{f(z)}$. The image of R under the reflection is the rectangle $R^* = [-a,0] \times [0,b]$ and it is mapped by the analytic continuation of f to $R'^* = [-a',0] \times [0,b']$. On the union $R \cup R^*$ the function f is still one-to-one, since it is on R and R^* is mapped onto R'^* in a one-to-one way and R'^* is disjoint from R'.

The reflection can be repeated along the sides $\{a\} \times [0,b]$ (resp. $\{a'\} \times [0,b']$), $[0,a] \times \{0\}$ (resp. $[0,a'] \times \{0\}$), $[0,a] \times \{b\}$ (resp. $[0,a'] \times \{b'\}$). After these reflections we keep reflecting along the sides of the new rectangles we created: this is possible, say, along the side $\{-a\} \times [0,b]$ since the values of f on it are the reflected values of f on $\{a\} \times [0,b]$, which lie on $\{a'\} \times [0,b']$. The reflection of this last side is $\{-a'\} \times [0,b']$. The rectangles form a tesselation of the z and of the w planes, and every rectangle in the z-plane corresponds to a unique one in the w-plane. See Figure 1. The analytic continuation of f on $\mathbb{C}$ is still one-to-one, since the values on any rectangle in the z plane lie inside only one of the rectangles in the w plane. This way we get a one-to-one mapping on $\mathbb{C}$, which is holomorphic. We need to examine what happens at ∞. We claim that it is a pole. We need to show that $\lim_{z\to\infty} f(z) = \infty$. Let R be given. Let r be chosen in such a way that every rectangle that intersects $|z| > r$ corresponds to a rectangle inside $|w| > R$. The correspondence is achieved with the help of f. As a result f is a polynomial which is one-to-one, so it is a linear function $f(z) = cz+d$. Since $f(0) = 0$, we get $d = 0$. Now $f(a) = a' \Longrightarrow c = a'/a$, $f(b) = b' \Longrightarrow c = b'/b$. This gives $a/b = a'/b'$.

(b) Let $A = \{z, R_1 \le |z| \le R_2\}$ and $B = \{w, r_1 \le |w| \le r_2\}$ be two annuli and $f : A \to B$ be a holomorphic one-to-one and onto map that maps $|z| = R_i$ to $|w| = r_i$, $i = 1,2$. Show that there exists a $c \in \mathbb{C}$, $|c| = 1$ such that

$$\frac{R_1}{R_2} = \frac{r_1}{r_2}$$

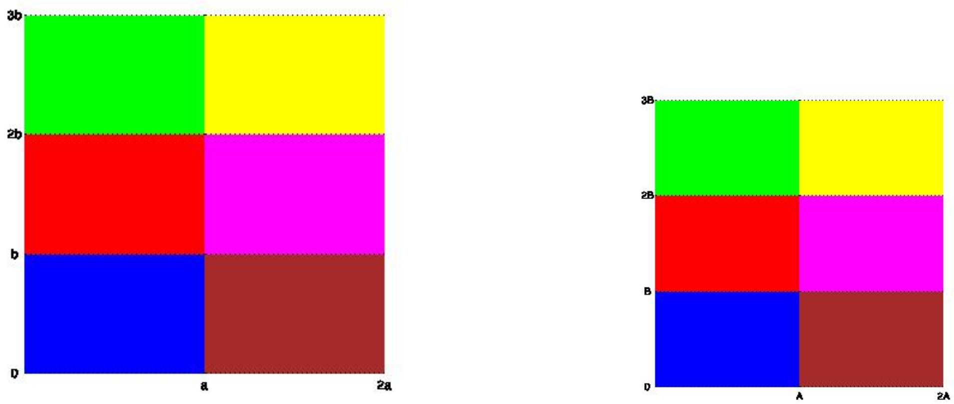

Figure 1: The tesselations on the z- and w-planes. The function f maps the rectangles on the left to the rectangles on the right with the same color.

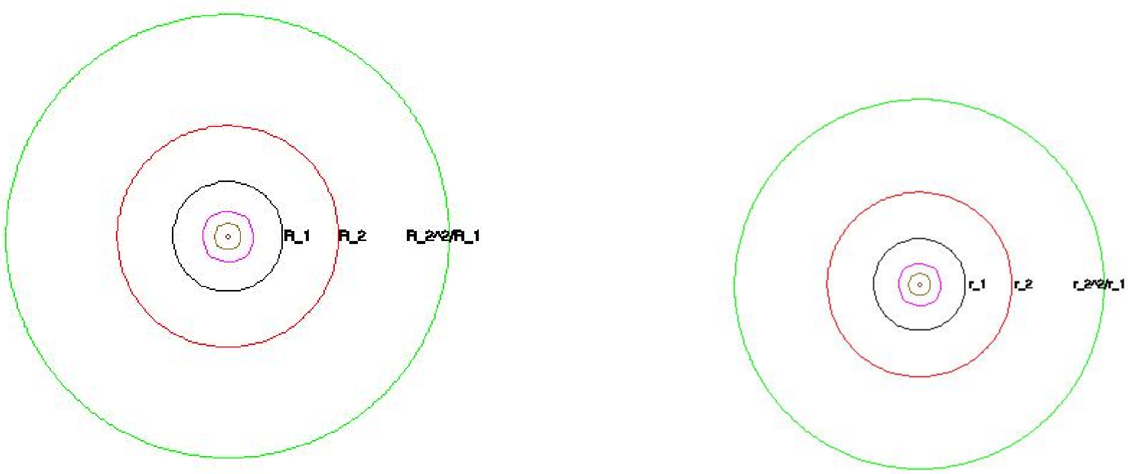

Figure 2: The function f maps the circles on the left to the circles on the right with the same color.

and

$$f(z) = c\frac{r_1}{R_1}z.$$

We apply the Schwarz reflection principle. We reflect as first step along $|z| = R_2$ and $|w| = r_2$. The reflection of the annulus $R_1 < |z| < R_2$ will be mapped into an annulus determined by the reflection of $|w| = r_1$ along $|w| = r_2$. The reflection maps circles to circles according to the formula

$$(w^* - a)\overline{w - a} = r_2^2, \quad a = 0.$$

For $|w| = r_1$ we get a circle of radius r_2^2/r_1. So the reflection of the annulus $r_1 < |w| < r_2$ is the annulus $r_2 < |w| < r_2^2/r_1$. Similarly the reflection of the annulus $R_1 < |z| < R_2$ along $|z| = R_2$ is the annulus $R_2 < |z| < R_2^2/R_1$. Because $0 \notin \{w, r_1 < |w| < r_2\}$, the analytic continuation of f is a mapping that is still holomorphic (under a reflection 0 is mapped to ∞). We keep reflecting outwards and analytically continue f on the annulus $R_2^2/R_1 < |z| < R_2^3/R_1^2$ with values in the annulus $r_2^2/r_1 < |w| < r_2^3/r_1^2$ and continue until we cover the whole region $R_1 < |z|$ on the domain and the region $r_1 < |w|$ in the image of f. But we can also reflect inwards: this continues $f(z)$ analytically on the annulus $R_1^2/R_2 < |z| < R_1$ with values in the annulus $r_1^2/r_2 < |w| < r_1$ and we continue until we cover the region $0 < |z| < R_1$ with values in $0 < |w| < r_1$. Notice that the radii are a geometric progression with ratios R_2/R_1 and r_2/r_1 respectively. So they tend to infinity outwards and 0 inwards as we apply repeatedly the Schwarz reflection principle. See Figure 2.

We study the behavior of f at 0 and ∞. The function has a removable singularity at 0, since it tends to 0 at 0 (look at the images of the shrinking annuli, they are shrinking annuli as well). We set $f(0) = 0$. The function has a pole at ∞, since $\lim_{z\to\infty} f(z) = \infty$ (look at the images of expanding annuli, they are expanding annuli as well). Such an entire function is a polynomial. Moreover, since f was one-to-one inside the annulus $R_1 < |z| < R_2$, and because of the images we get by reflection, it is clear that the analytic continuation of $f(z)$ is also one-to-one on $\mathbb{C}$. This forces the polynomial to be of degree 1. Moreover, $f(0) = 0$, so $f(z) = dz$. Since $|z| = R_1$ is mapped to $|w| = r_1$, this gives $|d| = r_1/R_1$. Similarly, $|d| = r_2/R_2$. This gives $R_1/R_2 = r_1/r_2$ and $d = cr_1/R_1$ with $|c| = 1$.

8. (a) Let f be analytic in a bounded region D and its boundary C, such that $|f(z)| = 1$ on C. Show that f has at least one zero inside D, unless f is a constant.

   By the maximum modulus principle, $|f(z)| \le 1$ inside D. If $f(z)$ does not have a zero inside D, then the function $g(z) = 1/f(z)$ is holomorphic in D and also satisfies $|g(z)| = 1$ on the boundary. By the maximum modulus principle we get $|g(z)| \le 1$ inside D, i.e. $|f(z)| \ge 1$. This gives $|f(z)| = 1$ inside D and as a result f is constant.

   (b) Let $f(z)$ be an analytic function in a region D except for one simple pole and assume $|f(z)| = 1$ on the boundary of D. Prove that every value a with $|a| > 1$ is taken by $f(z)$ inside D once and once only.

Let $\Gamma_a = (f-a)(\partial D) = \{w, |w+a| = 1\}$ the image of the boundary curve of D under $f(z) - a$. By the argument principle

$$\frac{1}{2\pi i}\int_{\partial D} \frac{f'(z)}{f(z)-a}dz = n(\Gamma_a, 0) = \#\{a_j \in D, f(a_j) - a = 0\} - \#\{b_j \in D, b_j \text{ pole of } f(z) - a\}.$$

Clearly $f(z) - a$ has one simple pole inside D, exactly where f does. The a_j's are the points where $f(z)$ assumes the value a. If $|a| > 1$, 0 is not inside the circle $|z+a| = 1$, so $n(\Gamma_a, 0) = 0$. This gives

$$\#\{a_j \in D, f(a_j) = a\} = 1.$$

9. (a) How many roots of the equation $z^4 - 6z + 3 = 0$ have their modulus between 1 and 2?

Let $f(z) = z^4$ and $g(z) = -6z + 3$. On the circle $|z| = 2$ we have

$$|f(z)| = |z|^4 = 2^4 = 16, \quad |g(z)| = |-6z + 3| \leq 6|z| + 3 = 6 \cdot 2 + 3 = 15 < |f(z)|.$$

By Rouché's theorem, f has the same number of zeros inside the circle $|z| = 2$ as $f(z) + g(z) = z^4 - 6z + 3$. Since $f(z)$ has a fourth order zero at 0 and no other zeros, we get for the given polynomial 4 zeros inside $|z| = 2$. We need to subtract the number of zeros inside $|z| = 1$. Now we set $f(z) = -6z$, $g(z) = z^4 + 3$. We get on $|z| = 1$:

$$|f(z)| = |-6z| = 6|z| = 6, \quad |g(z)| = |z^4 + 3| \leq |z|^4 + 3 = 1 + 3 = 4 < 5 = |f(z)|.$$

Since $f(z)$ has one simple zero at 0, we get that $f(z) + g(z) = -6z + z^4 + 3$ has one zero inside $|z| = 1$. As a result there are 3 zeros inside the annulus.

(b) Find the number of the roots of the equation

$$z^6 - 5z^4 + 8z - 1 = 0$$

in the annulus $\{z : 1 < |z| < 2\}$.

For the disc $|z| \leq 1$ we use $f(z) = 8z$, $g(z) = z^6 - 5z^4 - 1$, so that on $|z| = 1$ we have

$$|f(z)| = |8z| = 8, \quad |g(z)| = |z^6 - 5z^4 - 1| \leq |z|^6 + 5|z|^4 + 1 = 1 + 5 + 1 = 7 < 8 = |f(z)|.$$

Since $f(z)$ has one zero at 0, the function $f(z) + g(z) = z^6 - 5z^4 + 8z - 1$ has the same number of zeros inside $|z| = 1$, i.e. 1. (Rouché's theorem).

For the disc $|z| = 2$, the above technique does not work. With any choice of splitting in one plus three or two plus two terms of the polynomial, we cannot apply Rouché's theorem. Here are the calculations:

$$|z^6| = 2^6 = 64, \quad |-5z^4 + 8z - 1| \leq 5 \cdot 2^4 + 8 \cdot 2 + 1 = 80 + 16 + 1 > 64,$$

$$|-5z^4| = 5 \cdot 16 = 80, \quad |z^6 + 8z - 1| \leq 2^6 + 8 \cdot 2 + 1 = 64 + 16 + 1 = 81 > 80,$$

$$|8z| = 16, \quad |z^6 - 5z^4 - 1| \le 2^6 + 5\cdot 2^4 + 1 = 64 + 80 + 1 > 16,$$
$$|1| = 1, \quad |z^6 - 5z^4 + 8z| \le 2^6 + 5\cdot 16 + 16 > 1,$$
$$|z^6 - 5z^4| = |z|^4|z^2 - 5| \ge 16\cdot 1 = 16, \quad |8z - 1| \le 8|z| + 1 = 17 > 16,$$
$$|z^6 + 8z| = |z||z^5 + 8| \ge 2\cdot(32 - 8) = 48, \quad |-5z^4 - 1| \le 80 + 1 > 48,$$
$$|z^6 - 1| \ge 63, \quad |-5z^5 + 8z| = |z||-5z^4 + 8| \le 2\cdot(80 + 8) > 63.$$

Here is a trick. We choose a smaller disc of radius 3/2 and find the number of zeros inside $|z| = 3/2$. We set $f(z) = -5z^4$, $g(z) = z^6 + 8z - 1$, so that on $|z| = 3/2$ we have

$$|f(z)| = 5(3/2)^4 = 405/16, \quad |g(z)| \le (3/2)^6 + 8\cdot 3/2 + 1 = (3/2)^6 + 13 < 405/16,$$

as

$$(3/2)^6 < 197/16 \Leftrightarrow (81/16)(9/4) < 197/16 \Leftrightarrow 729/4 < 197 \Leftrightarrow 729 < 788.$$

By Rouché's theorem $5z^4$ has the same number of zeros as $z^6 - 5z^4 + 8z - 1$ inside $|z| = 3/2$, i.e. 4. However, this does not suffice to guarantee that inside the given annulus $1 < |z| < 2$ there are 3 zeros. It is possible that we have one or two more zeros in the annulus $3/2 < |z| < 2$. We use the intermediate value theorem on the real axis:

$$f(-3) = 299, \quad f(-2) = -33, \quad f(2) = -1, \quad f(3) = 347.$$

By the intermediate value theorem there exist one solution in $(-3, -2)$ and another in $(2, 3)$. This accounts for all 6 roots. With Maple one can compute the roots to be

$$0.1251528553, 1.296061165, 2.023328803, -0.5358533086 + 0.9984804513i,$$
$$-2.372836206, -0.5358533086 - 0.9984804513i.$$

10. Let λ be real and $\lambda > 1$, Show that the equation

$$ze^{\lambda - z} = 1$$

has exactly one solution in the disc $|z| = 1$, which is real and positive.

We rewrite the equation as

$$ze^{\lambda} = e^z \Leftrightarrow ze^{\lambda} - e^z = 0,$$

so that we choose $f(z) = ze^{\lambda}$ and $g(z) = -e^z$ and apply Rouché's theorem on $|z| = 1$.

$$|f(z)| = |z|e^{\lambda} = e^{\lambda} > e, \quad \text{as } \lambda > 1,$$
$$|g(z)| = |-e^z| = e^{\Re z} \le e^1 < |f(z)|,$$

so the function $f(z) + g(z) = ze^{\lambda} - e^z$ has the same number of zeros inside $|z| = 1$ as $f(z)$, i.e. 1 solution. To prove that it is real and positive we apply the intermediate value theorem on $[0, 1]$:

$$f(0) + g(0) = 0 - e^0 = -1, \quad f(1) + g(1) = e^{\lambda} - e^1 > 0$$

as $\lambda > 1$. So there exists a solution in $[0, 1]$.