Math 70300

Homework 5

Due: November 14

1. Calculate the integrals using contour integration. Complete explanations are required.

$$\text{(i)} \int_0^\infty \frac{dx}{x^3+1} \quad \text{(ii)} \int_0^\infty \frac{\cos x}{x^2+1} dx, \quad \text{(iii)} \int_0^\infty \frac{(\log x)^2}{1+x^2} dx$$

$$\text{(iv)} \int_0^\infty \frac{\cos(ax)}{(x^2+b^2)^2}\, dx \quad (a,b>0), \quad \text{(v)} \int_0^{\pi/2} \frac{d\theta}{a+\sin^2\theta} \quad (a>0), \quad \text{(vi)} \int_0^\infty \frac{x^2 dx}{x^4+5x^2+6}$$

$$\text{(vii)} \int_{-\infty}^\infty \frac{dx}{(1+x^2)^{n+1}} \quad \text{(viii)} \int_0^\infty \frac{\log x}{(1+x^2)^2} dx \quad \text{(ix)} \int_0^{2\pi} \frac{d\theta}{(a+\cos\theta)^2} \quad (a>1).$$

Answers: (i) $2\pi/(3\sqrt{3})$, (ii) $\pi e^{-1}/2$, (iii) $\pi^3/8$, (iv) $\pi(1+ab)e^{-ab}/(4b^3)$, (v) $\pi/(2\sqrt{a^2+a})$, (vi) $(\sqrt{3}-\sqrt{2})\pi/2$, (vii) $1{\cdot}3{\cdot}5\cdots(2n-1)\pi/(2{\cdot}4{\cdot}6\cdots(2n))$, (viii) $-\pi/4$, (ix) $2\pi a/(a^2-1)^{3/2}$.

(i) We consider the contour in Figure 1 and $f(z) = 1/(z^3+1)$. The only pole of the denominator $x^3+1 = (x+1)(x^2-x+1) = (x+1)(x-e^{\pi i/3})(x-e^{5\pi i/3})$ inside the contour is $e^{\pi i/3}$.

$$\operatorname{Res}(f, e^{\pi i/3}) = \lim_{z\to e^{\pi i/3}} (z-e^{\pi i/3})\frac{1}{z^3+1} = \lim_{z\to e^{\pi i/3}} \frac{1}{3z^2} = \frac{1}{3e^{2\pi i/3}}$$

by L'Hôpital's rule. We get on the horizontal segment $[0,R]$ that $z(x)=x$, on the segment from 0 to $Re^{2\pi i/3}$ that $z(x) = xe^{2\pi i/3}, 0\le x\le R$. The part of the contour on a circle of radius R is denoted by γ_R. These give:

$$\int_0^R \frac{dx}{x^3+1} + \int_{\gamma_R} \frac{dz}{z^3+1} - \int_0^R \frac{e^{2\pi i/3}dx}{x^3 e^{2\pi i}+1} = 2\pi i \operatorname{Res}(f, e^{\pi i/3}).$$

Notice that on the slanted segment $z^3 = (xe^{2\pi i/3})^3 = x^3e^{2\pi i} = x^3$. This is what makes this the right contour. So we get

$$\int_0^R \frac{dx}{x^3+1} + \int_{\gamma_R} \frac{dz}{z^3+1} - e^{2\pi i/3}\int_0^R \frac{dx}{x^3+1} = \frac{2\pi i}{3e^{2\pi i/3}}.$$

The integral over γ_R tends to 0 as $R\to\infty$. This is because on γ_R for $R>1$ we have

$$\left|\frac{1}{z^3+1}\right| \le \frac{1}{R^3-1}$$

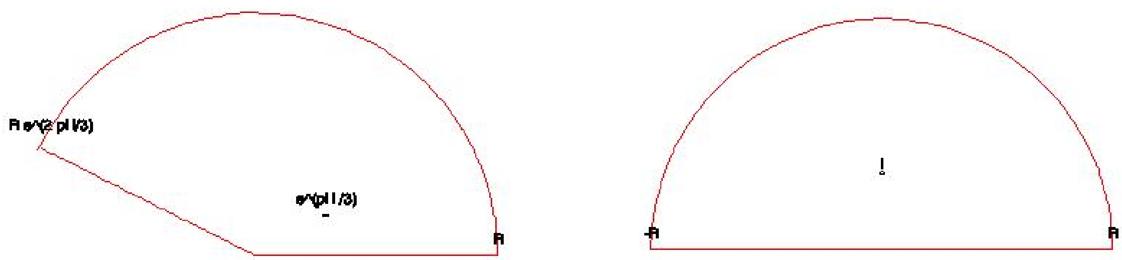

Figure 1: Contours for (i) and (ii)

by the triangle inequality ($|z^3+1| \geq |z|^3 - 1 = R^3 - 1$). On the other hand the length of the arc is $2\pi R/3$. This gives

$$\int_{\gamma_R} \frac{dz}{z^3+1} \leq \frac{2\pi R/3}{R^3-1} \to 0, \quad R \to \infty.$$

The result is

$$\int_0^\infty \frac{dx}{x^3+1} - e^{2\pi i/3} \int_0^\infty \frac{dx}{x^3+1} = \frac{2\pi i}{3e^{2\pi i/3}}$$

$$\implies \int_0^\infty \frac{dx}{x^3+1} = \frac{2\pi i}{3e^{2\pi i/3}(1-e^{2\pi i/3})} = \frac{2\pi i}{3(e^{2\pi i/3}-e^{4\pi i/3})} = \frac{2\pi i}{3\cdot 2i\sin(2\pi/3)} = \frac{2\pi}{3\sqrt{3}}.$$

(ii) We use the function $f(z) = e^{iz}/(1+z^2)$ inside the contour in Figure 1. The only pole is at i with residue

$$\mathrm{Res}(f,i) = \lim_{z\to i}(z-i)\frac{e^{iz}}{(z-i)(z+i)} = \frac{e^{ii}}{2i} = \frac{e^{-1}}{2i}.$$

We notice that

$$\int_{-R}^{R} \frac{e^{iz}}{z^2+1}dz = \int_0^R \frac{e^{ix}}{x^2+1}dx + \int_{-R}^0 \frac{e^{ix}}{x^2+1}dx = \int_0^R \frac{e^{ix}}{x^2+1}dx + \int_0^R \frac{e^{-ix}}{x^2+1}dx$$

$$= \int_0^R \frac{e^{ix}+e^{-ix}}{x^2+1}dx = 2\int_0^R \frac{\cos x}{x^2+1}dx,$$

with a change on variable $x = -y$ on $[-R, 0]$. The semicircle is denoted by γ_R. On it $|e^{iz}| = |e^{iRe^{i\theta}}| = e^{-R\sin\theta} \leq 1$, because $\theta \in [0, \pi]$. This gives

$$\left|\int_{\gamma_R} \frac{e^{iz}}{z^2+1}dz\right| \leq \frac{\pi R}{R^2-1} \to 0, \quad R \to \infty,$$

using the triangle inequality in the denominator ($|z^2+1| \geq |z|^2 - 1 = R^2 - 1$). The result is that

$$2\int_0^\infty \frac{\cos x}{x^2+1}dx = 2\pi i e^{-1}/(2i) = \pi e^{-1} \Longrightarrow \int_0^\infty \frac{\cos x}{x^2+1}dx = \frac{\pi e^{-1}}{2}.$$

(iii) We use the function

$$f(z) = \frac{(\log z)^2}{z^2+1}$$

with $\log z = \log|z| + i\arg z, \quad -\pi/2 < \arg z < 3\pi/2$ (non-principal branch of log). This works on the upper-half plane, where our contour is located, with the exception of 0. This is why to make the small semicircle γ_ϵ part of our contour. There are two poles of the integrand, but only one i lies inside the contour.

$$\text{Res}(f,i) = \lim_{z\to i}(z-i)\frac{(\log z)^2}{(z-i)(z+i)} = \frac{(\log i)^2}{2i} = \frac{(i\pi/2)^2}{2i} = \frac{-\pi^2}{8i}.$$

We call the large semicircle γ_R and the small one γ_r both traversed in the positive sense. The residue theorem gives

$$\int_r^R \frac{(\log x)^2}{x^2+1}dx + \int_{\gamma_R}\frac{(\log z)^2}{z^2+1}dz + \int_{-R}^{-r}\frac{(\log|x|+i\pi)^2}{x^2+1}dx - \int_{\gamma_r}\frac{(\log z)^2}{z^2+1}dz = 2\pi i\frac{-\pi^2}{8i} = -\pi^3/4,$$

since on the negative real axis $x<0$ the argument is π, which gives $\log x = \log|x| + i\pi$. On γ_R we have $|\log z| = |\log R + i\arg(z)| \leq \log R + \pi$, therefore,

$$\left|\int_{\gamma_R}\frac{(\log z)^2}{z^2+1}dz\right| \leq \pi R\frac{(\log R+\pi)^2}{R^2-1} \to 0, \quad R\to\infty,$$

since $\log R$ and its powers grow slower than any power of R. On γ_r we have $|\log z| = |\log r + i\arg(z)| \leq \log r + \pi$. This gives

$$\left|\int_{\gamma_r}\frac{(\log z)^2}{z^2+1}dz\right| \leq \pi r\frac{(\log r+\pi)^2}{1-r^2} \to 0, \quad r\to 0,$$

since the triangle inequality in the denominator gives $|z^2-1| \geq 1 - |z|^2 = 1 - r^2$, and $r\log r$, $r\log^2 r$ tend to 0 for $r\to 0$. If you are not familiar with this, consider (change variables $x = \log r$)

$$\lim_{r\to 0} r(\log r)^k = \lim_{x\to -\infty} e^x x^k = \lim_{x\to-\infty}\frac{x^k}{e^{-x}} = \lim_{y\to\infty}\frac{(-1)^k y^k}{e^y} = 0,$$

since e^y grows more quickly than any power of y. For the integral on the segment on the negative axis we have by expanding the numerator

$$\int_{-R}^{-r}\frac{(\log|x|+i\pi)^2}{1+x^2}dx = \int_{-R}^{-r}\frac{(\log|x|)^2}{x^2+1}dx + 2\pi i\int_{-R}^{-r}\frac{\log|x|}{x^2+1}dx - \pi^2\int_{-R}^{-r}\frac{1}{x^2+1}dx.$$

We now change variable in the first integral $x = -y$ to get

$$= \int_r^R \frac{(\log y)^2}{y^2+1} dy + 2\pi i \int_r^R \frac{\log y)}{y^2+1} dy - \pi^2 \int_r^R \frac{1}{y^2+1} dy$$

$$\longrightarrow \int_0^\infty \frac{(\log x)^2}{x^2+1} dx + 2\pi i \int_0^\infty \frac{\log y}{1+y^2} dy - \pi^2 \int_0^\infty \frac{1}{x^2+1} dx$$

as $r \to 0$, $R \to \infty$. The last integral is elementary as $\arctan(x)' = 1/(1+x^2)$ and gives

$$\int_0^\infty \frac{1}{x^2+1} dx = \frac{\pi}{2}.$$

We do not need to worry about the middle integral. It comes with an imaginary coefficient, while the integral is real. This means that we can take real parts and its contribution will be zero. Another approach is to notice that the integral is actually exactly zero:

$$\int_0^\infty \frac{\log x}{x^2+1} dx = \int_0^1 \frac{\log x}{x^2+1} dx + \int_1^\infty \frac{\log x}{x^2+1} dx = -\int_1^\infty \frac{\log(1/y)}{1/y^2+1} \frac{-dy}{y^2} + \int_1^\infty \frac{\log x}{1+x^2} dx,$$

where we changed variables $x = 1/y$, $dx = -dy/y^2$. Finally this gives

$$\int_0^\infty \frac{\log x}{x^2+1} dx = -\int_1^\infty \frac{\log y}{1+y^2} dy + \int_1^\infty \frac{\log x}{1+x^2} dx = 0.$$

Bringing all the results together we get

$$2\int_0^\infty \frac{(\log x)^2}{x^2+1} dx - \pi^2 \frac{\pi}{2} = -\frac{\pi^3}{4} \Longrightarrow \int_0^\infty \frac{(\log x)^2}{x^2+1} dx = \frac{1}{2}\frac{\pi^3}{4} = \frac{\pi^3}{8}.$$

(iv) We consider the function

$$f(z) = \frac{e^{iaz}}{(z^2+b^2)^2}$$

on the contour in Figure 2. There is only one pole inside the contour and this is at ib. It is a double pole. We calculate the residue

$$\mathrm{Res}(f, ib) = \lim_{z \to ib} \frac{d}{dz}(z-ib)^2 f(z) = \lim_{z \to ib} \frac{d}{dz} \frac{e^{iaz}}{(z+iab)^2} = \lim_{z \to ib} \frac{iae^{iaz}(z+ib) - 2e^{iaz}}{(z+ib)^3}$$

$$= \frac{iae^{iaib}(2ib) - 2e^{iaib}}{(2ib)^3} = \frac{iae^{-ab}2ib - 2e^{-ab}}{-i8b^3} = \frac{e^{-ab}(2ab+2)}{8ib^3}.$$

We call the semicircular contour γ_R and estimate the integral over it ($R > b$) as

$$\left| \int_{\gamma_R} \frac{e^{iaz}}{(z^2+b^2)^2} dz \right| \le \pi R \frac{1}{(R^2-b^2)^2} \to 0, \quad R \to \infty,$$

since $|e^{iaRe^{i\theta}}| = e^{-aR\sin\theta} \leq 1$ on the contour. We also have, after splitting the integral on the negative numbers and substituting $x = -y$,

$$\int_{-R}^{R} \frac{e^{iax}}{(x^2+b^2)^2}dx = \int_{-R}^{0} \frac{e^{iax}}{(x^2+b^2)^2}dx + \int_{0}^{R} \frac{e^{iax}}{(x^2+b^2)^2}dx = \int_{0}^{R} \frac{e^{-iay}}{(y^2+b^2)^2}dy + \int_{0}^{R} \frac{e^{iax}}{(x^2+b^2)^2}dx$$

$$= \int_{0}^{R} \frac{e^{-iax}+e^{iax}}{(x^2+b^2)^2}dx = 2\int_{0}^{R} \frac{\cos(ax)}{(x^2+b^2)^2}dx \to 2\int_{0}^{\infty} \frac{\cos(ax)}{(x^2+b^2)^2}dx, \quad R \to \infty.$$

We use the residue theorem in the contour of Figure 2, let $R \to \infty$ to get

$$\int_{0}^{\infty} \frac{\cos(ax)}{(x^2+b^2)^2}dx = 2\pi i \frac{e^{-ab}(2ab+2)}{8ib^3} = \frac{(ab+1)e^{-ab}\pi}{2b^3}.$$

(iv) Since

$$\int_{0}^{\pi/2} \frac{d\theta}{a+\sin^2\theta} = \frac{1}{4}\int_{0}^{2\pi} \frac{d\theta}{a+\sin^2\theta},$$

as $\sin^2\theta$ takes the same values in every quadrant, we consider the function

$$f(z) = \frac{1}{a + \left(\frac{z-z^{-1}}{2i}\right)^2}\frac{1}{z}$$

on the contour of Figure 3 $|z| = 1$. Notice that we substituted $\sin\theta = (z - z^{-1})/(2i)$, since on $|z| = 1$, i.e. $z = e^{i\theta}$ this holds. Moreover, notice that we multiplied with $1/z$, so that $dz/z = ie^{i\theta}d\theta/e^{i\theta} = id\theta$. We get

$$f(z) = \frac{1}{a - \frac{(z-z^{-1})^2}{4}}\frac{1}{z} = \frac{1}{a - z^2/4 - z^{-2}/4 + 2/4}\frac{1}{z} = \frac{4z}{(4a+2)z^2 - z^4 - 1}.$$

We solve the biquadratic equation $-z^4 + (4a+2)z^2 - 1 = 0$.

$$z^2 = 2a+1 \pm \sqrt{(2a+1)^2 - 1} = 2a+1 \pm 2\sqrt{a^2+a}.$$

There are two solutions for z^2 but only is inside the circle, since the solutions are real and positive and their product is 1. Clearly this is the smaller solution $b^2 = 2a+1-2\sqrt{a^2+a}$. This gives two solutions for z, $z = \pm b$. The poles are simple. We set $c^2 = 2a+1+2\sqrt{a^2+a}$.

$$\mathrm{Res}(f,b) = \lim_{z\to b} \frac{(z-b)4z}{-(z^2-b^2)(z^2-c^2)} = \lim_{z\to b} \frac{4z}{-(z+b)(z^2-c^2)} = \frac{4b}{-2b(b^2-c^2)} = \frac{1}{2\sqrt{a^2+a}}$$

$$\mathrm{Res}(f,-b) = \lim_{z\to -b} \frac{(z+b)4z}{-(z^2-b^2)(z^2-c^2)} = \lim_{z\to -b} \frac{4z}{-(z-b)(z^2-c^2)} = \frac{-4b}{2b(b^2-c^2)} = \frac{1}{2\sqrt{a^2+a}}$$

Using the parametrization $z = e^{i\theta}$ we get

$$\int_{|z|=1} f(z)dz = \int_{0}^{2\pi} f(e^{i\theta}ie^{i\theta}d\theta = \int_{0}^{2\pi} \frac{1}{a+\sin^2\theta}\frac{1}{e^{i\theta}}ie^{i\theta}d\theta = \int_{0}^{2\pi} \frac{1}{a+\sin^2\theta}id\theta$$

$$= 2\pi i \sum \operatorname{Res}(f, z_j) = 2\pi i \frac{1}{\sqrt{a^2+a}}$$

$$\Longrightarrow \int_0^{2\pi} \frac{1}{a+\sin^2\theta} d\theta = \frac{2\pi}{\sqrt{a^2+a}} \Longrightarrow \int_0^{\pi/2} \frac{1}{a+\sin^2\theta} d\theta = \frac{\pi}{2\sqrt{a^2+a}}.$$

Alternative method:

$$\int_0^{\pi/2} \frac{d\theta}{a+\sin^2\theta} = \int_0^{\pi/2} \frac{d\theta}{a+\frac{1-\cos(2\theta)}{2}} = 2\int_0^{\pi/2} \frac{d\theta}{(2a+1)-\cos(2\theta)}$$

$$= \int_0^{\pi} \frac{dx}{(2a+1)-\cos x} = \int_{-\pi}^{0} \frac{dy}{(2a+1)+\cos y},$$

with the change of variables $x = \pi - y$. Now we use the example in p. 155 of Ahlfors: we take for a the expression $2a+1$ to get

$$\int_{-\pi}^{0} \frac{dy}{(2a+1)+\cos y} = \frac{\pi}{\sqrt{(2a+1)^2-1}} = \frac{\pi}{\sqrt{4a^2+4a}}.$$

(vi) We take

$$f(z) = \frac{z^2}{z^4+5z^2+6}$$

on the contour in Figure 3 We have

$$z^4+5z^2+6 = (z^2+3)(z^2+2) = (z+i\sqrt{3})(z-i\sqrt{3})(z+i\sqrt{2})(z-i\sqrt{2}).$$

The poles inside the contour are $i\sqrt{2}$, and $i\sqrt{3}$.

$$\operatorname{Res}(f, i\sqrt{2}) = \frac{(i\sqrt{2})^2}{(2i\sqrt{2})((i\sqrt{2})^2+3)} = \frac{-2}{i2\sqrt{2}} = \frac{i\sqrt{2}}{2}.$$

$$\operatorname{Res}(f, i\sqrt{3}) = \frac{(i\sqrt{3})^2}{(2i\sqrt{3})((i\sqrt{3})^2+2)} = \frac{-3}{-i2\sqrt{3}} = \frac{-i\sqrt{3}}{2}.$$

The function is even so $\int_{-R}^{R} f(x)dx = 2\int_0^R f(x)dx$. The residue theorem gives

$$2\int_0^R \frac{x^2 dx}{x^4+5x^2+6} + \int_{\gamma_R} f(z)dz = 2\pi i \left(\frac{i\sqrt{2}}{2} - \frac{i\sqrt{3}}{2}\right) = 2\pi \left(\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\right).$$

The integral over γ_R tends to zero:

$$\left|\int_{\gamma_R} f(z)dz\right| \leq \pi R \frac{R^2}{R^4-5R^2-6} \to 0, \quad R \to \infty,$$

since $|z^4+5z^2+6| \geq |z^4| - |5z^2+6| \geq |z|^4 - 5|z|^2 - 6$. Taking the limit as $R \to \infty$ we get

$$\int_0^\infty \frac{x^2 dx}{x^4+5x^2+6} = \frac{\pi(\sqrt{3}-\sqrt{2})}{2}.$$

Alternative method: This integral can be computed without complex integration. We use partial fractions

$$\frac{x^2}{x^4+5x^2+6} = \frac{A}{x^2+3} + \frac{B}{x^2+2}$$

which gives $A+B=1$, $2A+3B=0$ and, therefore, $A=3$, $B=-2$.

$$\int_0^\infty x^2 dx x^4 + 5x^2 + 6 = \int_0^\infty \frac{3}{x^2+3} - \frac{2}{x^2+2} dx = \int_0^\infty \frac{3\sqrt{3}dy}{3(y^2+1)} dy - \int_0^\infty \frac{2\sqrt{2}dt}{2(t^2+1)} = \sqrt{3}\frac{\pi}{2} - \sqrt{2}\frac{\pi}{2},$$

where we substituted $x=\sqrt{3}y$ and $x=\sqrt{2}t$.

(vii) We consider $f(z) = 1/(z^2+1)^{n+1}$ and the contour of (ii). The only relevant pole is at i but it is of multiplicity $n+1$. We remark that

$$\frac{d^n}{dz^n}(z+i)^{-n-1} = (-n-1)(-n-2)\cdots(-n-n)(z+i)^{-n-1-n} = (n+1)(n+2)\cdots(2n)(-1)^n(z+i)^{-2n-1}.$$

This gives

$$\text{Res}(f,i) = \frac{1}{n!}\frac{d^n}{dz^n}\left((z-i)^{n+1}\frac{1}{(z-i)^{n+1}(z+i)^{n+1}}\right)_{|z=i} = \frac{1}{n!}(n+1)(n+2)\cdots(2n)(-1)^n(2i)^{-2n-1}$$

$$= \frac{(n+1)(n+2)\cdots(2n)}{n!}(-1)^n 2^{-2n-1}(-i)^{2n+1} = \frac{(n+1)(n+2)\cdots(2n)}{n!}(-1)^n\frac{1}{2^{2n+1}}(-i)(-1)^n$$

$$= -i\frac{(n+1)(n+2)\cdots(2n)}{n!2^{2n+1}}.$$

We estimate that the integral over the semicircle γ_R tends to 0 as $R\to\infty$:

$$\left|\int_{\gamma_R}\frac{dz}{(1+z^2)^{n+1}}\right| \le \pi R\frac{1}{(R^2-1)^{n+1}} \to 0, \quad R\to\infty.$$

The residue theorem gives

$$\int_{-R}^R \frac{dx}{(1+x^2)^{n+1}} + \int_{\gamma_R}\frac{dz}{(1+z^2)^{n+1}} = 2\pi i \text{Res}(f,i).$$

Taking the limit as $R\to\infty$ we get

$$\int_{-\infty}^\infty \frac{dx}{(1+x^2)^{n+1}} = 2\pi i(-i)\frac{(n+1)(n+2)\cdots 2n}{n!2^{n+1}} = 2\pi\frac{1\cdot 2\cdots n(n+1)(n+2)\cdots(2n)}{2^{2n}2\cdot 1\cdot 2\cdots n\cdot 1\cdot 2\cdots n}$$

$$= \pi\frac{1\cdot 2\cdots 2n}{2\cdot 4\cdots(2n)\cdot 2\cdot 4\cdots(2n)} = \pi\frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots 2n}.$$

Alternative method: The substitution $x=\tan\theta$ gives, as $1+x^2=(\cos\theta)^{-2}$, $dx = (\cos\theta)^{-2}d\theta$,

$$\int_{-\infty}^\infty \frac{dx}{(1+x^2)^{n+1}} = \int_{-\pi/2}^{\pi/2}(\cos^2\theta)^{n+1}\frac{d\theta}{\cos^2\theta} = \int_{-\pi/2}^{\pi/2}(\cos\theta)^{2n}d\theta = \pi\frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots(2n)}$$

by problem $2a$ in Homework 4. Notice that $\cos^2\theta$ takes the same value on any quadrant.

(viii) We use the same contour as in (iii) and the function

$$f(z) = \frac{\log z}{(1+z^2)^2}$$

with nonprincipal branch of $\log z$ given by

$$\log z = \log|z| + i\arg z, \quad -\pi/2 < \arg z < 3\pi/2.$$

Inside the contour there is a double pole at i. We compute the residue

$$\mathrm{Res}(f,i) = \lim_{z\to i}\frac{d}{dz}(z-i)^2\frac{\log z}{(z-i)^2(z+i)^2} = \lim_{z\to i}\frac{z^{-2}(z+i)-2\log z}{(z+i)^3} = \frac{i^{-1}2i-2\log i}{(2i)^3}$$

$$= \frac{2-2(i\pi/2)}{-8i} = \frac{2-i\pi}{-8i}.$$

For the integral over γ_R we have

$$\left|\int_{\gamma_R}\frac{\log z}{(1+z^2)^2}dz\right| \le \pi R\frac{\log R+\pi}{(R^2-1)^2} \to 0, \quad R\to\infty,$$

the same way as in (iii). For the integral over γ_r we have

$$\left|\int_{\gamma_r}\frac{\log z}{(1+z^2)^2}dz\right| \le \pi r\frac{\log r+\pi}{(1-r^2)^2} \to 0, \quad r\to 0.$$

The residue theorem gives

$$\int_r^R\frac{\log x dx}{(1+x^2)^2} + \int_{-R}^{-r}\frac{\log|x|+i\pi}{(1+x^2)^2}dx + \int_{\gamma_R}\frac{\log z}{(1+z^2)^2}dz + \int_{\gamma_r}\frac{\log z}{(1+z^2)^2}dz = 2\pi i\mathrm{Res}(f,i).$$

We take the limit as $R\to\infty$, $r\to 0$ to get

$$\int_0^\infty\frac{\log x dx}{(1+x^2)^2} + \int_{-\infty}^0\frac{\log|x|}{(1+x^2)^2}dx + i\pi\int_{-\infty}^0\frac{dx}{(1+x^2)^2} = 2\pi i\frac{2-i\pi}{-8i} = \frac{(i\pi-2)\pi}{4}.$$

We substitute $y=-x$ in the second integral, take real parts (notice that $\int_{-\infty}^0(1+x^2)^{-2}dx$ is real) to get

$$2\int_0^\infty\frac{\log x}{(1+x^2)^2}dx = \frac{-2\pi}{4} \Longrightarrow \int_0^\infty\frac{\log x}{(1+x^2)^2}dx = -\frac{\pi}{4}.$$

(ix) We use the same contour as in (v). The function to integrate is

$$f(z) = \frac{1}{(a+(z+z^{-1})/2)^2}\frac{1}{z} = \frac{4}{(2a+z+z^{-1})^2}\frac{1}{z} = \frac{4z}{(z+z^{-1}+2a)^2z^2} = \frac{4z}{(z^2+2az+1)^2}.$$

We solve $z^2 + 2az + 1 = 0$ by completing the square

$$(z+a)^2 + 1 - a^2 = 0 \Longrightarrow z + a = \pm\sqrt{a^-1} \Longrightarrow z_{1,2} = -a \pm \sqrt{a^2-1}.$$

The two roots are real, as $a > 1$ is given. They have negative sum and product 1. This means that only one is inside the circle $|z| = 1$. Call it $z_1 = -a + \sqrt{a^2-1}$, while $z_2 = -a - \sqrt{a^2-1} < -1$. The pole at z_1 is double. We calculate the residue

$$\text{Res}(f, z_1) = \lim_{z\to z_1} \frac{d}{dz} \frac{(z-z_1)^2 4z}{(z-z_1)^2(z-z_2)^2} = \lim_{z\to z_1} \frac{4(z-z_2) - 2\cdot 4z}{(z-z_2)^3} = \lim_{z\to z_1} \frac{-4(z+z_2)}{(z-z_2)^3}$$

$$= \frac{-4(z_1+z_2)}{(z_1-z_2)^3} = \frac{-4(-2a)}{(2\sqrt{a^2-1})^3} = \frac{a}{(a^2-1)^{3/2}}.$$

The residue theorem gives

$$\int_{|z|=1} f(z)dz = \int_0^{2\pi} \frac{1}{(a+\cos\theta)^2} \frac{1}{e^{i\theta}} ie^{i\theta} d\theta = 2\pi i \text{Res}(f, i) = \frac{2\pi i a}{(a^2-1)^{3/2}}$$

$$\Longrightarrow \int_0^{2\pi} \frac{1}{(a+\cos\theta)^2} d\theta = \frac{2\pi a}{(a^2-1)^{3/2}}.$$

2. Prove that

$$\sum_{n=-\infty}^{\infty} \frac{1}{(u+n)^2} = \frac{\pi^2}{\sin^2(\pi u)} \quad (u \notin \mathbb{Z}), \quad \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6},$$

using the function $f(z) = \dfrac{\pi \cot(\pi z)}{(u+z)^2}$ integrated over the boundary of the square $[-(N+1/2), N+1/2] \times [-(N+1/2), N+1/2]$, $N \geq |u|$, $N \in \mathbb{N}$. This is one of the many derivations of the value $\sum 1/n^2$, due originally to Euler. The function

$$g(z) = \pi \cot(\pi z) = \frac{\pi \cos(\pi z)}{\sin(\pi z)}$$

has poles at $n \in \mathbb{Z}$. Since $(\sin(\pi z))' = \pi \cos(\pi z)$, which does not vanish at the integers, the poles are simple. We calculate the residues

$$\text{Res}(g, n) = \lim_{z\to n} (z-n) \frac{\pi\cos(\pi z)}{\sin(\pi z)} = \lim_{z\to n} \frac{(z-n)\pi\cos(\pi z)}{\sin(\pi z) - \sin(\pi n)} = \frac{\pi\cos\pi n}{\pi \cos \pi n} = 1,$$

using the definition of the derivative of $\sin(\pi z)$ at n. If $u \notin \mathbb{Z}$, the function

$$f(z) = g(z) \frac{1}{(z+u)^2}$$

has poles at the integers and has an extra pole at $-u$. This is a double pole with residue

$$\text{Res}(f, -u) = \lim_{z\to -u} \frac{d}{dz} (z+u)^2 f(z) = \lim_{z\to -u} g'(z) = \lim_{z\to -u} \frac{-\pi^2}{\sin^2(\pi z)} = -\frac{\pi^2}{\sin^2(\pi u)}.$$

Moreover,

$$\operatorname{Res}(f,n) = \frac{1}{(n+u)^2}, \quad n \in \mathbb{Z}.$$

This formula is true for $n \neq 0$ and $u = 0$ as well.

There is a modification needed for $u = 0$: In this case $f(z)$ has a triple pole at 0. We have

$$\operatorname{Res}(f,0) = \frac{1}{2!}\lim_{z\to 0}\frac{d^2}{dz^2}z^3\frac{\pi\cot(\pi z)}{z^2} = \frac{\pi}{2}\lim_{z\to 0}\frac{z\cos\pi z}{\pi z - (\pi z)^3/6 + \cdots} = \frac{1}{2}\lim_{z\to 0}\frac{\cos\pi z}{1-\pi^2 z^2/6+\cdots}$$

using the Taylor series of $\sin\pi z$. The easiest way to proceed is to find the reciprocal the power series in the denominator:

$$\left(1 - \frac{\pi^2 z^2}{6} + \frac{\pi^4 z^4}{5!} - \cdots\right)^{-1} = 1 + \frac{\pi^2}{6}z^2 + \cdots$$

The technique is to assume that the reciprocal has a power series $\sum a_n z^n$ and match coefficients.

$$\left(1 - \frac{\pi^2 z^2}{6} + \frac{\pi^4 z^4}{5!} - \cdots\right)\left(a_0 + a_1 z + a_2 z^2 + \cdots\right) = 1$$

$$\Longrightarrow 1\cdot a_0 = 1, \quad 1\cdot a_1 + 0\cdot a_0 = 0, \quad 1\cdot a_2 - \frac{\pi^2}{6}a_0 = 0, \ldots$$

Now we multiply with the Taylor series of $\cos(\pi z)$ and differentiate twice to get

$$\operatorname{Res}(f,0) = \lim_{z\to 0}\frac{1}{2}\frac{d^2}{dz^2}\left(1 - \frac{\pi^2 z^2}{2} + \cdots\right)\left(1 + \frac{\pi^2 z^2}{6} + \cdots\right)$$

$$= \lim_{z\to 0}\frac{1}{2}\frac{d^2}{dz^2}\left(1 + \pi^2(-1/2 + 1/6)z^2 + \cdots\right) = \frac{1}{2}\cdot 2\pi^2(1/6 - 1/2) = -\frac{\pi^2}{3}.$$

The alternative would be successive differentiation of $z^3 f(z)$, which is not easier.

If we can show that the integrals over the boundary of the square S_R tend to 0, then the residue theorem gives ($R > |u|$):

$$\int_{\partial S_R} f(z)dz = 2\pi i\left(\sum_{|n|<R}\frac{1}{(n+u)^2} - \frac{\pi^2}{\sin^2(\pi u)}\right)$$

$$\Longrightarrow 0 = \sum_{n\in\mathbb{Z}}\frac{1}{(n+u)^2} - \frac{\pi^2}{\sin^2(\pi u)},$$

which gives the result for $u \notin \mathbb{Z}$. If $u = 0$, the residue theorem gives:

$$\int_{\partial S_R} f(z)dz = 2\pi i\left(\sum_{|n|<R, n\neq 0}\frac{1}{n^2} - \frac{\pi^2}{3}\right)$$

and finally

$$2\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{3}\Longrightarrow\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}.$$

On the line $z = N + 1/2 + iy$ we have

$$\left|\frac{\cos\pi z}{\sin\pi z}\right| = \left|\frac{\cos(N+1/2+iy)\pi}{\sin(N+1/2+iy)\pi}\right| = \left|\frac{\cos(N+1/2)\pi\cos(\pi iy)-\sin(N+1/2)\pi\sin(\pi iy)}{\sin(N+1/2)\pi\cos(\pi iy)+\sin(\pi iy)\cos(N+1/2)\pi}\right|$$

$$= \left|\frac{\sin\pi iy}{\cos\pi iy}\right| = \frac{|e^{-\pi y}-e^{\pi y}|}{e^{-\pi y}+e^{\pi y}}$$

which is bounded. On the line $z = x \pm i(N+1/2)$ we have

$$\left|\frac{\cos\pi z}{\sin\pi z}\right|^2 = \left|\frac{\cos(x\pm i(N+1/2))\pi}{\sin(x\pm i(N+1/2))\pi}\right|^2 = \left|\frac{\cos x\pi\cos(\pi i(N+1/2))\mp\sin x\pi\sin(\pi i(N+1/2))}{\sin x\pi\cos(\pi i(N+1/2))\pm\sin(\pi i(N+1/2))\cos x\pi}\right|^2$$

Since $\sin\pi i(N+1/2)$ is purely imaginary and

$$\cos\pi i(N+1/2) = \frac{e^{-\pi(N+1/2)}+e^{\pi(N+1/2)}}{2} = \cosh(\pi(N+1/2))$$

$$\sin\pi i(N+1/2) = \frac{e^{-\pi(N+1/2)}-e^{\pi(N+1/2)}}{2i} = \frac{-1}{i}\sinh(\pi(N+1/2))$$

we get

$$\left|\frac{\cos\pi z}{\sin\pi z}\right|^2 = \frac{\cos^2(x\pi)\cosh^2(\pi(N+1/2))+\sin^2(\pi x)\sinh^2(\pi(N+1/2))}{\sin^2(\pi x)\cosh^2(\pi(N+1/2))+\cos^2(x\pi)\sinh^2(\pi(N+1/2))}$$

and this expression is bounded for $-(N+1) \le x \le N+1/2$ and $N \to \infty$. Let K be an upper bound of $|\cot(\pi z)|$ on the sides of all the squares $S_R = S_{R_N}$. The length of the boundary of the square is $4 \cdot 2(N+1/2)$. This gives

$$\left|\int_{\partial S_{R_N}} f(z)dz\right| \le 8(N+1/2)\pi\frac{K}{(N+1/2-|u|)^2} \to 0, \quad N\to\infty,$$

since on ∂S_{R_N} we have $|z+u| \ge |z|-|u| \ge (N+1/2)-|u|$ (the closest points to the origin on the sides of the square are the points on the real and imaginary axes).

3. In this problem $\int_{c-i\infty}^{c+i\infty}$ denotes a contour integral along the vertical line $\Re(s) = c$ traversed upwards.

(a) Prove that for $c > 0$ we have $\displaystyle\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{x^s}{s^2}ds = \begin{cases}\log x, & x>1,\\ 0, & 0<x\le 1.\end{cases}$

For $x > 1$ we use the contour with the semicircle γ_R on the left in Figure 5. If R is sufficiently large, the contour contains the double pole at $s = 0$. We compute the residue:

$$\operatorname{Res}(f,0) = \lim_{s\to 0}\frac{d}{ds}\left(s^2\frac{x^s}{s^2}\right) = \lim_{s\to 0}x^s\log x = x^0\log x = \log x.$$

On γ_R we have
$$\left|\int_{\gamma_R} \frac{x^s}{s^2} ds\right| \leq \pi R \frac{1}{(R-c)^2} \to 0, \quad R \to \infty,$$
since on $\gamma_R(t) = c + Re^{it}$, $\pi/2 \leq t \leq 3\pi/2$, $|s^2| = |c + Re^{it}|^2 \geq (R-c)^2$ and $|x^s| = |e^{s \log x}| = e^{\log x \Re(s)} = e^{\log x (c + R\cos t)} \leq x^c$ since $\cos t \leq 0$ on the left semicircle, while $\log x \geq 0$. The choice of this contour depends exactly on this fact that. We apply the residue theorem gives
$$\int_{\gamma_R} \frac{x^s}{s^2} ds + \int_{c-iR}^{c+iR} \frac{x^s}{s^2} ds = 2\pi i \text{Res}(f,0) = 2\pi i \log x \Longrightarrow \int_{c-i\infty}^{c+i\infty} \frac{x^s}{s^2} ds = 2\pi i \log x,$$
by letting $R \to \infty$.

For $0 < x \leq 1$ we use the contour with the semicircle γ_R on the right in Figure 5. On γ_R we have
$$\left|\int_{\gamma_R} \frac{x^s}{s^2} ds\right| \leq \pi R \frac{1}{(R-c)^2} \to 0, \quad R \to \infty,$$
since on $\gamma_R(t) = c + Re^{it}$, $-\pi/2 \leq t \leq \pi/2$, $|s^2| = |c + Re^{it}|^2 \geq (R-c)^2$ and $|x^s| = |e^{s \log x}| = e^{\log x \Re(s)} = e^{\log x (c + R\cos t)} \leq x^c$ since $\cos t \geq 0$ on the right semicircle, while $\log x \leq 0$. The choice of this contour depends exactly on this fact that.

Inside the contour there is no pole, so Cauchy's theorem gives
$$\int_{\gamma_R} \frac{x^s}{s^2} ds + \int_{c-iR}^{c+iR} \frac{x^s}{s^2} ds = 0 \Longrightarrow \int_{c-i\infty}^{c+\infty} \frac{x^s}{s^2} ds = 0,$$
by taking $R \to \infty$.

(b) Prove that, for $c > 0$, $\displaystyle \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{x^s}{s} ds = \begin{cases} 1, & x > 1, \\ 1/2, & x = 1, \\ 0, & 0 < x < 1. \end{cases}$ (Perron formula)

We use the same contours as in (a) for the same x with the exception of $x = 1$. However, the argument is a bit trickier because we have one power less in the denominator.

For $x > 1$, inside the contour there is a pole at $s = 0$, which is simple:
$$\text{Res}(f,0) = \lim_{s \to 0} s \frac{x^s}{s} = x^0 = 1.$$

We need to control the integral on γ_R. As above, on the left semicircle $s(t) = c + Re^{it}$, $\pi/2 \leq t \leq 3\pi/2$.
$$|x^s| = |e^{s \log x}| = e^{\log x (c + R\cos t)}$$

We also remark that the inequality $\sin y \geq 2y/\pi$ holds for $0 \leq y \leq \pi/2$. This follows from the concavity of $\sin y$ on $[0, \pi/2]$. The secant line $2y/\pi$ from $(0,0)$ to $(\pi/2, 1)$ is below the graph. Now
$$\left|\int_{\gamma_R} \frac{x^s}{s} ds\right| = \left|\int_{\pi/2}^{3\pi/2} \frac{x^c x^{R\cos t}}{c + Re^{it}} iRe^{it} dt\right| = \left|\int_0^{\pi} \frac{x^c x^{-R\sin y}}{c + Re^{i(y+\pi/2)}} Re^{iy} dy\right| \leq \int_0^{\pi} \frac{x^c x^{-R\sin y}}{R-c} R dy$$

with the substitution $t = y + \pi/2$. The last integral can be split into two equal integrals over $[0, \pi/2]$ and $[\pi/2, \pi]$, since $\sin y$ takes the same values in both. We get

$$\left|\int_{\gamma_R} \frac{x^s}{s} ds\right| \leq 2\int_0^{\pi/2} \frac{Rx^c x^{-R\sin y}}{R-c} dy \leq 2\int_0^{\pi/2} \frac{Rx^c x^{-R2y/\pi}}{R-c} dy = \frac{Rx^c}{R-c}\left[\frac{x^{-2Ry/\pi}}{-R(\log x)2/\pi}\right]_0^{\pi/2}$$

$$= \frac{\pi x^c}{(R-c)(\log x)2}\left(-x^{-R}+1\right) \to 0, \quad R \to \infty,$$

as $x > 1$.

For $0 < x < 1$ the parametrization of the right semicircle is $s(t) = c + Re^{it}$, $-\pi/2 \leq t \leq \pi/2$. We substitute $t = y - \pi/2$:

$$\left|\int_{\gamma_R} \frac{x^s}{s} ds\right| = \left|\int_{-\pi/2}^{\pi/2} \frac{x^c x^{R\cos t}}{c + Re^{it}} iRe^{it} dt\right| \leq \int_{-\pi/2}^{\pi/2} R\frac{x^c x^{R\cos t}}{R-c} dt = 2\int_0^{\pi/2} \frac{Rx^c x^{R\sin y}}{R-c} dy$$

Now $x < 1$, so $\log x < 0$ and

$$x^{R\sin y} = e^{\log x R \sin y} \leq e^{\log x R2y/\pi} = x^{2Ry/\pi}.$$

$$\left|\int_{\gamma_R} \frac{x^s}{s} ds\right| \leq 2\int_0^{\pi/2} \frac{Rx^c x^{R2y/\pi}}{R-c} dy = \frac{Rx^c}{R-c}\left[\frac{x^{2Ry/\pi}}{R(\log x)2/\pi}\right]_0^{\pi/2}$$

$$= \frac{\pi x^c}{(R-c)(\log x)2}\left(x^R - 1\right) \to 0, \quad R \to \infty,$$

as $x < 1$.

For $x = 1$ we compute the integral directly:

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{1}{s} ds = \frac{1}{2\pi i}\int_{\infty}^{\infty} \frac{idt}{c+it} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{c}{c^2+t^2} - \frac{it}{c^2+t^2} dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{c}{c^2+t^2} dt$$

as the function $t/(c^2+t^2)$ is odd. This integral is elementary: substitute $t = cu$ to get

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{1}{s} ds = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{c \cdot cdt}{c^2 + c^2u^2} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{du}{1+u^2} = \left[\frac{\arctan(u)}{2\pi}\right]_{-\infty}^{\infty} = \frac{1}{2\pi}\pi = \frac{1}{2}.$$

(c) Let the function $f(s)$ be defined by the absolutely convergent series

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}, \quad \Re(s) > a \geq 0.$$

Show that for $x \notin \mathbb{Z}$

$$\sum_{n \leq x} a_n = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} f(s)\frac{x^s}{s} ds, \quad c > a.$$

Since $|n^s| = n^{\Re(s)}$ and the series converges absolutely for $\Re(s) > a$, we have that the series

$$\sum_{n=1}^{\infty} \frac{|a_n|}{n^{\Re(s)}} < \infty, \quad \Re(s) > a.$$

This implies that, if we fix $\Re(s) = c > a$, then the convergence of the series in the s variable is uniform (Weierstraß test). Moreover, on the vertical line $\Re(s) = c$ we have

$$\left|\frac{x^s}{s}\right| = \frac{x^c}{|s|},$$

which is bounded so we get uniform convergence of the series even when multiplied by x^s/s. This means we can interchange summation and integration to get

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} f(s)\frac{x^s}{s}ds = \sum_{n=1}^{\infty} a_n \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{(x/n)^s}{s}ds = \sum_{n=1}^{\infty} a_n \left\{ \begin{array}{ll} 0, & x < n, \\ 1, & x > n \end{array} \right\} = \sum_{n<x} a_n.$$

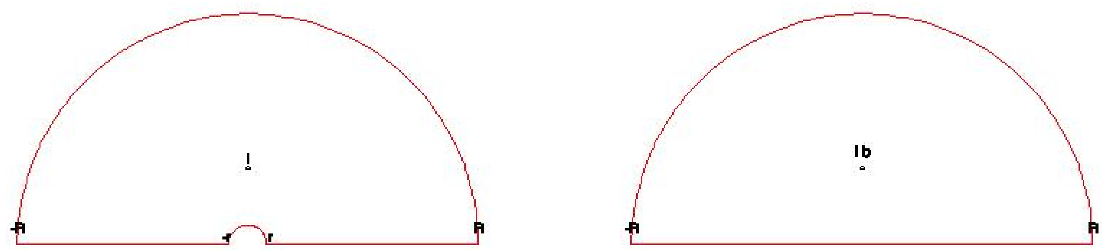

Figure 2: Contours for (iii) and (iv)

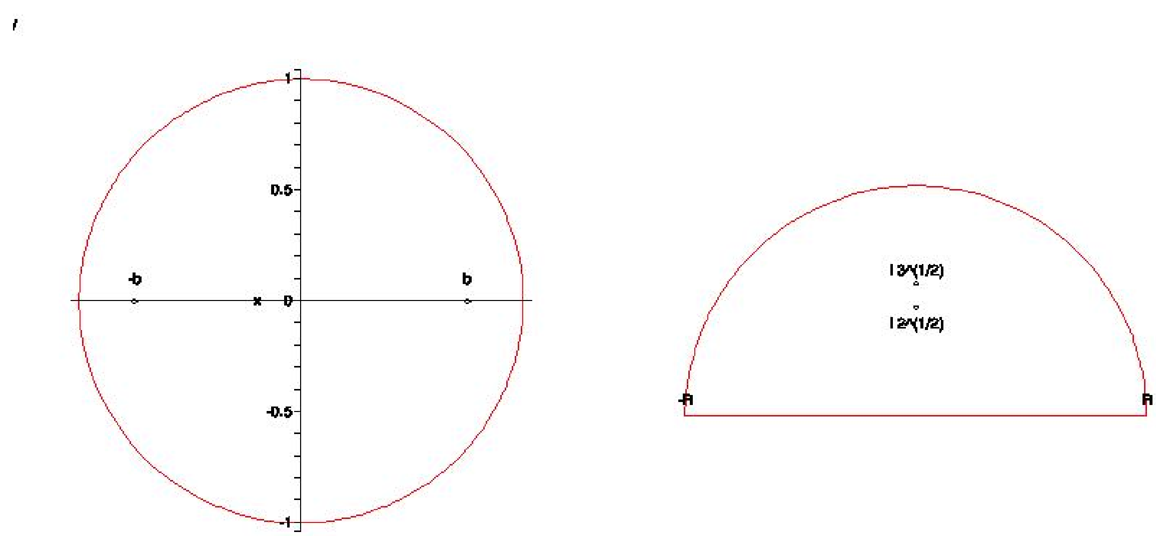

Figure 3: Contours for (v) and (vi)

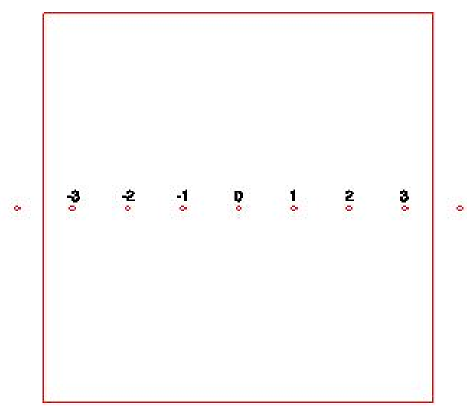

Figure 4: Contour for Problem 2

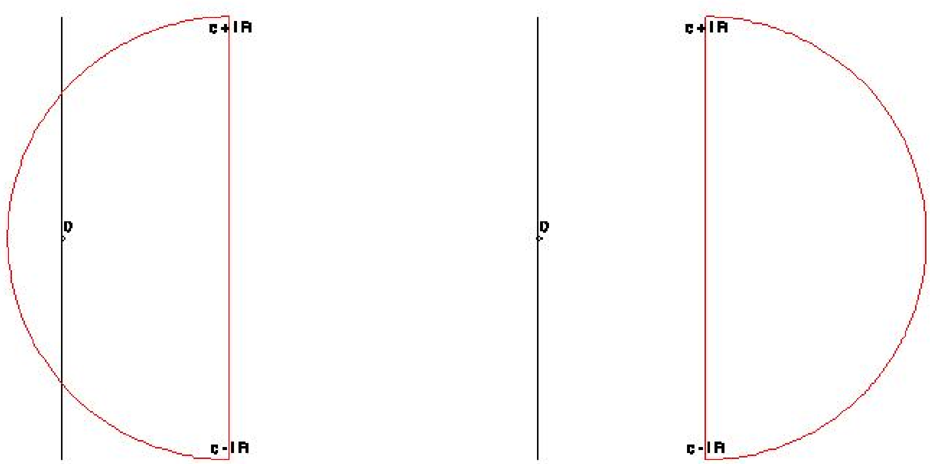

Figure 5: Contours for Problem 3: on the left $x > 1$, on the right $x < 1$.