# Math 70300

## Homework 4

**Due: within 72 hours**

1. (a) Let $z_1$ and $z_2$ be two points on a circle $C$. Let $z_3$ and $z_4$ be symmetric with respect to the circle. Show that the cross ratio $(z_1, z_2, z_3, z_4)$ has absolute value 1.

   (b) Let $ad - bc = 1$, $c \neq 0$ and consider $T(z) = \dfrac{az+b}{cz+d}$. Show that it increases lengths and areas inside the circle $|cz + d| = 1$ and decreases lengths and areas outside the circle $|cz + d| = 1$.

2. (a) By considering the contour integral

$$\int_{|z|=1} \left(z + \frac{1}{z}\right)^{2n} \frac{dz}{z}$$

   prove that

$$\int_0^{2\pi} \cos^{2n} \theta \, d\theta = 2\pi \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)}.$$

   (b) Prove that

$$\int_0^{\pi/2} \sin^{2n} \theta \, d\theta = \frac{\pi}{2} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)}.$$

3. Map conformally the region inside both circles $|z - 1| < 1$ and $|z + i| < 1$ to the upper-half plane $\mathbb{H} = \{z \in \mathbb{C}, \Im(z) > 0\}$.

   *Hint:* Use first the map $w = z^{-1}$ and at some later stage use $w = z^2$.

4. Let $f(z)$ be holomorphic on the unit disc and $f(0) = 1$. By working with

$$\frac{1}{2\pi i} \int_{|z|=1} \left[2 \pm (z + \frac{1}{z})\right] f(z) \frac{dz}{z}$$

   prove that

$$\frac{2}{\pi} \int_0^{2\pi} f(e^{i\theta}) \cos^2 \frac{\theta}{2} \, d\theta = 2 + f'(0), \quad \frac{2}{\pi} \int_0^{2\pi} f(e^{i\theta}) \sin^2 \frac{\theta}{2} \, d\theta = 2 - f'(0).$$

   (b) If $f(z)$ is holomorphic on $|z| \leq 1$, $f(0) = 1$, and for all $|z| \leq 1$ we have $\Re(f(z)) \geq 0$, then $-2 \leq \Re(f'(0)) \leq 2$.

5. Let $f(z)$ be holomorphic in the region $|z| \leq R$ with power series expansion $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Let the partial sum of the series be defined as

$$s_N(z) = \sum_{n=0}^{N} a_n z^n.$$

Show that for $|z| < R$ we have

$$s_N(z) = \frac{1}{2\pi i} \int_{|w|=R} f(w) \frac{w^{N+1} - z^{N+1}}{w - z} \frac{dw}{w^{N+1}}.$$

6. Let $C$ be a circle enclosing the distinct points $z_1, z_2, \ldots z_n$. Let

$$p(z) = (z - z_1)(z - z_2) \cdots (z - z_n)$$

be (the) polynomial of degree $n$ with roots at these points. Let $f(z)$ be holomorphic in a disc that includes $C$. Show that

$$P(z) = \frac{1}{2\pi i} \int_C \frac{f(w)}{p(w)} \frac{p(w) - p(z)}{w - z} dw$$

is a polynomial of degree $n - 1$, with the property

$$P(z_i) = f(z_i), \quad i = 1, 2, \ldots n.$$

7. Let $f(z)$ be holomorphic on $|z| < 1$ and $|f(z)| < \dfrac{1}{1 - |z|}$ for $|z| < 1$. Show that the Taylor coefficients $a_n$ of $f(z)$ satisfy

$$|a_n| \leq (n + 1) \left(1 + \frac{1}{n}\right)^n < e(n + 1).$$

8. Calculate the Fresnel integrals

$$\int_0^{\infty} \cos(x^2)\, dx = \int_0^{\infty} \sin(x^2)\, dx = \sqrt{2\pi}/4.$$

*Hint:* Consider the function $f(z) = e^{iz^2}$ and a contour that includes the line segment from 0 to $Re^{i\pi/4}$ and the circular arc from $R$ to $Re^{i\pi/4}$.

9. Let $f(z) = u + iv$ be an analytic function, $\psi(u, v)$ any function with second order partial derivatives and $g(u, v)$ any function with first partial derivatives.

(a) Let $\Delta_{x,y}$ be the Laplace operator in $x, y$ coordinates, i.e. $\Delta_{x,y} = \partial_x^2 + \partial_y^2$, and $\Delta_{u,v}$ be the Laplace operator in $u, v$ coordinates, i.e. $\Delta_{u,v} = \partial_u^2 + \partial_v^2$. Show that

$$\Delta_{x,y}(\psi \circ f) = \Delta_{u,v}\psi \cdot |f'(z)|^2.$$

(b) Let $\nabla_{u,v} = \partial_u \mathbf{i} + \partial_v \mathbf{j}$ be the gradient vector in $u, v$, and $\nabla_{x,y} = \partial_x \mathbf{i} + \partial_y \mathbf{j}$ the gradient vector in $x, y$ coordinates. Show that

$$|\nabla_{x,y}(g \circ f)|^2 = |\nabla_{u,v} g|^2 \cdot |f'(z)|^2,$$

where $|\cdot|$ is the euclidean norm in $\mathbb{C}$.