Math 70300

Homework 3

Due: October 19, 2006

1. Let $f(z)$ be an analytic function with nonzero derivative. Let $f(z) = u(x, y) + iv(x, y)$ and consider the level curves of u and v, i.e., the sets

$$\{z = x + iy \in \mathbb{C} : u(x, y) = u_0\}, \quad \{z = x + iy \in \mathbb{C} : v(x, y) = v_0\}$$

for fixed numbers u_0, v_0. Prove that the set of level curves of u and the set of level curves of v are orthogonal to each other.

If $\gamma(t) = x(t) + iy(t)$ is a curve of the level set $u(x, y) = u_0$, then the chain rule gives at $t = t_0$, $x_0 + iy_0 = \gamma(t_0)$

$$u_x(x_0, y_0)x'(t_0) + u_y(x_0, y_0)y'(t_0) = 0,$$

which gives that the slope of the tangent vector is $y'(t_0)/x'(t_0) = -u_x(x_0, y_0)/u_y(x_0, y_0)$.

Similarly if $z(t) = a(t) + ib(t)$ is a curve of the level set $v(x, y) = v_0$, then the chain rule gives at $t = t_0$, $a_0 + ib_0 = z(t_0)$

$$v_x(a_0, b_0)a'(t_0) + v_y(a_0, b_0)b'(t_0) = 0,$$

which gives that the slope of the tangent vector is $b'(t_0)/a'(t_0) = -v_x(a_0, b_0)/v_y(a_0, b_0)$.

At a common point of the two level curves $a_0 = x_0$ and $y_0 = b_0$ the Cauchy-Riemann equations give:

$$u_x(a_0, b_0) = v_y(a_0, b_0), \quad u_y(a_0, b_0) = -v_x(a_0, b_0).$$

Then the product of the slopes of the tangent vectors is

$$\lambda_1\lambda_2 = -\frac{u_x(a_0, b_0)}{u_y(a_0, b_0)} \cdot \frac{-v_x(a_0, b_0)}{v_y(a_0, b_0)} = -1.$$

So the curves are orthogonal at (a_0, b_0).

Second method: One needs to know the local mapping theorem, that in the case of nonzero derivative implies that the holomorphic map is locally one-to-one with inverse function analytic, therefore conformal. So let $f : U \to V$ be one-to-one. Notice that

$$(f^{-1})'(w) = \frac{1}{f'(z)}, \quad f(z) = w.$$

We apply this to the point of intersection of two level curves (a_0, b_0). Then

$$\{z = x + iy \in U : u(x, y) = u_0\} = f^{-1}(\{w, \Re w = u_0, w \in V\}),$$

$$\{z = x + iy \in U : v(x, y) = v_0\} = f^{-1}(\{w, \Im w = v_0, w \in V\}).$$

Since the two lines are perpendicular their image under the conformal map f^{-1} are orthogonal.

2. (a) Let z_1, z_2, z_3, z_4 lie on a circle. Show that z_1, z_3, z_4 and z_2, z_3, z_4 determine the same orientation iff $(z_1, z_2, z_3, z_4) > 0$.

We can find a linear fractional transformation T with $T(z_2) = 0, T(z_3) = 1, T(z_4) = 2$, so that $(z_1, z_2, z_3, z_4) = (T(z_1), T(z_2), T(z_3), T(z_4))$ by the invariance of the cross ratio. By definition

$$(T(z_1), T(z_2), T(z_3), T(z_4)) = \frac{Tz_1 - Tz_3}{Tz_1 - Tz_4} : \frac{Tz_2 - Tz_3}{Tz_2 - Tz_4},$$

In this way Tz_2, Tz_3, Tz_4 determine an orientation from left to right, so that

$$\frac{Tz_2 - Tz_3}{Tz_2 - Tz_4} > 0$$

as quotient of negative numbers. The cross ratio (z_1, z_2, z_3, z_4) is positive iff

$$\frac{Tz_1 - Tz_3}{Tz_1 - Tz_4} > 0$$

i.e. either both numerator and denominator are positive or both are negative. In the second case Tz_1 is left from Tz_3, which is left from Tz_4, i.e. the orientation is the same, while in the first case

$$Tz_1 > Tz_4 > Tz_3.$$

so we go from Tz_1 to infinity to Tz_3 to Tz_4, which is the same orientation. The l.f.t. T preserves the orientation, so the result follows.

(b) Let z_1, z_2, z_3, z_4 lie on a circle and be the consecutive vertices of a quadrilateral. Prove that

$$|z_1 - z_3| \cdot |z_2 - z_4| = |z_1 - z_2| \cdot |z_3 - z_4| + |z_2 - z_3| \cdot |z_1 - z_4|.$$

Interpret the result geometrically.

We know that $(z_1, z_2, z_3, z_4) > 0$ by (a) and the choice of labeling of the consecutive vertices of the quadrilateral. Also we know that $(z_1, z_3, z_2, z_4) < 0$, since the orientation of z_3, z_2, z_4 is the opposite of z_1, z_2, z_4. This gives

$$\frac{z_1 - z_2}{z_1 - z_4}\frac{z_3 - z_4}{z_3 - z_2} < 0 \Leftrightarrow \frac{z_1 - z_2}{z_2 - z_3}\frac{z_3 - z_4}{z_1 - z_4} > 0.$$

The equality

$$(z_1 - z_3)(z_2 - z_4) = (z_1 - z_2)(z_3 - z_4) + (z_2 - z_3)(z_1 - z_4)$$

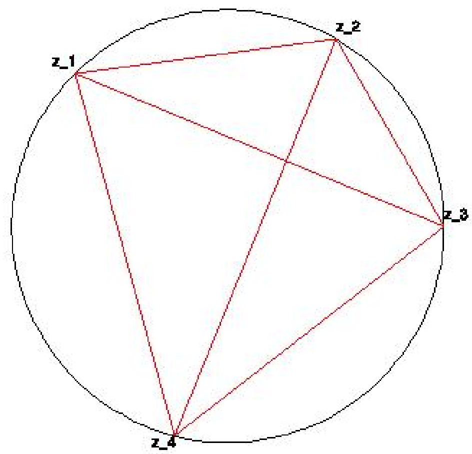
follows from elementary algebra, as the left-hand side is $z_1 z_2 - z_1 z_4 - z_3 z_2 + z_3 z_4$, while the right-hand side is $z_1 z_3 - z_2 z_3 - z_1 z_4 + z_2 z_4 + z_2 z_1 - z_2 z_4 - z_3 z_1 + z_3 z_4$. In this identity we divide by $(z_1 - z_4)(z_2 - z_3)$ to get

$$\frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)} = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_2 - z_3)} + 1.$$

Since the fractions represent (z_1, z_2, z_3, z_4) and $-(z_1, z_3, z_2, z_4)$ which are both positive, this equality gives an equality of absolute values:

$$\frac{|(z_1 - z_3)(z_2 - z_4)|}{|(z_1 - z_4)(z_2 - z_3)|} = \frac{|(z_1 - z_2)(z_3 - z_4)|}{|(z_1 - z_4)(z_2 - z_3)|} + 1$$

and this is equivalent to the result.



The geometric interpretation is that the product of the diagonals of the quadrilateral is equal to the sum of the products of the opposite sides.

3. Let $T(z) = \dfrac{az+b}{cz+d}$. Assume that it maps the real line to the real line. Show that we can choose a, b, c, d to be real numbers. The converse is obvious.

   Set $\mu = T(0) = b/d$, $\nu = T(\infty) = a/c$ and $s = T(1) = (a+b)/(c+d)$, which are real by assumption. This gives

$$a + b = s(c + d) = \nu c + \mu d \Longrightarrow (s - \nu)c = (\mu - s)d.$$

   If $(s-\nu)(\mu-s) \neq 0$, then $c = d(\mu - s)/(s - \nu) = \rho d$, with ρ real. Also $a = \nu c = \nu\rho d$, $b = \mu d$. These imply

$$Tz = \frac{az+b}{cz+d} = \frac{\nu\rho dz + \mu d}{\rho dz + d} = \frac{\nu\rho z + \mu}{\rho z + 1},$$

   which has real coefficients. If $s = \nu = \mu$, $b = sd$, $a = sc$ and

$$Tz = \frac{az+b}{cz+d} = \frac{scz + sd}{cz+d} = s \in \mathbb{R}.$$

   The case $d = 0$ is even easier: $Tz = (a/d)z + (b/d)$ with $b/d \in \mathbb{R}$ ($T(0)$ is real), and $T(1) = a/d + b/d \in \mathbb{R}$. So $a/d \in \mathbb{R}$.

4. (a) Let $-\infty < a < b < \infty$ and set $M(z) = \dfrac{z - ia}{z - ib}$. Define the lines $L_1 = \{z : \Im(z) = b\}$, $L_2 = \{z : \Im(z) = a\}$ and $L_3 = \{z : \Re(z) = 0\}$. The three lines split the complex plane into 6 regions. Determine the image of them in the complex plane.

Since ib is mapped to ∞ the lines L_1, L_3 are mapped to lines, while the line L_2 is mapped to the circle through $0 = M(ia)$ and $1 = M(\infty)$. This is the circle $C : |w - 1/2| = 1/2$. So $C = M(L_2)$. Since $M(ia) = 0$ the image of L_3 should contain 0, so it is the horizontal axis. This leaves $M(L_1) = \{w \in \mathbb{C} : \Re(w) = 1\}$.

We need to determine also the orientation of traversing the images, as we traverse the three lines on the z-plane. We check that $M(i(a + b)/2) = -1$, so as we go on the line L_3 from ia towards ib we traverse the horizontal axis from 0 to ∞ going left. At ia the lines L_2 and L_3 intersect perpendicularly. If we go from left to right on L_2, we should go from left to right w.r.t. $M(L_3)$ on $M(L_2)$. So we traverse $C = M(L_2)$ clockwise. If we go from left to right on L_1, i.e. $z = x + ib$, then $M(z) = 1 + i(b-a)/x$, which that for $x > 0$, $M(z)$ lies on the upper-half plane, while for $x < 0$ on the lower half-plane. We take into account that ∞ is mapped to 1, so as we go from left to right on L_1, we must traverse $M(L_1)$ from up to down. We now follow the directions, making sure that the regions on the left (resp. right) of a line are mapped to the left (resp. right) of the image curve and we get figure

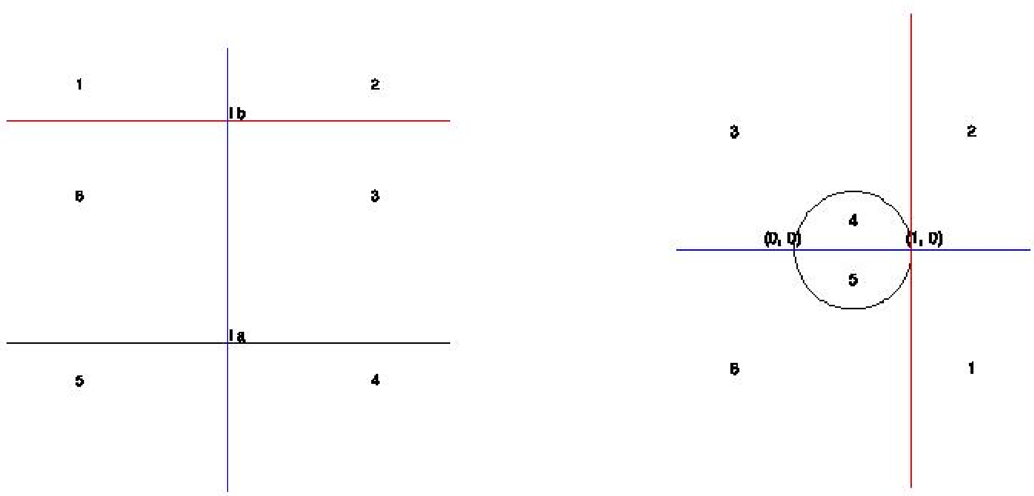


Figure 1: The regions with the same number are mapped to each other

(b) Let log be the principal branch of the logarithm. Show that $\log(M(z))$ is defined for all $z \in \mathbb{C}$ with the exception of the line segment from ia to ib.

From the figure we see that outside the segment from ia to ib the value $M(z)$ is not a negative real number. So the principal branch of log can be applied to Mz.

(c) Define $h(z) = \Im(\log(M(z)))$ for $\Re(z) > 0$. Show that h is harmonic and that $0 < h(z) < \pi$.

The imaginary part of the holomorphic function $\log M(z)$ is a harmonic function. For $\Re(z) > 0$ we get that Mz lies on the upper-half plane, consequently the imaginary part of $\log Mz$, which is the argument, is in $(0, \pi)$ (use of the principal branch of log).

(d) Show that $\log(z - ic)$ is defined for $\Re(z) > 0$ and any real number c. Prove that $|\Im(\log(z - ic))| < \pi/2$ in this region.

We have for $\Re(z) > 0$ and c real that $z - ic$ is not a negative number, as its real part $\Re(z - ic)$ is positive. So we can define $\log(z - ic)$ for the principal branch of the logarithm. Moreover, since $\Re(z - ic) > 0$, its argument is in $(-\pi/2, \pi/2)$.

(e) Prove that $h(z) = \Im(\log(z - ia) - \log(z - ib))$.

We have $h(z) = \Im(\log M(z)) = \arg M(z) = \arg(z - ia) - \arg(z - ib)$ up to $2\pi ik$, $k \in \mathbb{Z}$.

By (d) $|\arg(z - ia)| < \pi/2$ and $|\arg(z - ib)| < \pi/2$. So when we subtract the arguments we get an angle in $(-\pi, \pi)$, i.e. the principal argument, and this agrees with the argument of $\log M(z)$ by (c), as this is the principal argument as well. Consequently we do not have to add $2\pi ik$.

(f) Use the fundamental theorem of calculus to show that

$$\int_a^b \frac{dt}{z - it} = i(\log(z - ib) - \log(z - ia)).$$

We have

$$\int_a^b \frac{dt}{z - it} = \left[\frac{\log(z - it)}{-i}\right]_a^b = i(\log(z - ib) - \log(z - ia)).$$

(g) Combine (e) and (f) to show that

$$h(x + iy) = \int_a^b \frac{xdt}{x^2 + (y - t)^2} = \arctan((y - a)/x) - \arctan((y - b)/x).$$

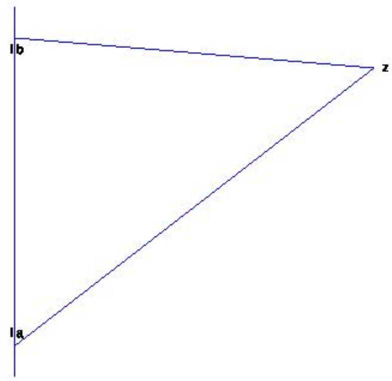
$$h(x + iy) = \Im(\log(z - ia) - \log(z - ib)) = \Im(i \int_a^b \frac{dt}{z - it})$$

$$= \Re \int_a^b \frac{dt}{z - it} = \Re \int_a^b \frac{dt}{x + i(y - t)} = \Re \int_a^b \frac{xdt}{x^2 + (y - t)^2} - i\frac{y - t}{x^2 + (y - t)^2}dt = \int_a^b \frac{xdt}{x^2 + (y - t)^2}.$$

Using the substitution $t - y = xu$ we get

$$\int_a^b \frac{xdt}{x^2 + (y - t)^2} = \int_{(a-y)/x}^{(b-y)/x} \frac{x^2du}{x^2 + x^2u^2} = \int_{(a-y)/x}^{(b-y)/x} \frac{du}{1 + u^2} = \arctan((b-y)/x) - \arctan((a-y)/x).$$

It remains to use the fact that arctan is an odd function.



(h) Interpret (g) geometrically by showing that $h(z)$ measures (with sign) the interior angle of the triangle with vertices ia, ib and z at the vertex z. What are the limits of $h(z)$ as $\Re(z) \to 0$ for $\Im(z) \in (a,b)$ and for $\Im(z) \notin [a,b]$?

We have that $\arg(z - ia)$ represent the angle from the positive real axis towards the ray from ia to z, while $\arg(z - ib)$ represent the angle from the positive real axis towards the ray from ib to z. The oriented angle with vertex z in the counterclockwise direction represents $h(z)$: Use the triangle with vertices ia, ib and z to see that $\hat{z} = \pi - (\pi/2 - \arg(z - ia)) - (\pi/2 + \arg(z - ib)) = \arg(z - ia) - \arg(z - ib)$. Pay attention to the $\pm$ according to the positive or negative orientation for the angles.

We have

$$\lim_{\Re z \to 0} h(z) = \pi, \quad \Im(z) \in (a,b), \qquad \lim_{\Re z \to 0} h(z) = 0, \quad \Im(z) \notin [a,b].$$

5. Suppose that C_1 and C_2 are two circles with real centers, tangent to each other at $a \in \mathbb{R}$. Assume that the one is contained inside the other. Call G the region between the two circles. Map conformally G to the unit disc $\mathbb{D}$.

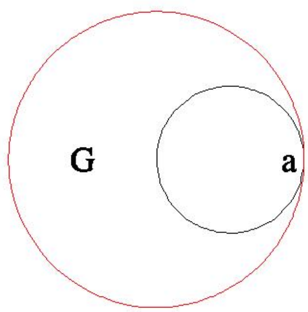
   Hint: First try $(z - a)^{-1}$.

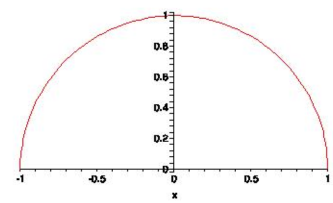
   We can assume that $a > 0$ and that the center of one circle is at 0. Let c be the point where the small circle meets the real line. The real line is mapped to itself by $T(z) = (z - a)^{-1}$. Since C_1 and C_2 are perpendicular to $\mathbb{R}$ at $-a$ and c respectively, the images by T, i.e. $T(C_1)$ and $T(C_2)$ are perpendicular to $\mathbb{R}$. Also they contain ∞ as $T(a) = \infty$. So they are lines perpendicular to $\mathbb{R}$ at $-1/(2a) = T(-a)$, $1/(c - a) = T(c)$ respectively. Then $T(G)$ is the region between the two parallel line. We map it conformally to a horizontal strip of width π around the real axis. This is done by a linear map

$$U(w_1) = i(\pi/2 + l(w_1 + 1/(2a))), \quad l = \frac{\pi}{-1/(2a) - 1/(c-a)}.$$

Here l is the slope of a line that maps $-1/(2a)$ to $\pi/2$ and $1/(c-a)$ to $-\pi/2$. The factor i rotates the picture. As in Ahlfors p. 93, $S(z) = \tanh(z/2)$ maps the horizontal strip $|\Im z| < \pi/2$ to the unit disc. The composition $S \circ U \circ T$ is the desired map.

Alternatively, $\exp(z)$ maps the two parallel lines $\Im(z) = \pm\pi/2$ to the rays $i[0,\infty)$ and $i[0,-\infty)$ on the imaginary axis. The horizontal strip is mapped to the right-half plane, as $\exp(0) = 1$. Now use $V(z) = (z-1)/(z+1)$ to map the right-half plane to the unit disc. The desired map is $V \circ \exp \circ U \circ T$.





6. Let Ω be the upper half of the unit disc $\mathbb{D}$. Find a conformal mapping $f : \Omega \to \mathbb{D}$ that maps $\{-1, 0, 1\}$ to $\{-1, -i, 1\}$. Find $z \in \Omega$ with $f(z) = 0$.

   Hint: $f = T_1 \circ S \circ T_2$, where T_i are linear fractional transformations and $S(z) = z^2$.

   We start with the map $U(z) = 1/(z+1)$. The idea is to map -1 to ∞. Since U maps the real line to the real line and this is perpendicular to the circle $|z| = 1$ at -1 and 1, $U(\partial D)$ is a line perpendicular to the real axis at $U(1) = 1/2$. Since $U(0) = 1$ we easily check that $U(\Omega) = \{z : \Re(z) > 1/2), \Im(z) < 0\}$ (a quarter plane). We map $U(\Omega)$ to the first quadrant by the map $V(w_1) = 2i(w_1 - 1/2)$ (shift by $-1/2$, rotate by $\pi/2$ and scale by 2 to make things easier later), so that

$$T_2(z) = V \circ U(z) = 2i\left(\frac{1}{z+1} - \frac{1}{2}\right)$$

   maps Ω to the first quadrant and

$$T_2(-1) = \infty, \quad T_2(0) = i, \quad T_2(1) = 0.$$

   The map $S(z) = z^2$ doubles the argument so it maps the first quadrant to the upper half-plane. We have

$$S \circ T_2(-1) = \infty, \quad S \circ T_2(0) = -1, \quad S \circ T_2(1) = 0.$$

   Now we use a map from the upper-half-plane to the unit disc that send the points ∞, -1 and 0 to -1, $-i$, 1. This is

$$T_1(w) = -\frac{w-i}{w+i}.$$

The desired map is $f = T_1 \circ S \circ T_1$. We solve the equation $f(z) = 0$. First $T_1(w) = 0$ gives $w = i$. Now we solve $S \circ T_1(z) = i$. We get $T_1(z) = e^{\pi i/4}$. Finally

$$T_1(z) = e^{\pi i/4} \Leftrightarrow 2i\left(\frac{1}{z+1} - 1/2\right) = e^{\pi i/4} \Leftrightarrow \frac{1}{z+1} = \frac{e^{\pi i/4}}{2i} + \frac{1}{2} = \frac{1}{2}e^{-\pi i/4} + \frac{1}{2}$$

$$\Leftrightarrow z+1 = \frac{2}{e^{-\pi i/4}+1} \Leftrightarrow z = -1 + \frac{2}{e^{-\pi i/4}+1} = \frac{1-e^{-\pi i/4}}{1+e^{-\pi i/4}} = \frac{(1-e^{-\pi i/4})(1+e^{\pi i/4})}{|1+e^{-\pi i/4}|^2}$$

$$z = \frac{-e^{-\pi i/4}+e^{\pi i/4}}{(1+1/\sqrt{2})^2+1/2} = \frac{2i/\sqrt{2}}{2+2/\sqrt{2}} = \frac{i}{\sqrt{2}+1}.$$

7. Let z and z' be points in $\mathbb{C}$ with corresponding points on the unit sphere Z and Z' by stereographic projection. Let N be the north pole $N(0,0,1)$.

(a) Show that Z and Z' are diametrically opposite on the unit sphere iff $z\bar{z}' = -1$.

Let $Z(x_1, x_2, x_3)$, $Z'(x_1', x_2', x_3')$, where (see Ahlfors p. 18, Conway p. 9)

$$x_1 = \frac{z+\bar{z}}{1+|z|^2}, x_2 = \frac{-i(z-\bar{z})}{1+|z|^2}, x_3 = \frac{|z|^2-1}{|z|^2+1},$$

$$x_1' = \frac{z'+\bar{z}'}{1+|z'|^2}, x_2' = \frac{-i(z'-\bar{z}')}{1+|z'|^2}, x_3' = \frac{|z'|^2-1}{|z'|^2+1}.$$

The points Z and Z' are diametrically opposite on the unit sphere iff $Z = -Z'$ which gives

$$\frac{|z'|^2-1}{|z'|^2+1} = -\frac{|z|^2-1}{|z|^2+1} \Leftrightarrow (|z|^2-1)(1+|z'|^2) = -(|z|^2+1)(|z'|^2-1) \Leftrightarrow$$

$$|z|^2|z'|^2 - |z'|^2 + |z|^2 - 1 = -|z|^2|z'|^2 - |z'|^2 + |z|^2 + 1 \Leftrightarrow 2|z|^2|z'|^2 = 2 \Leftrightarrow |z'| = |z|^{-1}.$$

Now

$$x_1 = -x_1' \Leftrightarrow \frac{2\Re z}{1+|z|^2} = -\frac{2\Re z'}{1+|z|^{-2}} = -\frac{|z|^2\Re z'}{|z|^2+1} \Leftrightarrow \Re z = -\Re z'|z|^2,$$

and similarly

$$x_2 = -x_2' \Leftrightarrow \Im z = -\Im z'|z|^2.$$

The last two equations give

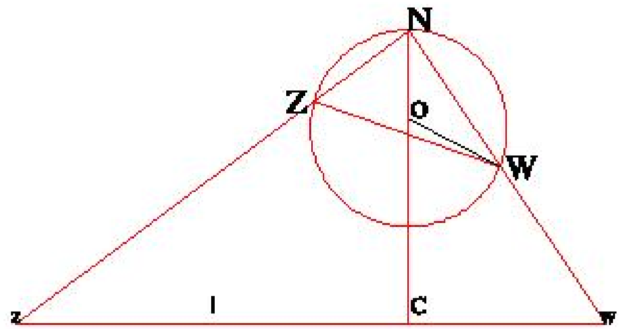
$$z = -z'|z|^2.$$

Now

$$z\bar{z}' = -z'\bar{z}'|z|^2 = -|z'|^2|z|^2 = -1.$$

(b) Show that the triangles $Nz'z$ and NZZ' are similar. The order of the vertices is important and is as given. Use this to derive the formula for the euclidean distance in $\mathbb{R}^3$

$$d(Z, Z') = \frac{2|z-z'|}{\sqrt{1+|z|^2}\sqrt{1+|z'|^2}}.$$



Note: Ahlfors and Conway denote this distance $d(z, z')$.

We change notation so that $z' = w$ and $Z' = W$. The three points N, z and w define a plane P that intersects the Riemann sphere. This intersection has to be a circle, the one passing through N, Z, W. If we fix w but move z on the line l, which is the intersection of the plane P and the complex plane, the slanted plane P does not change position, and the circle remains the same. The effect moves the stereographic projection Z on the circle. The further away we move z, the closer to N the point Z gets. If z gets the closest to the circle as possible while moving on the line l (this means at C), then Z has to be as far away from N as possible, this means diametrically opposite to N. In this case ZN is a diameter. The point C is then the foot of the height of the triangle. This means that the center of the circle lies on the height from N to the line l.

Since the angle NZW extends on the arc NW, we have

$$\widehat{NZW} = \frac{1}{2}\widehat{NOW}.$$

As complimentary angles we get

$$\widehat{NwC} = \pi/2 - \widehat{WNO}$$

while

$$\widehat{WNO} = \widehat{NWO}.$$

They all give

$$\widehat{WON} = \pi - 2\widehat{ONW} = \pi - 2(\pi/2 - \widehat{WwC}) = 2\widehat{WwC}.$$

This implies

$$\widehat{NwC} = \widehat{NZW}$$

and the triangles NZW and Nwz are similar. This implies

$$\frac{|ZW|}{|zw|} = \frac{|NW|}{|Nz|} \Leftrightarrow \frac{d(Z,W)}{|z-w|} = \frac{|NW|}{|Nz|}.$$

So it suffices to prove that

$$\frac{|NW|}{|Nz|} = \frac{2}{\sqrt{1+|z|^2}\sqrt{1+|z'|^2}}.$$

We have

$$|Nz| = |(x, y, -1)| = \sqrt{x^2+y^2+1} = \sqrt{1+|z|^2},$$

while

$$|NW|^2 = |(x_1', x_2', x_3'-1)|^2 = (x_1')^2+(x_2')^2+(x_3'-1)^2 = (x_1')^2+(x_2')^2+(x_3')^2+1-2x_3' = 2-2x_3'$$

$$= 2\left(1 - \frac{|z'|^2-1}{|z'|^2+1}\right) = \frac{4}{|z'|^2+1}.$$

(c) Show that the stereographic projection preserves angles by looking at two lines l_1 and l_2 through the point z in the complex plane and their images of the Riemann sphere, which are two arcs through the north pole. Compare the angle between l_1 and l_2 with the angle of the arcs at N and at the image Z of z under the projection. This part can be done solely with geometry.

From the figure we see that the tangent line to the stereographic projection of a line in the complex plane (this is a circle through the north pole) is parallel to the line. The argument in (b) actually proves this.

If we now have two intersecting lines l_1 and l_2 in the complex plane, their projections are two arcs, which have two tangent lines at N lying on the horizontal tangent plane at N. Their angle is equal to the angle between l_1 and l_2. The arcs meet at N but also Z. Since they are arcs of circles, the angle between NZ and each arc is the same at both endpoints N and Z of the arc. This implies that the angle between the tangent lines to the arcs at Z is equal to the angle of the tangent lines to the arcs at N, which is equal to the angle between l_1 and l_2. (The picture at N and Z is the same in terms of angles and tangent lines)

8. Consider the function $f(z) = e^z$ and the set

$$D_\epsilon = \{z \in \mathbb{C}, a-\epsilon \le \Re(z) \le a+\epsilon, -\epsilon \le \Im(z) \le \epsilon\},$$

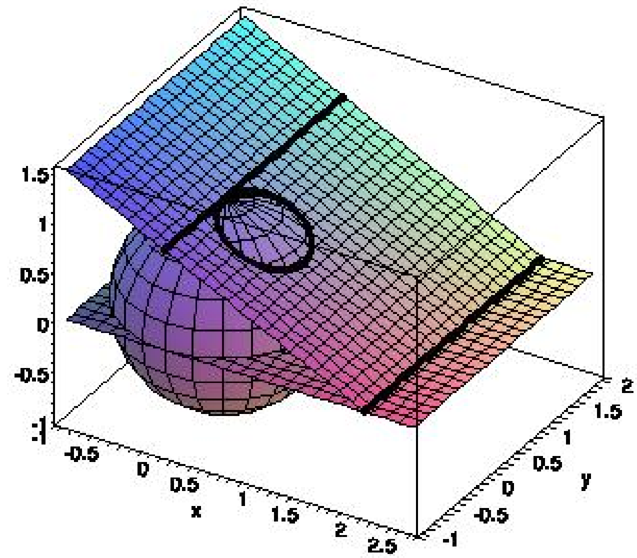


Figure 2: One line on $\mathbb{C}$, its projection and the tangent at N

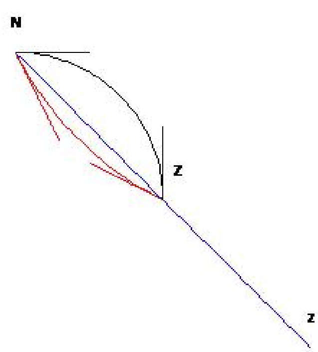


Figure 3: The two arcs on the sphere and the tangent lines at N and Z

where $\epsilon \in (0, \pi)$.

(a) Compute the area of $f(D_\epsilon)$ in two ways: First geometrically and second using the formula

$$A(f(D_\epsilon)) = \int_{D_\epsilon} |f'(z)|^2 \, dx \, dy.$$

We have, since $e^z = e^x(\cos y + i \sin y)$,

$$f(D_\epsilon) = \{w \in \mathbb{C}, e^{a-\epsilon} \leq |w| \leq e^{a+\epsilon}, |\arg w| \leq \epsilon\},$$

i.e., it is a sector in an annulus. The inner and outer radii are $e^{a-\epsilon}$ and $e^{a+\epsilon}$ respectively and the sector angle is 2ϵ. Elementary geometry gives us the total area of the annulus to be

$$\pi(e^{2a+2\epsilon} - e^{2a-2\epsilon})$$

and for the sector we multiple with $2\epsilon/2\pi$ to get

$$A(f(D_\epsilon)) = \epsilon(e^{2a+2\epsilon} - e^{2a-2\epsilon}).$$

Second method: Clearly $f'(z) = e^z$ and $|f'(z)|^2 = e^{2x}$. So

$$A(f(D_\epsilon)) = \int_{a-\epsilon}^{a+\epsilon} \int_{-\epsilon}^{\epsilon} e^{2x} \, dy \, dx = 2\epsilon \left[\frac{e^{2x}}{2}\right]_{a-\epsilon}^{a+\epsilon} = \epsilon(e^{2a+2\epsilon} - e^{2a-2\epsilon}).$$

(b) Compute the limit

$$\lim_{\epsilon \to 0} \frac{A(f(D_\epsilon))}{A(D_\epsilon)}.$$

Interpret the result.

We notice that $A(D_\epsilon) = (2\epsilon)^2$. We compute the limit

$$\lim_{\epsilon \to 0} \frac{A(f(D_\epsilon))}{A(D_\epsilon)} = \lim_{\epsilon \to 0} \frac{\epsilon(e^{2a+2\epsilon} - e^{2a-2\epsilon})}{4\epsilon^2} = \lim_{\epsilon \to 0} \frac{e^{2a+2\epsilon} - e^{2a-2\epsilon}}{4\epsilon} = \lim_{\epsilon \to 0} e^{2a} \frac{e^{2\epsilon} - e^{-2\epsilon}}{4\epsilon}$$

$$= \lim_{\epsilon \to 0} e^{2a} \frac{1 + 2\epsilon + 4\epsilon^2/2 + 8\epsilon^3/3! + \cdots - (1 - 2\epsilon + 4\epsilon^2/2 - 8\epsilon^3/3! + \cdots)}{4\epsilon}$$

$$= e^{2a} \lim_{\epsilon \to 0} \frac{4\epsilon + 16\epsilon^3/3! + \cdots}{4\epsilon} = e^{2a}.$$

(Alternatively use L'Hôpital's rule). The limit is the infinitesimal magnification factor for the area at the point a. The general theory says that this is $\det J_F = |f'(a)|^2 = e^{2a}$.