Math 70300

Homework 3

Due: October 19, 2006

1. Let $f(z)$ be an analytic function with nonzero derivative. Let $f(z) = u(x,y)+iv(x,y)$ and consider the level curves of u and v, i.e., the sets

$$\{z = x + iy \in \mathbb{C} : u(x,y) = u_0\}, \quad \{z = x + iy \in \mathbb{C} : v(x,y) = v_0\}$$

for fixed numbers u_0, v_0. Prove that the set of level curves of u and the set of level curves of v are orthogonal to each other.

2. (a) Let z_1, z_2, z_3, z_4 lie on a circle. Show that z_1, z_3, z_4 and z_2, z_3, z_4 determine the same orientation iff $(z_1, z_2, z_3, z_4) > 0$.

(b) Let z_1, z_2, z_3, z_4 lie on a circle and be the consecutive vertices of a quadrilateral. Prove that

$$|z_1 - z_3| \cdot |z_2 - z_4| = |z_1 - z_2| \cdot |z_3 - z_4| + |z_2 - z_3| \cdot |z_1 - z_4|.$$

Interpret the result geometrically.

3. Let $T(z) = \dfrac{az+b}{cz+d}$. Assume that it maps the real line to the real line. Show that we can choose a, b, c, d to be real numbers. The converse is obvious.

4. (a) Let $-\infty < a < b < \infty$ and set $M(z) = \dfrac{z - ia}{z - ib}$. Define the lines $L_1 = \{z : \Im(z) = b\}$, $L_2 = \{z : \Im(z) = a\}$ and $L_3 = \{z : \Re(z) = 0\}$. The three lines split the complex plane into 6 regions. Determine the image of them in the complex plane.

(b) Let log be the principal branch of the logarithm. Show that $\log(M(z))$ is defined for all $z \in \mathbb{C}$ with the exception of the line segment from ia to ib.

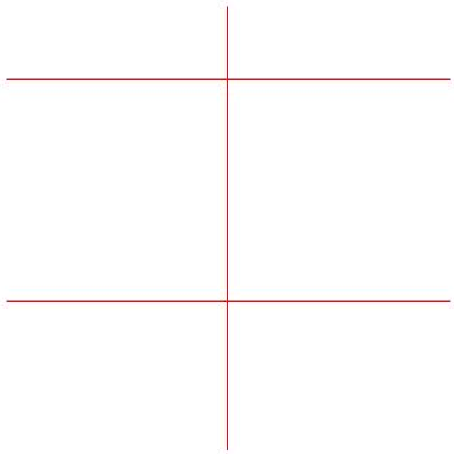
(c) Define $h(z) = \Im(\log(M(z)))$ for $\Re(z) > 0$. Show that h is harmonic and that $0 < h(z) < \pi$.

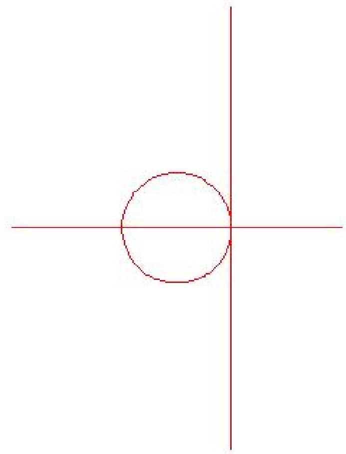
(d) Show that $\log(z - ic)$ is defined for $\Re(z) > 0$ and any real number c. Prove that $|\Im(\log(z - ic))| < \pi/2$ in this region.

(e) Prove that $h(z) = \Im(\log(z - ia) - \log(z - ib))$.

(f) Use the fundamental theorem of calculus to show that

$$\int_a^b \frac{dt}{z - it} = i(\log(z - ib) - \log(z - ia)).$$

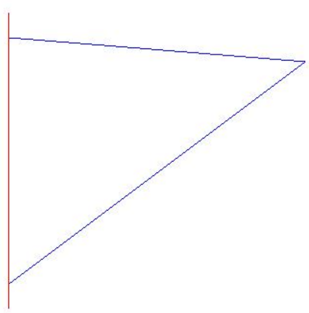




(g) Combine (e) and (f) to show that

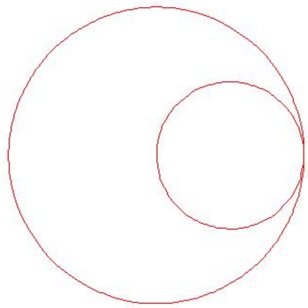
$$h(x+iy) = \int_a^b \frac{x\,dt}{x^2+(y-t)^2} = \arctan((y-a)/x) - \arctan((y-b)/x).$$

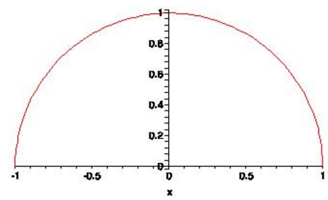
(h) Interpret (g) geometrically by showing that $h(z)$ measures (with sign) the interior angle of the triangle with vertices ia, ib and z at the vertex z. What are the limits of $h(z)$ as $\Re(z) \to 0$ for $\Im(z) \in (a,b)$ and for $\Im(z) \notin [a,b]$?



5. Suppose that C_1 and C_2 are two circles with real centers, tangent to each other at $a \in \mathbb{R}$. Assume that the one is contained inside the other. Call G the region between the two circles. Map conformally G to the unit disc $\mathbb{D}$.

   Hint: First try $(z-a)^{-1}$.





6. Let Ω be the upper half of the unit disc $\mathbb{D}$. Find a conformal mapping $f : \Omega \to \mathbb{D}$ that maps $\{-1, 0, 1\}$ to $\{-1, -i, 1\}$. Find $z \in \Omega$ with $f(z) = 0$.

   Hint: $f = T_1 \circ S \circ T_2$, where T_i are linear fractional transformations and $S(z) = z^2$.

7. Let z and z' be points in $\mathbb{C}$ with corresponding points on the unit sphere Z and Z' by stereographic projection. Let N be the north pole $N(0, 0, 1)$.

   (a) Show that Z and Z' are diametrically opposite on the unit sphere iff $z\bar{z}' = -1$.

   (b) Show that the triangles $Nz'z$ and NZZ' are similar. The order of the vertices is important and is as given. Use this to derive the formula for the euclidean distance in $\mathbb{R}^3$

$$d(Z, Z') = \frac{2|z - z'|}{\sqrt{1 + |z|^2}\sqrt{1 + |z'|^2}}.$$

   Note: Ahlfors and Conway denotes this distance $d(z, z')$.

   (c) Show that the stereographic projection preserves angles by looking at two lines l_1 and l_2 through the point z in the complex plane and their images of the Riemann sphere, which are two arcs through the north pole. Compare the angle between l_1 and l_2 with the angle of the arcs at N and at the image Z of z under the projection. This part can be done solely with geometry.

8. Consider the function $f(z) = e^z$ and the set

$$D_\epsilon = \{z \in \mathbb{C}, a - \epsilon \le \Re(z) \le a + \epsilon, -\epsilon \le \Im(z) \le \epsilon\},$$

   where $\epsilon \in (0, \pi)$.

   (a) Compute the area of $f(D_\epsilon)$ in two ways: First geometrically and second using the formula

$$A(f(D_\epsilon)) = \int_{D_\epsilon} |f'(z)|^2 \, dx \, dy.$$

   (b) Compute the limit

$$\lim_{\epsilon \to 0} \frac{A(f(D_\epsilon))}{A(D_\epsilon)}.$$

   Interpret the result.