1. Suppose that \( f \) is holomorphic in a region \( \Omega \), i.e. an open connected set. Prove that in any of the following cases
(a) \( \Re(f) \) is constant; (b) \( \Im(f) \) is constant; (c) \( |f| \) is constant; (d) \( \arg(f) \) is constant;
we can conclude that \( f \) is a constant.

See also Ahlfors, p. 72.

If \( f'(z) = 0 \) for all \( z \in \Omega \), we fix a point \( z_0 \in \Omega \) and join it with any path \( \gamma \) to \( z \).
Then
\[
f(z) - f(z_0) = \int_\gamma f'(z) \, dz = 0.
\]

So \( f(z) \) is a constant. We set \( f(z) = u + iv \). For (a) we note that if \( u \) is constant, then \( u_x = 0 \) and \( u_y = 0 \). But \( f'(z) = u_x + iv_x = u_x - iu_y = 0 \). So the previous statement gives that \( f \) is constant. For (b) we note that if \( v \) is constant, then \( v_x = v_y = 0 \). But \( f'(z) = v_y + iv_x = 0 \). So again \( f \) is constant. For (c) we argue as follows: If \( u^2 + v^2 = k \), then we get by differentiation
\[
2uu_x + 2vv_x = 0, \quad 2uu_y + 2vv_y = 0.
\]

Using the Cauchy-Riemann equations we get:
\[
2uu_x - 2vu_y = 0, \quad 2uu_y + 2vu_x = 0.
\]

We now have a homogeneous system with unknowns \( u_x \) and \( u_y \). The determinant is
\[
\begin{vmatrix}
2u & -2v \\
2v & 2u
\end{vmatrix} = 4(u^2 + v^2).
\]

If \( u^2 + v^2 \neq 0 \) the system has a unique solution \( u_x = u_y = 0 \), which implies that \( u \) is constant and we can use (a). If \( u^2 + v^2 = 0 \) at some point, then automatically \( u^2 + v^2 = 0 \) for all points, and the function \( f \) vanishes.

For (d) we set \( u = kv \) for fixed \( k \). Consider the analytic function \( g(z) = (1+ki)(u+iv) \), which has real part \( u - kv = 0 \). Then by (a), \( g(z) \) is constant, which implies that \( f \) is constant.
2. Show that if \( \{a_n\}_{n=0}^{\infty} \) is a sequence of non-zero complex numbers such that
\[
\lim_{n \to \infty} \frac{|a_{n+1}|}{|a_n|} = L,
\]
then
\[
\lim_{n \to \infty} |a_n|^{1/n} = L.
\]
This is the ratio test and it can be used for the calculation of the radius of convergence of a power series.

*Hint:* Show that
\[
\liminf \frac{|a_{n+1}|}{|a_n|} \leq \liminf |a_n|^{1/n} \leq \limsup |a_n|^{1/n} \leq \limsup \frac{|a_{n+1}|}{|a_n|}.
\]

We only need to prove the hint, since
\[
\lim x_n = L \iff \liminf x_n = \limsup x_n = L.
\]

Also since we work with absolute values, we might as well prove the result for the sequence \( b_n = |a_n| \). Set
\[
l = \liminf \frac{b_{n+1}}{b_n}, \quad L = \liminf \sqrt[n]{b_n}, \quad M = \limsup \sqrt[n]{b_n}, \quad m = \limsup \frac{b_{n+1}}{b_n}.
\]
Fix \( \epsilon > 0 \). Since \( m = \limsup b_{n+1}/b_n = \inf_N \sup_{n \geq N} b_{n+1}/b_n \), the number \( m + \epsilon \) is not a lower bound, i.e., we can find a \( N \) such that \( \sup_{n \geq N} b_{n+1}/b_n < m + \epsilon \), i.e. for all \( n \geq N \) we have
\[
\frac{b_{n+1}}{b_n} < m + \epsilon.
\]
We apply this successively for \( n = N, N + 1, \ldots, n - 1 \) to get
\[
b_{N+1} < (m + \epsilon)b_N,
b_{N+2} < (m + \epsilon)b_{N+1},
\vdots
\]
\[
b_n < (m + \epsilon)b_{n-1}.
\]
We multiply together to get, after cancellations,
\[
b_n < (m + \epsilon)^{n-N}b_N.
\]
We take \( n \)-th roots to get
\[
\sqrt[n]{b_n} < (m + \epsilon) \left( \frac{b_N}{(m + \epsilon)^N} \right)^{1/n}, \quad n \geq N.
\]
We take lim sup of both sides, taking into account that this does not depend on a finite number of initial terms and that $c^{1/n} \to 1$ for $c > 0$. This gives:
\[
\limsup \sqrt[n]{b_n} \leq (m + \epsilon).
\]
Since this is true for all $\epsilon > 0$, we get $\limsup \sqrt[n]{b_n} = M \leq m$.

For the lim inf we work analogously: Fix $\epsilon > 0$. Find $N$ such that $\inf_{n \geq N} b_{n+1}/b_n > l - \epsilon$, which implies, for $n \geq N$, $b_{n+1}/b_n > l - \epsilon$. We apply this successively for $n = N, N + 1, \ldots, n - 1$ to get
\[
\begin{align*}
b_{N+1} &> (l - \epsilon)b_N \\
b_{N+2} &> (l - \epsilon)b_{N+1} \\
 &\vdots \\
b_n &> (l - \epsilon)b_{n-1}.
\end{align*}
\]
We multiply together to get, after cancellations,
\[
b_n > (l - \epsilon)^{n-N}b_N.
\]
We take $n$-th roots to get
\[
\sqrt[n]{b_n} > (l - \epsilon) \left( \frac{b_N}{(l - \epsilon)^N} \right)^{1/n}, \quad n \geq N.
\]
We take lim inf of both sides, taking into account that this does not depend on a finite number of initial terms and that $c^{1/n} \to 1$ for $c > 0$. This gives:
\[
\liminf \sqrt[n]{b_n} \geq (l - \epsilon).
\]
Since this is true for all $\epsilon > 0$, we get $\liminf \sqrt[n]{b_n} = L \geq l$.

3. (a) Find the radius of convergence of the hypergeometric series
\[
F(\alpha, \beta, \gamma; z) = 1 + \sum_{n=1}^{\infty} \frac{\alpha(\alpha + 1) \cdots (\alpha + n - 1) \beta(\beta + 1) \cdots (\beta + n - 1) \gamma(\gamma + 1) \cdots (\gamma + n - 1) z^n}{n!}.
\]
Here $\alpha, \beta \in \mathbb{C}$ and $\gamma \neq 0, -1, -2, \ldots$. We use the ratio test. We must assume that the terms are nonzero. This happens as long as $\alpha$ and $\beta$ are not negative integers (or zero). If either is a negative integer (or zero), the power series terminates and we get a polynomial with infinite radius of convergence. We get in the general case
\[
\frac{|a_{n+1}|}{|a_n|} = \frac{|\alpha(\alpha + 1) \cdots (\alpha + n - 1) (\alpha + n) \beta(\beta + 1) \cdots (\beta + n - 1) (\beta + n)|}{|\alpha(\alpha + 1) \cdots (\alpha + n - 1) \beta(\beta + 1) \cdots (\beta + n - 1)|} \frac{n! |\gamma(\gamma + 1) \cdots (\gamma + n - 1) (\gamma + n)|}{(n+1)! |\gamma(\gamma + 1) \cdots (\gamma + n - 1)|} = \frac{(\alpha + n)(\beta + n)}{(n+1)(\gamma + n)} \to 1, n \to \infty.
\]
This implies that $R = 1$.

(b) Find the radius of convergence of the Bessel function of order $r$:

$$J_r(z) = \left(\frac{z}{2}\right)^r \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+r)!} \left(\frac{z}{2}\right)^{2n}, \quad r \in \mathbb{N}.$$ 

We use the ratio test

$$\frac{|a_{n+1}|}{|a_n|} = \frac{2^{2n}n!(n+r)!}{2^{2n+2}(n+1)!(n+1+r)!} = \frac{1}{4(n+1)(n+1+r)} \to 0, \quad n \to \infty.$$ 

So the radius of convergence is $\infty$.

4. Prove that, although all the following power series have $R = 1$,

(a) $\sum n z^n$ does not converge on any point of the unit circle,

(b) $\sum z^n / n^2$ converges at every point of the unit circle,

(c) $\sum z^n / n$ converges at every point of the unit circle except $z = 1$. (Hint: Use summation by parts.)

Since $\lim \sqrt[n]{n} = 1$ it is easy to see that $R = 1$. For (a) we notice that on $|z| = 1$, $|nz^n| = n \to \infty$, so the series cannot converge, as the general term does not go to $0$. For (b) we notice that we can apply the comparison test with $b_n = n^{-2}$, where $\sum b_n < \infty$. We have on $|z| = 1$ the bound $|z^n / n^2| = b_n$. The comparison test implies that the series converges.

For (c) and $z = 1$ we notice that we get the harmonic series $\sum n^{-1}$, which diverges by the integral test: compare with the integral

$$\int_1^{\infty} \frac{1}{x} \, dx = \infty.$$ 

For $z \neq 1$ and $|z| = 1$ we get by summation by parts, setting $s_N = \sum_{n=1}^{N} z^n = (1 - z^{N+1})/(1 - z)$, $s_0 = 0$

$$\sum_{n=1}^{M} \frac{z^n}{n} = \sum_{n=1}^{M} \frac{s_n - s_{n-1}}{n} = \sum_{n=1}^{M} \frac{s_n}{n} - \sum_{n=0}^{M-1} \frac{s_n}{n+1} = \frac{s_M}{M} + \sum_{n=1}^{M-1} s_n \left(\frac{1}{n} - \frac{1}{n+1}\right) = \frac{s_M}{M} + \sum_{n=1}^{M-1} \frac{s_n}{n(n+1)}.$$ 

Since $|s_M| \leq 2/|1 - z|$ (note that it is exactly in this geometric series that we use $z \neq 1$), the first term goes to zero, while the series can be compared with

$$\frac{1}{|1 - z|} \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{|1 - z|}.$$
5. The Fibonacci numbers are defined by \( c_0 = 1, c_1 = 1, \)
\[
c_n = c_{n-1} + c_{n-2}, \quad n = 2, 3, \ldots.
\]
Define their generating function as
\[
F(z) = \sum_{n=0}^{\infty} c_n z^n.
\]
(a) Find a quadratic polynomial \( Az^2 + Bz + C \) such that
\[
(Az^2 + Bz + C)F(z) = 1.
\]
We get
\[
\infty \sum_{n=0}^{\infty} Ac_n z^{n+2} + \infty \sum_{n=0}^{\infty} Bc_n z^{n+1} + \infty \sum_{n=0}^{\infty} Cc_n z^n = 1,
\]
\[
\infty \sum_{n=2}^{\infty} Ac_{n-2} z^n + \infty \sum_{n=1}^{\infty} Bc_{n-1} z^n + \infty \sum_{n=0}^{\infty} Cc_n z^n = 1,
\]
\[
\infty \sum_{n=2}^{\infty} (Ac_{n-2} + Bc_{n-1} + Cc_n) z^n + (Bc_0 + Cc_1) z + Cc_0 = 1.
\]
By the uniqueness of the coefficients of the power series, we get
\[
Ac_{n-2} + Bc_{n-1} + Cc_n = 0, \quad Bc_0 + Cc_1 = 0, \quad Cc_0 = 1.
\]
Using \( c_0 = c_1 = 1 \) we get \( C = 1, B = -1, \) while the first equation becomes \( Ac_{n-2} - c_{n-1} + c_n = 0. \) The recurrence formula \( c_n = c_{n-1} + c_{n-2} \) implies \( A = -1. \) So the quadratic polynomial is \(-z^2 - z + 1.\)
(b) Use partial fractions to determine the following closed expression for \( c_n.\)
\[
c_n = \frac{(1+\sqrt{5})^{n+1} - (1-\sqrt{5})^{n+1}}{\sqrt{5}}.
\]
(Qualifying exam Sept. 2006)
The roots of the quadratic polynomial are
\[
\rho_{1,2} = \frac{-1 \pm \sqrt{5}}{2}
\]
with \( \rho_1 \rho_2 = -1. \) We set
\[
\frac{1}{-z^2 - z + 1} = \frac{A}{z - \rho_1} + \frac{B}{z - \rho_2}
\]
to get
\[ -1 = A(z - \rho_2) + B(z - \rho_1) \implies A = \frac{1}{\rho_2 - \rho_1}, \quad B = \frac{1}{\rho_1 - \rho_2} = -A. \]

This gives \( A = -1/\sqrt{5}, \ B = 1/\sqrt{5}. \) We expand the fractions into geometric series valid for \(|z/\rho_1| < 1\) and \(|z/\rho_2| < 1\) respectively to get
\[
\frac{A}{z - \rho_1} = \frac{1}{\sqrt{5}} \frac{1}{\rho_1 (1 - z/\rho_1)} = \frac{1}{\sqrt{5}} \sum_{n=0}^{\infty} \frac{z^n}{\rho_1^{n+1}},
\]
and
\[
\frac{B}{z - \rho_1} = -\frac{1}{\sqrt{5}} \frac{1}{\rho_2 (1 - z/\rho_2)} = -\frac{1}{\sqrt{5}} \sum_{n=0}^{\infty} \frac{z^n}{\rho_2^{n+1}}.
\]

This gives, by the uniqueness of the power series
\[
c_n = \frac{1}{\sqrt{5}} \left( \frac{1}{\rho_1^{n+1}} - \frac{1}{\rho_2^{n+1}} \right)
\]
\[
c_n = \frac{1}{\sqrt{5}} \left( (-\rho_2)^{n+1} - (-\rho_1)^{n+1} \right)
\]
using \( \rho_1 \rho_2 = -1. \)

6. Expand \((1 - z)^{-m}\) in powers of \(z\), for \(m \in \mathbb{N}\). Let
\[
(1 - z)^{-m} = \sum_{n=0}^{\infty} a_n z^n,
\]
then
\[
a_n \sim \frac{1}{(m-1)!} n^{m-1}, \quad n \to \infty,
\]
where \(\sim\) means that the quotient of the expressions to the left and the right of it tends to 1.

The geometric series is
\[
(1 - z)^{-1} = \sum_{n=0}^{\infty} z^n, \quad |z| < 1.
\]

We differentialte \(m - 1\)-times to get
\[
\frac{(m - 1)!}{(1 - z)^m} = \sum_{n=m-1}^{\infty} n(n-1)(n-2) \cdots (n-m+2)z^{n-m+1} = \sum_{n=0}^{\infty} (n+m-1)(n+m-2) \cdots (n+1)z^n.
\]
So
\[
a_n = \frac{(n + m - 2) \cdots (n + 1)}{(m - 1)!}.
\]
Since \( m \) is fixed and \( n \to \infty \) we easily see that
\[
\frac{a_n}{n^{m-1}/(m-1)!} \to 1, \quad n \to \infty
\]
as \( a_n \) has \( m - 1 \) factors in the numerator.

7. Show that for \(|z| < 1\) we have
\[
\frac{z}{1 - z^2} + \frac{z^2}{1 - z^4} + \cdots + \frac{z^{2^n}}{1 - z^{2^{n+1}}} + \cdots = \frac{z}{1 - z}, \quad (1)
\]
and
\[
\frac{z}{1 + z} + \frac{2z^2}{1 + z^2} + \cdots + \frac{2^k z^{2^k}}{1 + z^{2^k}} + \cdots = \frac{z}{1 - z}, \quad (2)
\]
Justify any change in the order of summation.

*Hint:* Use the dyadic expansion of an integer and the sum \( 1 + 2 + 2^2 + \cdots + 2^k = 2^{k+1} - 1 \).

We first prove the Lemma:

*Every positive integer \( n \) has a unique expression of the \( 2^t + k_t 2^{t+1} \), with \( t \geq 0 \), \( k_t \geq 0 \).*

We use for this the dyadic expansion of \( n \). If
\[
n = a_0 + a_1 2 + a_2 2^2 + \cdots + a_s 2^s, \quad a_i \in \{0, 1\}
\]
we pick \( t \) to be the first nonzero \( a_i \): i.e. \( a_t = 1 \), while \( a_0 = a_1 = \cdots = a_{t-1} = 0 \). Then
\[
n = 2^t + a_{t+1} 2^{t+1} + \cdots + a_s 2^s = 2^t + 2^{t+1}(a_{t+1} + \cdots + a_s 2^{s-t-1}) = 2^t + 2^{t+1}k_t.
\]

This expression is unique as
\[
2^t + k_t 2^{t+1} = 2^j + 2^{j+1}k_j, \quad j \neq t
\]
implies that either \( j > t \) or \( j < t \). By symmetry we assume \( j > t \), i.e., \( j \geq t + 1 \).

Look at the remainder when we divide by \( 2^{t+1} \). The left-hand side gives \( 2^t \), while the right-hand side gives 0.

We expand the left-side of (1) using the geometric series to get
\[
\sum_{k_1=0}^{\infty} z^{2k_1+1} + \sum_{k_2=0}^{\infty} z^{4k_2+2} + \cdots + \sum_{k_m=0}^{\infty} z^{2^{m+1}k_m+2^m} + \cdots
\]
while the right-hand side is just \( \sum_{n=1}^{\infty} z^n \). The lemma proves the identity, provided that we have absolute convergence of the double series
\[
\sum_{m=0}^{\infty} \sum_{k_m=0}^{\infty} z^{2^{m+1}k_m+2^m},
\]
which would mean that we can rearrange the terms in whatever way we would like. But the lemma shows that the series

\[ \sum_{m=0}^{\infty} \sum_{k_m=0}^{\infty} |z|^{2m+1}k_m+2^m \]

is exactly the geometric series \( \sum |z|^n \), \(|z| < 1\), which converges.

For (2) we expand the left-hand side to get

\[
z \sum_{n=0}^{\infty} (-1)^n z^n + 2z^2 \sum_{n=0}^{\infty} (-1)^n z^{2n} + \cdots + 2^k z^{2^k} \sum_{n=0}^{\infty} (-1)^n z^{2kn} + \cdots
\]

Looking at the dyadic expansion of \( n \) we pick the first \( a_t = 1 \), such that \( a_0 = a_1 = \cdots = a_{t-1} = 0 \). Then \( 2^i |n| \) for \( i \leq t \), while \( 2^{t+1} \) does not divide \( n \) and the same is true for higher powers of 2. For \( i \leq t - 1 \), \( n = 2^i \ast l \) with \( l \) even, so the coefficient it picks from the \( i \)-series in the sum is \( 2^i (-1)^{i+1} = -2^i \). On the other hand for \( i = t \), \( n = 2^t \ast l \) with \( l \) odd, so that it picks a coefficient \( 2^t (-1)^{t+1} = 2^t \). For \( i > t \) it picks nothing. So the coefficient of \( z^n \) is

\[-1 + (-2) + (-2^2) + \cdots + (-2^{t-1}) + 2^t = 1 \]

by the identity \( 1 + 2 + \cdots + 2^k = 2^{k+1} - 1 \).

**Example:** \( n = 4 + \) higher powers of 2. We write the first 4 series in the sum:

\[
z - z^2 + z^3 - z^4 + \cdots \\
2(z^2 - z^4 + z^6 - z^8 + \cdots) \\
4(z^4 - z^8 + z^{12} - z^{16} + \cdots) \\
8(z^8 - z^{16} + z^{24} - z^{32} + \cdots)
\]

The first three give to \( z^4 \) coefficients \(-1\), \(-2\), \(4\), while the last and the subsequent ones give 0.

We need to explain the interchange of summations. We look at the series with absolute values on the terms. This is

\[
\sum_{k=0}^{\infty} \sum_{m=1}^{\infty} 2^k |z|^{2km}.
\]

If this converges, then the original double sum converges absolutely and we can sum in whatever order we like. In this series the coefficient of \( |z|^n \) is equal to

\[
\sum_{2^k|m} 2^k = \sum_{k \leq t} 2^k = 2^{t+1} - 1 < 2^{t+1} \leq 2n, \quad n = 2^t + a_{t+1}2^{t+1} + \cdots.
\]

So we compare with the series \( \sum 2n |z|^n \), which has radius of convergence 1. This suffices.
8. Find the holomorphic function of $z$ that vanishes at $z = 0$ and has real part

$$u(x, y) = \frac{x(1 + x^2 + y^2)}{1 + 2x^2 - 2y^2 + (x^2 + y^2)^2}.$$ 

We substitute $z + \bar{z} = 2x$, $x^2 - y^2 = \Re(z^2) = (z^2 + \bar{z}^2)/2$ and $z\bar{z} = x^2 + y^2$ to get

$$u(x, y) = \frac{1}{2} \frac{(z + \bar{z})(1 + z\bar{z})}{1 + z^2 + z^2 + z^2\bar{z}^2} = \frac{1}{2} \frac{z + \bar{z} + z^2\bar{z} + z\bar{z}z}{(1 + z^2)(1 + \bar{z}^2)} = \frac{1}{2} \left( \frac{z}{1 + z^2} + \frac{\bar{z}}{1 + \bar{z}^2} \right).$$

We set $f(z) = z/(1 + z^2)$, which is holomorphic, to get

$$u(x, y) = \frac{1}{2} (f(z) + \bar{f}(z))$$

i.e., $u(x, y)$ is the real part of $f(z)$, and $f(z) = 0$, this gives the answer to be $f(z)$.

Second method: (Look at p. 27 in Ahlfors) Since $f(z)$ is holomorphic, the conjugate function $g(x, y) = \overline{f(z)}$ depends only on $\bar{z}$, so we write $g(z) = \overline{f(z)}$ to get

$$u(x, y) = \frac{1}{2} (f(z) + g(\bar{z})) = \frac{1}{2} (f(x + iy) + g(x - iy)).$$

This is true for $x$ and $y$ real. Formally we can plug $x = z$ and $y = z/(2i)$, so that

$$u(z/2, z/(2i)) = \frac{1}{2} (f(z) + g(0)).$$

This determines $f(z)$ up to a constant. If $u(0, 0) = 0$ and we ask for $f(0) = 0$, then $g(0) = 0$.

In our case

$$f(z) = 2u(z/2, z/(2i)) = 2\frac{(z/2)(1 + (z^2/4) + (-z^2/4))}{1 + 2(z^2/4 + z^2/4) + (z^2/4 - z^2/4)^2} = \frac{z}{1 + z^2}.$$ 

9. (i) Show that the Laplace operator can be calculated as

$$\Delta = 4 \frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}} = 4 \frac{\partial}{\partial \bar{z}} \frac{\partial}{\partial z}.$$ 

(ii) Show that for any analytic function $f(z)$ we have

$$\Delta |f(z)|^2 = 4|f'(z)|^2, \quad \text{and} \quad \Delta \log(1 + |f(z)|^2) = \frac{4|f'(z)|^2}{(1 + |f(z)|^2)^2}.$$ 

(i) We have for a function $f$ with continuous second partial derivatives

$$\frac{\partial f}{\partial z} = \frac{1}{2} \left( \frac{\partial f}{\partial x} - i \frac{\partial f}{\partial y} \right), \quad \frac{\partial f}{\partial \bar{z}} = \frac{1}{2} \left( \frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} \right)$$
which gives
\[4 \frac{\partial^2 f}{\partial z \partial \bar{z}} = \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) (f_x + i f_y) = \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) f_x + i \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) f_y\]
\[= f_{xx} - if_{yx} + i(f_{xy} - if_{yy}) = f_{xx} - if_{yx} + if_{xy} + f_{yy} = f_{xx} + f_{yy} = \Delta f,\]
since the mixed partials \(f_{xy}, f_{yx}\) are equal for functions with continuous second partial derivatives. Reversing the order of the calculation gives \(4 \frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}} f = \Delta f\).

(ii) For an analytic function \(\partial_z f(z) = f'(z),\) while \(\partial_{\bar{z}} f(z) = \overline{f'(z)},\) since \(f\) does not depend on \(\bar{z},\) so the result from applying \(\partial_z\) on its conjugate can be calculated from \(f'(z):\) e.g. \(f(z) = z^2,\) \(\overline{f(z)} = \bar{z}^2,\) \(\partial_{\bar{z}} f(z) = 2\bar{z} = \overline{\partial_z f(z)}\). This gives:
\[\Delta |f(z)|^2 = 4 \partial_z \partial_{\bar{z}} (f(z) \overline{f(z)}) = 4 \partial_z (f(z) \overline{f'(z)}) = 4 \overline{f'(z)} \partial_z f(z) = 4 \overline{f'(z)} f'(z) = 4|f'(z)|^2,\]
since \(f(z)\) is a constant when we differentiate in \(\bar{z},\) and \(\overline{f'(z)}\) is a constant when we differentiate in \(z,\) as it does not depend on \(z.\)

For the second part we will use the chain rule as expressed in \(z\) and \(\bar{z}\) coordinates (exercise 13 in homework 1). We use \(g(w) = \log(1 + w),\) which is holomorphic in \(w\) for \(\Re w > -1\) (there is no difficulty in defining the argument of \(1 + w\) in this domain. For \(f\) we have \(|f(z)|^2 \geq 0.\) We have
\[g_\bar{w} = 0,\quad \text{and} \quad g_w = \frac{1}{1 + w},\]
\[\partial_z \log(1 + |f(z)|^2) = \frac{1}{1 + |f(z)|^2} (1 + |f(z)|^2)_z = \frac{\overline{f(z)} f'(z)}{1 + |f(z)|^2}.\]

We now apply the quotient rule
\[4 \partial_z \partial_{\bar{z}} \log(1 + |f(z)|^2) = 4 \partial_z \frac{\overline{f(z)} f'(z)}{1 + |f(z)|^2} = 4 \partial_{\bar{z}} \frac{\overline{f(z)} f'(z)}{1 + f(z) \overline{f(z)}}\]
\[= 4 \frac{\partial_{\bar{z}} (\overline{f(z)} f'(z)) (1 + |f(z)|^2) - f(z) \overline{f'(z)} \partial_{\bar{z}} (1 + f(z) \overline{f(z)})}{(1 + |f(z)|^2)^2}\]
\[= 4 \frac{f'(z) \overline{f'(z)} (1 + |f(z)|^2) - f(z) \overline{f'(z)} \overline{f'(z)} f(z)}{(1 + |f(z)|^2)^2} = \frac{4 f'(z) \overline{f'(z)} f'(z)}{(1 + |f(z)|^2)^2}.\]

10. Let \(f(z)\) be holomorphic and one-to-one on a set containing the unit disc \(\mathbb{D}.\) Let \(D' = f(\mathbb{D}).\) Then the area \(A(D')\) of \(D'\) is given by
\[A(D') = \pi \sum_{n=1}^{\infty} n|a_n|^2,\]
where \(f(z) = \sum_{n=0}^{\infty} a_n z^n.\) (Qualifying exam September 2006)
The Jacobian of the mapping \( w = f(z) \) as a mapping from \( \mathbb{D} \to \mathbb{C} \) is \(|f'(z)|^2\). So by the change of variables formula

\[
A(D') = \int_{D'} 1 \, du \, dv = \int_{\mathbb{D}} |f'(z)|^2 \, dx \, dy.
\]

If \( f(z) = \sum_{n=0}^{\infty} a_n z^n \), then

\[
f'(z) = \sum_{n=1}^{\infty} na_n z^{n-1} \implies |f'(z)|^2 = \sum_{n,m=1}^{\infty} nma_n \bar{a}_m z^{n-1} \bar{z}^{m-1}.
\]

We switch to polar coordinates \( z = re^{i\theta} \) to get \((dx \, dy = rdr \, d\theta)\)

\[
A(D') = \int_{0}^{2\pi} \int_{1}^{\infty} \sum_{n,m=1}^{\infty} nma_n \bar{a}_m r^{n+m-2} e^{i(n-m)\theta} \, r \, dr \, d\theta
\]

If \( n - m \neq 0 \), the integral \( \int_{0}^{2\pi} e^{i(n-m)\theta} \, d\theta = 0 \), while for \( n = m \) we get \(2\pi\). This gives

\[
A(D') = \int_{1}^{\infty} \sum_{n=1}^{\infty} n^2 |a_n|^2 r^{2n-1} 2\pi = 2\pi \sum_{n=1}^{\infty} n^2 |a_n|^2 \left[ \frac{r^{2n}}{2n} \right]_0^\infty = \pi \sum_{n=1}^{\infty} |a_n|^2 n.
\]

11. (a) Compute the integral

\[
\int_{|z|=r} x \, dz
\]

for the positive sense of the circle, in two ways: first by using a parametrization, and second, by observing that \( x = (1/2)(z + \bar{z}) = (1/2)(z + r^2/z) \) on the circle.

First Method: Parametrization of the circle as \( z = re^{i\theta} \), \( x = r \cos \theta \), \( dz = rie^{i\theta} \). We get

\[
\int_{|z|=r} x \, dz = \int_{0}^{2\pi} r \cos \theta rie^{i\theta} \, d\theta = \int_{0}^{2\pi} r^2 i \cos \theta (\cos \theta + i \sin \theta) \, d\theta = \int_{0}^{2\pi} r^2 i (\cos^2 \theta + \sin^2 \theta) \, d\theta
\]

\[
= r^2 i \int_{0}^{2\pi} \frac{1 + \cos(2\theta)}{2} \, d\theta = r^2 i \left[ \theta + \frac{\sin 2\theta}{4} \right]_0^{2\pi} = r^2 i \pi.
\]

Second method: On the circle \( |z| = r \), we have \( z\bar{z} = r^2 \implies \bar{z} = r^2/z \), so that \( x = (z + \bar{z})/2 = (1/2)(z + r^2/z) \). We plug into the integral to get

\[
\int_{|z|=r} x \, dz = \int_{|z|=r} \frac{z}{2} + \frac{r^2}{2z} \, dz = \frac{r^2}{2} 2\pi i = r^2 i \pi,
\]

since we know that \( \int_{|z|=r} z^n \, dz = 0 \) for \( n \neq -1 \) and for \( n = -1 \) we get \(2\pi i\).
(b) Compute the integral
\[ \int_{|z|=2} \frac{dz}{z^2 - 1} \]
for the positive sense of the circle. Hint: Find a primitive function of the integrand. We use partial fractions to find that
\[ \frac{1}{z^2 - 1} = \frac{1}{2} \left( \frac{1}{z - 1} - \frac{1}{z + 1} \right). \]
We would like to integrate to get \((\log(z - 1) - \log(z + 1))/2\). However, this cannot make sense in the region we are working: The \(\log(z - 1)\) requires a cut from, say \(-\infty\) to 1 and \(\log(z + 1)\) requires a cut from \(-\infty\) to \(-1\). These cuts meet our domain, since the circle \(|z| = 2\) contains both points \(-1\) and 1. Make sense can be thought of being single-valued holomorphic functions that give us the primitive of the terms we consider. However, if we write the expected answer as
\[ F(z) = \frac{1}{2} \log \left( \frac{z - 1}{z + 1} \right) \]
things are much better: First of all, \((z - 1)/(z + 1)\) is real only for real \(z\). This is seen as follows:
\[ \frac{z - 1}{z + 1} \in \mathbb{R} \iff \frac{z - 1}{\bar{z} + 1} = \frac{\bar{z} - 1}{z + 1} \iff z\bar{z} + z - \bar{z} - 1 = z\bar{z} + \bar{z} - z - 1 \iff 2(z - \bar{z}) = 0 \iff \Im(z) = 0. \]
Moreover, it is a negative number exactly between \(-1\) and 1. So on the complement of \([-1, 1]\) the function takes nonnegative values. If we define the principal branch of the logarithm to be
\[ \log w = \log |w| + i \arg w, \quad -\pi < \arg w < \pi, \]
then \(F(z)\) as the composition of holomorphic maps is holomorphic and is the primitive of \(1/(z^2 - 1)\) in a region that contains our contour. Consequently \(dz/(z^2 - 1)\) is exact and the integral is 0.

Remark: One can solve this problem with the residue theorem, which will be seen later in the course.