Math 70300

Homework 2

Due: September 28, 2006

1. Suppose that f is holomorphic in a region Ω, i.e. an open connected set. Prove that in any of the following cases

   (a) $\Re(f)$ is constant; (b) $\Im(f)$ is constant; (c) $|f|$ is constant; (d) $\arg(f)$ is constant;

   we can conclude that f is a constant.

   See also Ahlfors, p. 72.

   If $f'(z) = 0$ for all $z \in \Omega$, we fix a point $z_0 \in \Omega$ and join it with any path γ to z. Then

$$f(z) - f(z_0) = \int_\gamma f'(z)\,dz = 0.$$

   So $f(z)$ is a constant. We set $f(z) = u + iv$. For (a) we note that if u is constant, then $u_x = 0$ and $u_y = 0$. But $f'(z) = u_x + iv_x = u_x - iu_y = 0$. So the previous statement gives that f is constant. For (b) we note that if v is constant, then $v_x = v_y = 0$. But $f'(z) = v_y + iv_x = 0$. So again f is constant. For (c) we argue as follows: If $u^2 + v^2 = k$, then we get by differentiation

$$2uu_x + 2vv_x = 0, \quad 2uu_y + 2vv_y = 0.$$

   Using the Cauchy-Riemann equations we get:

$$2uu_x - 2vu_y = 0, \quad 2uu_y + 2vu_x = 0.$$

   We now have a homogeneous system with unknowns u_x and u_y. The determinant is

$$\begin{vmatrix} 2u & -2v \\ 2v & 2u \end{vmatrix} = 4(u^2 + v^2).$$

   If $u^2 + v^2 \neq 0$ the system has a unique solution $u_x = u_y = 0$, which implies that u is constant and we can use (a). If $u^2 + v^2 = 0$ at some point, then automatically $u^2 + v^2 = 0$ for all points, and the function f vanishes.

   For (d) we set $u = kv$ for fixed k. Consider the analytic function $g(z) = (1+ki)(u+iv)$, which has real part $u - kv = 0$. Then by (a), $g(z)$ is constant, which implies that f is constant.

2. Show that if $\{a_n\}_{n=0}^{\infty}$ is a sequence of non-zero complex numbers such that

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = L,$$

then

$$\lim_{n\to\infty} |a_n|^{1/n} = L.$$

This is the ratio test and it can be used for the calculation of the radius of convergence of a power series.

Hint: Show that

$$\liminf \frac{|a_{n+1}|}{|a_n|} \leq \liminf |a_n|^{1/n} \leq \limsup |a_n|^{1/n} \leq \limsup \frac{|a_{n+1}|}{|a_n|}.$$

We only need to prove the hint, since

$$\lim x_n = L \Leftrightarrow \liminf x_n = \limsup x_n = L.$$

Also since we work with absolute values, we might as well prove the result for the sequence $b_n = |a_n|$. Set

$$l = \liminf \frac{b_{n+1}}{b_n}, \quad L = \liminf \sqrt[n]{b_n}, \quad M = \limsup \sqrt[n]{b_n}, \quad m = \limsup \frac{b_{n+1}}{b_n}.$$

Fix $\epsilon > 0$. Since $m = \limsup b_{n+1}/b_n = \inf_N \sup_{n\geq N} b_{n+1}/b_n$, the number $m+\epsilon$ is not a lower bound, i.e., we can find a N such that $\sup_{n\geq N} b_{n+1}/b_n < m+\epsilon$, i.e. for all $n \geq N$ we have

$$\frac{b_{n+1}}{b_n} < m + \epsilon.$$

We apply this successively for $n = N, N+1, \ldots, n-1$ to get

$$\begin{aligned} b_{N+1} &< (m+\epsilon)b_N \\ b_{N+2} &< (m+\epsilon)b_{N+1} \\ \vdots & \quad \vdots \\ b_n &< (m+\epsilon)b_{n-1}. \end{aligned}$$

We multiply together to get, after cancellations,

$$b_n < (m+\epsilon)^{n-N} b_N.$$

We take n-th roots to get

$$\sqrt[n]{b_n} < (m+\epsilon)\left(\frac{b_N}{(m+\epsilon)^N}\right)^{1/n}, \quad n \geq N.$$

We take lim sup of both sides, taking into account that this does not depend on a finite number of initial terms and that $c^{1/n} \to 1$ for $c > 0$. This gives:

$$\limsup \sqrt[n]{b_n} \leq (m + \epsilon).$$

Since this is true for all $\epsilon > 0$, we get $\limsup \sqrt[n]{b_n} = M \leq m$.

For the lim inf we work analogously: Fix $\epsilon > 0$. Find N such that $\inf_{n \geq N} b_{n+1}/b_n > l - \epsilon$, which implies, for $n \geq N$, $b_{n+1}/b_n > l - \epsilon$. We apply this successively for $n = N, N+1, \ldots, n-1$ to get

$$\begin{array}{rcl} b_{N+1} & > & (l-\epsilon) b_N \\ b_{N+2} & > & (l-\epsilon) b_{N+1} \\ \vdots & & \vdots \\ b_n & > & (l-\epsilon) b_{n-1}. \end{array}$$

We multiply together to get, after cancellations,

$$b_n > (l-\epsilon)^{n-N} b_N.$$

We take n-th roots to get

$$\sqrt[n]{b_n} > (l-\epsilon) \left(\frac{b_N}{(l-\epsilon)^N} \right)^{1/n}, \quad n \geq N.$$

We take lim inf of both sides, taking into account that this does not depend on a finite number of initial terms and that $c^{1/n} \to 1$ for $c > 0$. This gives:

$$\liminf \sqrt[n]{b_n} \geq (l - \epsilon).$$

Since this is true for all $\epsilon > 0$, we get $\liminf \sqrt[n]{b_n} = L \geq l$.

3. (a) Find the radius of convergence of the hypergeometric series

$$F(\alpha, \beta, \gamma; z) = 1 + \sum_{n=1}^{\infty} \frac{\alpha(\alpha+1)\cdots(\alpha+n-1)\beta(\beta+1)\cdots(\beta+n-1)}{n!\gamma(\gamma+1)\cdots(\gamma+n-1)} z^n.$$

Here $\alpha, \beta \in \mathbb{C}$ and $\gamma \neq 0, -1, -2, \ldots$.

We use the ratio test. We must assume that the terms are nonzero. This happens as long as α and β are not negative integers (or zero) . If either is a negative integer (or zero), the power series terminates and we get a polynomial with infinite radius of convergence. We get in the general case

$$\begin{aligned} \frac{|a_{n+1}|}{|a_n|} &= \frac{|\alpha(\alpha+1)\cdots(\alpha+n-1)(\alpha+n)\beta(\beta+1)\cdots(\beta+n-1)(\beta+n)|}{|\alpha(\alpha+1)\cdots(\alpha+n-1)\beta(\beta+1)\cdots(\beta+n-1)|} \\ &\cdot \frac{n!|\gamma(\gamma+1)\cdots(\gamma+n-1)(\gamma+n)|}{(n+1)!|\gamma(\gamma+1)\cdots(\gamma+n-1)|} = \frac{(\alpha+n)(\beta+n)}{(n+1)(\gamma+n)} \to 1, n \to \infty. \end{aligned}$$

This implies that $R = 1$.

(b) Find the radius of convergence of the Bessel function of order r:

$$J_r(z) = \left(\frac{z}{2}\right)^r \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+r)!} \left(\frac{z}{2}\right)^{2n}, \quad r \in \mathbb{N}.$$

We use the ratio test

$$\frac{|a_{n+1}|}{|a_n|} = \frac{2^{2n} n!(n+r)!}{2^{2n+2}(n+1)!(n+1+r)!} = \frac{1}{4(n+1)(n+1+r)} \to 0, \quad n \to \infty.$$

So the radius of convergence is ∞.

4. Prove that, although all the following power series have $R = 1$,

(a) $\sum nz^n$ does not converge on any point of the unit circle,

(b) $\sum z^n/n^2$ converges at every point of the unit circle,

(c) $\sum z^n/n$ converges at every point of the unit circle except $z = 1$. (Hint: Use summation by parts.)

Since $\lim \sqrt[n]{n} = 1$ it is easy to see that $R = 1$. For (a) we notice that on $|z| = 1$, $|nz^n| = n \to \infty$, so the series cannot converge, as the general term does not go to 0. For (b) we notice that we can apply the comparison test with $b_n = n^{-2}$, where $\sum b_n < \infty$. We have on $|z| = 1$ the bound $|z^n/n^2| = b_n$. The comparison test implies that the series converges.

For (c) and $z = 1$ we notice that we get the harmonic series $\sum n^{-1}$, which diverges by the integral test: compare with the integral

$$\int_1^{\infty} \frac{1}{x}\, dx = \infty.$$

For $z \neq 1$ and $|z| = 1$ we get by summation by parts, setting $s_N = \sum_{n=1}^N z^n = (1 - z^{N+1})/(1 - z)$, $s_0 = 0$

$$\sum_{n=1}^{M} \frac{z^n}{n} = \sum_{n=1}^{M} \frac{s_n - s_{n-1}}{n} = \sum_{1}^{M} \frac{s_n}{n} - \sum_{0}^{M-1} \frac{s_n}{n+1}$$

$$= \frac{s_M}{M} + \sum_{1}^{M-1} s_n \left(\frac{1}{n} - \frac{1}{n+1}\right) = \frac{s_M}{M} + \sum_{1}^{M-1} \frac{s_n}{n(n+1)}.$$

Since $|s_M| \leq 2/|1 - z|$ (note that it is exactly in this geometric series that we use $z \neq 1$), the first term goes to zero, while the series can be compared with

$$\frac{1}{|1 - z|} \sum_{1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{|1 - z|}.$$

5. The Fibonacci numbers are defined by $c_0 = 1$, $c_1 = 1$,

$$c_n = c_{n-1} + c_{n-2}, \quad n = 2, 3, \ldots.$$

Define their generating function as

$$F(z) = \sum_{n=0}^{\infty} c_n z^n.$$

(a) Find a quadratic polynomial $Az^2 + Bz + C$ such that

$$(Az^2 + Bz + C)F(z) = 1.$$

We get

$$\sum_{n=0}^{\infty} Ac_n z^{n+2} + \sum_{n=0}^{\infty} Bc_n z^{n+1} + \sum_{n=0}^{\infty} Cc_n z^n = 1,$$

$$\sum_{n=2}^{\infty} Ac_{n-2} z^n + \sum_{n=1}^{\infty} Bc_{n-1} z^n + \sum_{n=0}^{\infty} Cc_n z^n = 1,$$

$$\sum_{n=2}^{\infty}(Ac_{n-2} + Bc_{n-1} + Cc_n)z^n + (Bc_0 + Cc_1)z + Cc_0 = 1.$$

By the uniqueness of the coefficients of the power series, we get

$$Ac_{n-2} + Bc_{n-1} + Cc_n = 0, \quad Bc_0 + Cc_1 = 0, \quad Cc_0 = 1.$$

Using $c_0 = c_1 = 1$ we get $C = 1$, $B = -1$, while the first equation becomes $Ac_{n-2} - c_{n-1} + c_n = 0$. The recurrence formula $c_n = c_{n-1} + c_{n-2}$ implies $A = -1$. So the quadratic polynomial is $-z^2 - z + 1$.

(b) Use partial fractions to determine the following closed expression for c_n.

$$c_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}}{\sqrt{5}}.$$

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The roots of the quadratic polynomial are

$$\rho_{1,2} = \frac{-1 \pm \sqrt{5}}{2}$$

with $\rho_1\rho_2 = -1$. We set

$$\frac{1}{-z^2 - z + 1} = \frac{A}{z - \rho_1} + \frac{B}{z - \rho_2}$$

to get

$$-1 = A(z-\rho_2) + B(z-\rho_1) \Longrightarrow A = \frac{1}{\rho_2-\rho_1}, \quad B = \frac{1}{\rho_1-\rho_2} = -A.$$

This gives $A = -1/\sqrt{5}$, $B = 1/\sqrt{5}$. We expand the fractions into geometric series valid for $|z/\rho_1| < 1$ and $|z/\rho_2| < 1$ respectively to get

$$\frac{A}{z-\rho_1} = \frac{1}{\sqrt{5}} \frac{1}{\rho_1(1-z/\rho_1)} = \frac{1}{\sqrt{5}} \sum_0^\infty \frac{z^n}{\rho_1^{n+1}},$$

and

$$\frac{B}{z-\rho_1} = \frac{-1}{\sqrt{5}} \frac{1}{\rho_2(1-z/\rho_2)} = \frac{-1}{\sqrt{5}} \sum_0^\infty \frac{z^n}{\rho_2^{n+1}}.$$

This gives, by the uniqueness of the power series

$$c_n = \frac{1}{\sqrt{5}} \left(\frac{1}{\rho_1^{n+1}} - \frac{1}{\rho_2^{n+1}} \right)$$

$$c_n = \frac{1}{\sqrt{5}} \left((-\rho_2)^{n+1} - (-\rho_1)^{n+1} \right)$$

using $\rho_1\rho_2 = -1$.

6. Expand $(1-z)^{-m}$ in powers of z, for $m \in \mathbb{N}$. Let

$$(1-z)^{-m} = \sum_{n=0}^\infty a_n z^n,$$

then

$$a_n \sim \frac{1}{(m-1)!} n^{m-1}, \quad n \to \infty,$$

where $\sim$ means that the quotient of the expressions to the left and the right of it tends to 1.

The geometric series is

$$(1-z)^{-1} = \sum_{n=0}^\infty z^n, \quad |z| < 1.$$

We differentialte $m-1$-times to get

$$\frac{(m-1)!}{(1-z)^m} = \sum_{n=m-1}^\infty n(n-1)(n-2)\cdots(n-m+2)z^{n-m+1} = \sum_{n=0}^\infty (n+m-1)(n+m-2)\cdots(n+1)z^n.$$

So

$$a_n = \frac{(n+m-1)\cdots(n+1)}{(m-1)!}.$$

Since m is fixed and $n \to \infty$ we easily see that

$$\frac{a_n}{n^{m-1}/(m-1)!} \to 1, n \to \infty$$

as a_n has $m-1$ factors in the numerator.

7. Show that for $|z| < 1$ we have

$$\frac{z}{1-z^2} + \frac{z^2}{1-z^4} + \cdots + \frac{z^{2^n}}{1-z^{2^{n+1}}} + \cdots = \frac{z}{1-z}, \tag{1}$$

and

$$\frac{z}{1+z} + \frac{2z^2}{1+z^2} + \cdots + \frac{2^k z^{2^k}}{1+z^{2^k}} + \cdots = \frac{z}{1-z}. \tag{2}$$

Justify any change in the order of summation.

Hint: Use the dyadic expansion of an integer and the sum $1+2+2^2+\cdots+2^k = 2^{k+1}-1$.

We first prove the Lemma:

Every positive integer n has a unique expression of the $2^t + k_t 2^{t+1}$, with $t \geq 0$, $k_t \geq 0$.

We use for this the dyadic expansion of n. If

$$n = a_0 + a_1 2 + a_2 2^2 + \cdots + a_s 2^s, \quad a_i \in \{0, 1\}$$

we pick t to be the first nonzero a_i: i.e. $a_t = 1$, while $a_0 = a_1 = \cdots a_{t-1} = 0$. Then

$$n = 2^t + a_{t+1}2^{t+1} + \cdots + a_s 2^s = 2^t + 2^{t+1}(a_{t+1} + \cdots + a_s 2^{s-t-1}) = 2^t + 2^{t+1}k_t.$$

This expression is unique as

$$2^t + k_t 2^{t+1} = 2^j + 2^{j+1}k_j, j \neq t$$

implies that either $j > t$ or $j < t$. By symmetry we assume $j > t$, i.e., $j \geq t+1$. Look at the remainder when we divide by 2^{t+1}. The left-hand side gives 2^t, while the right-hand side gives 0.

We expand the left-side of (1) using the geometric series to get

$$\sum_{k_1=0}^{\infty} z^{2k_1+1} + \sum_{k_2=0}^{\infty} z^{4k_2+2} + \cdots + \sum_{k_m=0}^{\infty} z^{2^{m+1}k_m+2^m} + \cdots$$

while the right-hand side is just $\sum_{n=1}^{\infty} z^n$. The lemma proves the identity, provided that we have absolute convergence of the double series

$$\sum_{m=0}^{\infty} \sum_{k_m=0}^{\infty} z^{2^{m+1}k_m+2^m},$$

which would mean that we can rearrange the terms in whatever way we would like. But the lemma shows that the series

$$\sum_{m=0}^{\infty}\sum_{k_m=0}^{\infty}|z^{2^{m+1}k_m+2^m}|$$

is exactly the geometric series $\sum|z|^n$, $|z|<1$, which converges.

For (2) we expand the left-hand side to get

$$z\sum_0(-1)^n z^n + 2z^2\sum_0(-1)^n z^{2n} + \cdots + 2^k z^{2^k}\sum_0(-1)^n z^{2^k n} + \cdots$$
$$= \sum_1(-1)^{m+1}z^m + 2\sum_1(-1)^{m+1}z^{2m} + \cdots + 2^k\sum_1(-1)^{m+1}z^{2^k m} + \cdots.$$

Looking at the dyadic expansion of n we pick the first $a_t=1$, such that $a_0=a_1=\cdots=a_{t-1}=0$. Then $2^i|n$ for $i\le t$, while 2^{t+1} does not divide n and the same is true for higher powers of 2. For $i\le t-1$, $n=2^i*l$ with l even, so the coefficient it picks from the i-series in the sum is $2^i(-1)^{l+1}=-2^i$. On the other hand for $i=t$, $n=2^t*l$ with l odd, so that it picks a coefficient $2^t(-1)^{l+1}=2^t$. For $i>t$ it picks nothing. So the coefficient of z^n is

$$-1+(-2)+(-2^2)+\cdots+(-2^{t-1})+2^t=1$$

by the identity $1+2+\cdots+2^k=2^{k+1}-1$.

Example: $n=4$+higher powers of 2. We write the first 4 series in the sum:

$$\begin{aligned} z-z^2+z^3-z^4+\cdots \\ 2(z^2-z^4+z^6-z^8+\cdots) \\ 4(z^4-z^8+z^{12}-z^{16}+\cdots) \\ 8(z^8-z^{16}+z^{24}-z^{32}+\cdots) \end{aligned}$$

The first three give to z^4 coefficients -1, -2, 4, while the last and the subsequent ones give 0.

We need to explain the interchange of summations. We look at the series with absolute values on the terms. This is

$$\sum_{k=0}^{\infty}\sum_{m=1}^{\infty}2^k|z|^{2^k m}.$$

If this converges, then the original double sum converges absolutely and we can sum in whatever order we like. In this series the coefficient of $|z|^n$ is equal to

$$\sum_{2^k|m}2^k=\sum_{k\le t}2^k=2^{t+1}-1<2^{t+1}\le 2n, \quad n=2^t+a_{t+1}2^{t+1}+\cdots.$$

So we compare with the series $\sum 2n|z|^n$, which has radius of convergence 1. This suffices.

8. Find the holomorphic function of z that vanishes at $z = 0$ and has real part

$$u(x, y) = \frac{x(1 + x^2 + y^2)}{1 + 2x^2 - 2y^2 + (x^2 + y^2)^2}.$$

We substitute $z + \bar{z} = 2x$, $x^2 - y^2 = \Re(z^2) = (z^2 + \bar{z}^2)/2$ and $z\bar{z} = x^2 + y^2$ to get

$$u(x, y) = \frac{1}{2}\frac{(z + \bar{z})(1 + z\bar{z})}{1 + z^2 + \bar{z}^2 + z^2\bar{z}^2} = \frac{1}{2}\frac{z + \bar{z} + z^2\bar{z} + \bar{z}^2 z}{(1 + z^2)(1 + \bar{z}^2)} = \frac{1}{2}\left(\frac{z}{1 + z^2} + \frac{\bar{z}}{1 + \bar{z}^2}\right).$$

We set $f(z) = z/(1 + z^2)$, which is holomorphic, to get

$$u(x, y) = \frac{1}{2}(f(z) + \bar{f}(z))$$

i.e., $u(x, y)$ is the real part of $f(z)$, and $f(z) = 0$, this gives the answer to be $f(z)$.

Second method: (Look at p. 27 in Ahlfors) Since $f(z)$ is holomorphic, the conjugate function $g(x, y) = \overline{f(z)}$ depends only on $\bar{z}$, so we write $g(\bar{z}) = \overline{f(z)}$ to get

$$u(x, y) = \frac{1}{2}(f(z) + g(\bar{z})) = \frac{1}{2}(f(x + iy) + g(x - iy)).$$

This is true for x and y real. Formally we can plug $x = z$ and $y = z/(2i)$, so that

$$u(z/2, z/(2i)) = \frac{1}{2}(f(z) + g(0)).$$

This determines $f(z)$ up to a constant. If $u(0, 0) = 0$ and we ask for $f(0) = 0$, then $g(0) = 0$.

In our case

$$f(z) = 2u(z/2, z/(2i)) = 2\frac{(z/2)(1 + (z^2/4) + (-z^2/4))}{1 + 2(z^2/4 + z^2/4) + (z^2/4 - z^2/4)^2} = \frac{z}{1 + z^2}.$$

9. (i) Show that the Laplace operator can be calculated as

$$\Delta = 4\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}} = 4\frac{\partial}{\partial \bar{z}}\frac{\partial}{\partial z}.$$

(ii) Show that for any analytic function $f(z)$ we have

$$\Delta|f(z)|^2 = 4|f'(z)|^2, \quad \text{and} \quad \Delta\log(1 + |f(z)|^2) = \frac{4|f'(z)|^2}{(1 + |f(z)|^2)^2}.$$

(i) We have for a function f with continuous second partial derivatives

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right), \quad \frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)$$

which gives

$$4\frac{\partial^2 f}{\partial z\partial\bar z} = \left(\frac{\partial}{\partial x} - i\frac{\partial f}{\partial y}\right)(f_x + i f_y) = \left(\frac{\partial}{\partial x} - i\frac{\partial f}{\partial y}\right) f_x + i\left(\frac{\partial}{\partial x} - i\frac{\partial f}{\partial y}\right) f_y$$

$$= f_{xx} - i f_{yx} + i(f_{xy} - i f_{yy}) = f_{xx} - i f_{yx} + i f_{xy} + f_{yy} = f_{xx} + f_{yy} = \Delta f,$$

since the mixed partial f_{xy}, f_{yx} are equal for functions with continuous second partial derivatives. Reversing the order of the calculation gives $4\partial_{\bar z}\partial_z f = \Delta f$.

(ii) For an analytic function $\partial_z f(z) = f'(z)$, while $\partial_{\bar z}\overline{f(z)} = \overline{f'(z)}$, since f does not depend on $\bar z$, so the result from applying $\partial_{\bar z}$ on its conjugate can be calculated from $f'(z)$: e.g. $f(z) = z^2$, $\overline{f(z)} = \bar z^2$,$\partial_{\bar z}\overline{f(z)} = 2\bar z = \overline{2z} = \overline{\partial_z f(z)}$. This gives:

$$\Delta|f(z)|^2 = 4\partial_z\partial_{\bar z}(f(z)\overline{f(z)}) = 4\partial_z(f(z)\overline{f'(z)}) = 4\overline{f'(z)}\partial_z f(z) = 4\overline{f'(z)}f'(z) = 4|f'(z)|^2,$$

since $f(z)$ is a constant when we differentiate in $\bar z$, and $\overline{f'(z)}$ is a constant when we differentiate in z, as it does not depend on z.

For the second part we will use the chain rule as expressed in z and $\bar z$ coordinates (exercise 13 in homework 1). We use $g(w) = \log(1+w)$, which is holomorphic in w for $\Re w > -1$ (there is no difficulty in defining the argument of $1+w$ in this domain. For f we have $|f(z)|^2 \geq 0$. We have

$$g_{\bar w} = 0, \quad \text{and} \quad g_w = \frac{1}{1+w}.$$

$$\partial_z \log(1+|f(z)|^2) = \frac{1}{1+|f(z)|^2}(1+|f(z)|^2)_z = \frac{\overline{f(z)}f'(z)}{1+|f(z)|^2}.$$

We now apply the quotient rule

$$4\partial_{\bar z}\partial_z \log(1+|f(z)|^2) = 4\partial_{\bar z}\frac{\overline{f(z)}f'(z)}{1+|f(z)|^2} = 4\partial_{\bar z}\frac{\overline{f(z)}f'(z)}{1+f(z)\overline{f(z)}}$$

$$= 4\frac{\partial_{\bar z}(\overline{f(z)}f'(z))(1+|f(z)|^2) - \overline{f(z)}f'(z)\partial_{\bar z}(1+\overline{f(z)}f(z))}{(1+|f(z)|^2)^2}$$

$$= 4\frac{\overline{f'(z)}f'(z)(1+|f(z)|^2) - \overline{f(z)}f'(z)\overline{f'(z)}f(z)}{(1+|f(z)|^2)^2} = \frac{4\overline{f'(z)}f'(z)}{(1+|f(z)|^2)^2}.$$

10. Let $f(z)$ be holomorphic and one-to-one on a set containing the unit disc $\mathbb{D}$. Let $D' = f(\mathbb{D})$. Then the area $A(D')$ of D' is given by

$$A(D') = \pi\sum_{n=1}^{\infty} n|a_n|^2,$$

where $f(z) = \sum_{n=0}^{\infty} a_n z^n$. (Qualifying exam September 2006)

The Jacobian of the mapping $w = f(z)$ as a mapping from $\mathbb{D} \to \mathbb{C}$ is $|f'(z)|^2$. So by the change of variables formula

$$A(D') = \int\int_{D'} 1\, du\, dv = \int\int_{\mathbb{D}} |f'(z)|^2\, dx\, dy.$$

If $f(z) = \sum_{n=0}^{\infty} a_n z^n$, then

$$f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1} \Longrightarrow |f'(z)|^2 = \sum_{n,m=1}^{\infty} nm a_n \bar{a}_m z^{n-1} \bar{z}^{m-1}.$$

We switch to polar coordinates $z = re^{i\theta}$ to get ($dxdy = rdrd\theta$)

$$A(D') = \int_0^1 \int_0^{2\pi} \sum_{n,m=1}^{\infty} nm a_n \bar{a}_m r^{n+m-2} e^{i\theta(n-m)} r\, dr\, d\theta$$

If $n - m \neq 0$, the integral $\int_0^{2\pi} e^{i\theta(n-m)}\, d\theta = 0$, while for $n = m$ we get 2π. This gives

$$A(D') = \int_0^1 \sum_{n=1}^{\infty} n^2 |a_n|^2 r^{2n-1} 2\pi = 2\pi \sum_{n=1}^{\infty} n^2 |a_n|^2 \left[\frac{r^{2n}}{2n}\right]_0^1 = \pi \sum_{n=1}^{\infty} |a_n|^2 n.$$

11. (a) Compute the integral

$$\int_{|z|=r} x\, dz$$

for the positive sense of the circle, in two ways: first by using a parametrization, and second, by observing that $x = (1/2)(z + \bar{z}) = (1/2)(z + r^2/z)$ on the circle.

First Method: Parametrization of the circle as $z = re^{i\theta}$, $x = r\cos\theta$, $dz = rie^{i\theta}$. We get

$$\int_{|z|=r} x\, dz = \int_0^{2\pi} r\cos\theta r i e^{i\theta}\, d\theta = \int_0^{2\pi} r^2 i \cos\theta(\cos\theta + i\sin\theta) d\theta = \int_0^{2\pi} r^2 i(\cos^2\theta + i\cos\theta\sin\theta) d\theta$$

$$= r^2 i \int_0^{2\pi} \frac{1+\cos(2\theta)}{2} + \frac{i}{2}\sin(2\theta) d\theta = r^2 i \left[\frac{\theta}{2} + \frac{\sin 2\theta}{4} + \frac{-i\cos 2\theta}{4}\right]_0^{2\pi} = r^2 i\pi.$$

Second method: On the circle $|z| = r$, we have $z\bar{z} = r^2 \Longrightarrow \bar{z} = r^2/z$, so that $x = (z + \bar{z})/2 = (1/2)(z + r^2/z)$. We plug into the integral to get

$$\int_{|z|=r} x\, dz = \int_{|z|=r} \frac{z}{2} + \frac{r^2}{2z}\, dz = \frac{r^2}{2} 2\pi i = r^2 i\pi,$$

since we know that $\int_{|z|=r} z^n\, dz = 0$ for $n \neq -1$ and for $n = -1$ we get $2\pi i$.

(b) Compute the integral

$$\int_{|z|=2} \frac{dz}{z^2-1}$$

for the positive sense of the circle. Hint: Find a primitive function of the integrand.

We use partial fractions to find that

$$\frac{1}{z^2-1} = \frac{1}{2}\left(\frac{1}{z-1} - \frac{1}{z+1}\right).$$

We would like to integrate to get $(\log(z-1) - \log(z+1))/2$. However, this cannot make sense in the region we are working: The $\log(z-1)$ requires a cut from, say $-\infty$ to 1 and $\log(z+1)$ requires a cut from $-\infty$ to -1. These cuts meet our domain, since the circle $|z| = 2$ contains both points -1 and 1. Make sense can be thought of being single-valued holomorphic functions that give us the primitive of the terms we consider. However, if we write the expected answer as

$$F(z) = \frac{1}{2}\log\left(\frac{z-1}{z+1}\right)$$

things are much better: First of all, $(z-1)/(z+1)$ is real only for real z. This is seen as follows:

$$\frac{z-1}{z+1} \in \mathbb{R} \Leftrightarrow \frac{z-1}{z+1} = \frac{\bar{z}-1}{\bar{z}+1} \Leftrightarrow z\bar{z}+z-\bar{z}-1 = z\bar{z}+\bar{z}-z-1 \Leftrightarrow 2(z-\bar{z}) = 0 \Leftrightarrow \Im(z) = 0.$$

Moreover, it is a negative number exactly between -1 and 1. So on the complement of $[-1, 1]$ the function takes nonnegative values. If we define the principal branch of the logarithm to be

$$\log w = \log|w| + i\arg w, \quad -\pi < \arg w < \pi,$$

then $F(z)$ as the composition of holomorphic maps is holomorphic and is the primitive of $1/(z^2-1)$ in a region that contains our contour. Consequently $dz/(z^2-1)$ is exact and the integral is 0.

Remark: One can solve this problem with the residue theorem, which will be seen later in the course.