Math 70300

Homework 2

Due: September 28, 2006

1. Suppose that f is holomorphic in a region Ω, i.e. an open connected set. Prove that in any of the following cases

   (a) $\Re(f)$ is constant; (b) $\Im(f)$ is constant; (c) $|f|$ is constant; (d) $\arg(f)$ is constant;

   we can conclude that f is a constant.

2. Show that if $\{a_n\}_{n=0}^{\infty}$ is a sequence of non-zero complex numbers such that

   $$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = L,$$

   then

   $$\lim_{n\to\infty} |a_n|^{1/n} = L.$$

   This is the ratio test and it can be used for the calculation of the radius of convergence of a power series.

   Hint: Show that

   $$\liminf \frac{|a_{n+1}|}{|a_n|} \le \liminf |a_n|^{1/n} \le \limsup |a_n|^{1/n} \le \limsup \frac{|a_{n+1}|}{|a_n|}.$$

3. (a) Find the radius of convergence of the hypergeometric series

   $$F(\alpha,\beta,\gamma;z) = 1 + \sum_{n=1}^{\infty} \frac{\alpha(\alpha+1)\cdots(\alpha+n-1)\beta(\beta+1)\cdots(\beta+n-1)}{n!\gamma(\gamma+1)\cdots(\gamma+n-1)} z^n.$$

   Here $\alpha,\beta \in \mathbb{C}$ and $\gamma \ne 0,-1,-2,\ldots$.

   (b) Find the radius of convergence of the Bessel function of order r:

   $$J_r(z) = \left(\frac{z}{2}\right)^r \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+r)!} \left(\frac{z}{2}\right)^{2n}, \quad r \in \mathbb{N}.$$

4. Prove that, although all the following power series have $R = 1$,

   (a) $\sum nz^n$ does not converge on any point of the unit circle,

   (b) $\sum z^n/n^2$ converges at every point of the unit circle,

   (c) $\sum z^n/n$ converges at every point of the unit circle except $z = 1$. (Hint: Use summation by parts.)

5. The Fibonacci numbers are defined by $c_0 = 1$, $c_1 = 1$,

$$c_n = c_{n-1} + c_{n-2}, \quad n = 2, 3, \ldots.$$

Define their generating function as

$$F(z) = \sum_{n=0}^{\infty} c_n z^n.$$

(a) Find a quadratic polynomial $Az^2 + Bz + C$ such that

$$(Az^2 + Bz + C)F(z) = 1.$$

(b) Use partial fractions to determine the following closed expression for c_n.

$$c_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}}{\sqrt{5}}.$$

(Qualifying exam Sept. 2006)

6. Expand $(1-z)^{-m}$ in powers of z, for $m \in \mathbb{N}$. Let

$$(1-z)^{-m} = \sum_{n=0}^{\infty} a_n z^n,$$

then

$$a_n \sim \frac{1}{(m-1)!} n^{m-1}, \quad n \to \infty,$$

where $\sim$ means that the quotient of the expressions to the left and the right of it tends to 1.

7. Show that for $|z| < 1$ we have

$$\frac{z}{1-z^2} + \frac{z^2}{1-z^4} + \cdots + \frac{z^{2^n}}{1-z^{2^{n+1}}} + \cdots = \frac{z}{1-z},$$

and

$$\frac{z}{1+z} + \frac{2z^2}{1+z^2} + \cdots + \frac{2^k z^{2^k}}{1+z^{2^k}} + \cdots = \frac{z}{1-z}.$$

Justify any change in the order of summation.

Hint: Use the dyadic expansion of an integer and the sum $1+2+2^2+\cdots+2^k = 2^{k+1}-1$.

8. Find the holomorphic function of z that vanishes at $z = 0$ and has real part

$$u(x, y) = \frac{x(1 + x^2 + y^2)}{1 + 2x^2 - 2y^2 + (x^2 + y^2)^2}.$$

9. (i) Show that the Laplace operator can be calculated as

$$\Delta = 4\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}} = 4\frac{\partial}{\partial \bar{z}}\frac{\partial}{\partial z}.$$

(ii) Show that for any analytic function $f(z)$ we have

$$\Delta|f(z)|^2 = 4|f'(z)|^2, \quad \text{and} \quad \Delta \log(1 + |f(z)|^2) = \frac{4|f'(z)|^2}{(1 + |f(z)|^2)^2}.$$

10. Let $f(z)$ be holomorphic and one-to-one on a set containing the unit disc $\mathbb{D}$. Let $D' = f(\mathbb{D})$. Then the area $A(D')$ of D' is given by

$$A(D') = \pi \sum_{n=1}^{\infty} n|a_n|^2,$$

where $f(z) = \sum_{n=0}^{\infty} a_n z^n$. (Qualifying exam September 2006)

11. (a) Compute the integral

$$\int_{|z|=r} x\, dz$$

for the positive sense of the circle, in two ways: first by using a parametrization, and second, by observing that $x = (1/2)(z + \bar{z}) = (1/2)(z + r^2/z)$ on the circle.

(b) Compute the integral

$$\int_{|z|=2} \frac{dz}{z^2 - 1}$$

for the positive sense of the circle. Hint: Find a primitive function of the integrand.