Math 70300

Homework 1

September 12, 2006

The homework consists mostly of a selection of problems from the suggested books.

1. (a) Find the value of $(1+i)^n + (1-i)^n$ for every $n \in \mathbb{N}$.

   We will use the polar form of $1+i = \sqrt{2}(\cos(\pi/4) + i\sin(\pi/4))$ and De Moivre's formula. Since $\overline{1+i} = 1-i$, $\overline{(1+i)^n} = (\overline{1+i})^n = (1-i)^n$ we have

$$(1+i)^n + (1-i)^n = 2\Re(1+i)^n = 2\Re 2^{n/2}(\cos(n\pi/4) + i\sin(n\pi/4))$$

$$= 2 \cdot 2^{n/2}\cos(n\pi/4) = 2^{n/2+1}\cos(n\pi/4).$$

   (b) Show that

$$\left(\frac{-1 \pm i\sqrt{3}}{2}\right)^3 = 1, \quad \left(\frac{\pm 1 \pm i\sqrt{3}}{2}\right)^6 = 1.$$

   Since $x^3 - 1 = (x-1)(x^2+x+1)$ it suffices to prove that

$$\rho_{1,2} = \frac{-1 \pm i\sqrt{3}}{2}$$

   satisfies $\rho_{1,2}^2 + \rho_{1,2} + 1 = 0$. We have

$$\left(\frac{-1 \pm i\sqrt{3}}{2}\right)^2 = \frac{1 - 3 \mp i2\sqrt{3}}{4} = \frac{-1 \mp i\sqrt{3}}{2}.$$

   From this $\rho_{1,2}^2 + \rho_{1,2} + 1 = 0$ follows easily.

   For $\dfrac{\pm 1 \pm i\sqrt{3}}{2}$ we argue as follows: if the signs are opposite we are considering $\pm\rho_{1,2}$. Since $\rho_{1,2}^3 = 1$, $\rho_{1,2}^6 = 1$. For the other choices

$$r_1 = \frac{1 + i\sqrt{3}}{2}, \quad r_2 = \frac{-1 - i\sqrt{3}}{2}, \quad r_2 = -r_1$$

   which are called the sixth primitive roots of unity, we argue as follows: $x^6 - 1 = (x^3-1)(x+1)(x^2-x+1)$ implies that we should prove that $r_1^2 - r_1 + 1 = 0$. We calculate

$$r_1^2 = \frac{1 - 3 + i2\sqrt{3}}{4} = \frac{-1 + i\sqrt{3}}{2}$$

from which the result follows. This way we avoid calculating sixth powers.

(c) Find the fourth roots of -1. We write $-1 = 1(\cos\pi + i\sin\pi)$ (polar form) and apply De Moivre's formulas. The fourth roots are:

$$r_k = \sqrt[4]{1}\left(\cos\left(\frac{\pi+2\pi k}{4}\right) + i\sin\left(\frac{\pi+2\pi k}{4}\right)\right), \quad k = 0, 1, 2, 3.$$

This gives

$$r_0 = \cos(\pi/4) + i\sin(\pi/4) = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}, \quad r_1 = \cos(3\pi/4) + i\sin(3\pi/4) = -\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$$

$$r_2 = \cos(5\pi/4) + i\sin(5\pi/4) = -\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}, \quad r_3 = \cos(7\pi/4) + i\sin(7\pi/4) = \frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}.$$

2. Find the conditions under which the equation $az + b\bar{z} + c = 0$ has exactly one solution and compute the solution.

We write the equation and its conjugate as a system

$$\begin{aligned} az + b\bar{z} + c &= 0 \\ \bar{b}z + \bar{a}\bar{z} + \bar{c} &= 0. \end{aligned}$$

We think of the variables z and $\bar{z}$ as independent variables. This system has exactly one solution when its determinant is nonzero, i.e.,

$$\begin{vmatrix} a & b \\ \bar{b} & \bar{a} \end{vmatrix} = |a|^2 - |b|^2 \neq 0.$$

In this case the solution is given by Cramer's rule as

$$z = \frac{\begin{vmatrix} -c & b \\ -\bar{c} & \bar{a} \end{vmatrix}}{|a|^2 - |b|^2} = \frac{\bar{c}b - c\bar{a}}{|a|^2 - |b|^2}.$$

We can also solve for $\bar{z}$ and we see that indeed it is conjugate of z:

$$\bar{z} = \frac{\begin{vmatrix} a & -c \\ \bar{b} & -\bar{c} \end{vmatrix}}{|a|^2 - |b|^2} = \frac{\bar{b}c - a\bar{c}}{|a|^2 - |b|^2}.$$

Alternatively we set the real and imaginary parts of the equation to be 0. We set $a = \alpha_1 + i\alpha_2$, $b = \beta_1 + i\beta_2$, $c = \gamma_1 + i\gamma_2$, $z = x + iy$ to get

$$\alpha_1 x - \alpha_2 y + \beta_1 x + \beta_2 y + \gamma_1 = 0,$$

$$\alpha_2 x + \alpha_1 y + \beta_2 x - \beta_1 y + \gamma_2 = 0,$$

which gives the system

$$\begin{aligned}(\alpha_1+\beta_1)x+(-\alpha_2+\beta_2)y+\gamma_1 &= 0\\ (\alpha_2+\beta_2)x+(-\beta_1+\alpha_1)y+\gamma_2 &= 0.\end{aligned}$$

The determinant is

$$\begin{vmatrix}\alpha_1+\beta_1 & -\alpha_2+\beta_2\\ \alpha_2+\beta_2 & -\beta_1+\alpha_1\end{vmatrix} = \alpha_1^2-\beta_1^2-\beta_2^2+\alpha_2^2 = |a|^2-|b|^2.$$

The advantage of the complex method is obvious.

3. Describe geometrically the sets of points z in the complex plane defined by the following relations.

(a) $|z-z_1| = |z-z_2|$, where z_1, z_2 are fixed points in $\mathbb{C}$.

This is the midpoint-perpendicular to the segment form z_1 to z_2.

In you do not like this characterization of the points equidistant from z_1 and z_2, notice first that the midpoint $(z_1+z_2)/2$ belongs to the locus we are investigating:

$$\left|\frac{z_1+z_2}{2}-z_1\right| = \left|\frac{z_2-z_1}{2}\right| = \left|\frac{z_1+z_2}{2}-z_2\right|.$$

Moreover, setting $z_1=\alpha_1+i\alpha_2$, $z_2=\beta_1+i\beta_2$ we get for $z=x+iy$

$$\begin{aligned}|z-z_1|^2=|z-z_2|^2 &\Leftrightarrow (x-\alpha_1)^2+(y-\alpha_2)^2=(x-\beta_1)^2+(y-\beta_2)^2\\ &\Leftrightarrow \alpha_1^2+\alpha_2^2-2\alpha_1x-2\alpha_2y=\beta_1^2+\beta_2^2-2\beta_1x-2\beta_2y\\ &\Leftrightarrow (\beta_1-\alpha_1)x+(\beta_2-\alpha_2)y=\frac{1}{2}(\beta_1^2+\beta_2^2-\alpha_1^2-\alpha_2^2).\end{aligned}$$

This is the equation of a line with slope

$$\lambda = -\frac{\beta_1-\alpha_1}{\beta_2-\alpha_2},$$

while the vector $z_1-z_2=(\alpha_1-\beta_1,\alpha_2-\beta_2)$ has slope $(\alpha_2-\beta_2)/(\alpha_1-\beta_1)$. Their product is -1. So they represent perpendicular lines.

(b) $1/z=\bar{z}$.

$$1/z=\bar{z} \Leftrightarrow z\bar{z}=1 \Leftrightarrow |z|^2=1 \Leftrightarrow |z|=1.$$

This is a circle centered at the origin with radius 1.

(c) $|z|=\Re(z)+1$.

We set $z=x+iy$ to get

$$|z|=\Re(z)+1 \Leftrightarrow |z|^2=(x+1)^2 \Leftrightarrow x^2+y^2=x^2+2x+1 \Leftrightarrow y^2=2x+1=2(x+1/2).$$

This is the equation of a parabola with axis the real axis and vertex at $(-1/2,0)$.

4. Prove the Lagrange identity

$$\left|\sum_{i=1}^n a_i b_i\right|^2 = \sum_{i=1}^n |a_i|^2 \sum_{i=1}^n |b_i|^2 - \sum_{1\le i<j\le n} |a_i\bar{b}_j - a_j\bar{b}_i|^2.$$

We need to show the equivalent formula

$$\left|\sum_{i=1}^n a_i b_i\right|^2 + \sum_{1\le i<j\le n} |a_i\bar{b}_j - a_j\bar{b}_i|^2 = \sum_{i=1}^n |a_i|^2 \sum_{i=1}^n |b_i|^2.$$

We expand out using $|z|^2 = z\bar{z}$ to get

$$\left(\sum_{i=1}^n a_i b_i\right)\left(\sum_{j=1}^n \bar{a}_j\bar{b}_j\right) + \sum_{1\le i<j\le n} (a_i\bar{b}_j - a_j\bar{b}_i)(\bar{a}_i b_j - \bar{a}_j b_i)$$

$$= \sum_{i,j=1}^n a_i b_i \bar{a}_j\bar{b}_j + \sum_{i<j}(|a_i|^2|b_j|^2 + |a_j|^2|b_i|^2 - a_i\bar{a}_j b_i\bar{b}_j - a_j\bar{a}_i b_j\bar{b}_i)$$

$$= \sum_{i\neq j} a_i b_i\bar{a}_j\bar{b}_j + \sum_{i=j=1}^n |a_i|^2|b_i|^2 + \sum_{i<j}(|a_i|^2|b_j|^2 + |a_j|^2|b_i|^2) - \sum_{i<j}(a_i\bar{a}_j b_i\bar{b}_j + a_j\bar{a}_i b_j\bar{b}_i).$$

The first and the last sum are equal and, therefore, cancel. This can be seen as follows: In the last sum we have the condition $i<j$ while in the first only $i\neq j$. A pair (i,j) of unequal integers has either $i<j$ or $j<i$. In the second case $a_i b_i\bar{a}_j\bar{b}_j = a_{j'}b_{j'}\bar{a}_{i'}\bar{b}_{i'}$ with $j = i' < j' = i$, so we get the term $a_j b_j \bar{a}_i\bar{b}_i$ with $i<j$. The two summands in the middle give exactly

$$\sum_{i=1}^n |a_i|^2 \sum_{j=1}^n |b_j|^2$$

by the distributive law and the same thinking about $i\neq j$ vs $i<j$ and $j<i$.

5. Prove that

$$\left|\frac{a-b}{1-\bar{a}b}\right| < 1$$

if $|a|<1$ and $|b|<1$.

We have

$$\left|\frac{a-b}{1-\bar{a}b}\right| < 1 \Leftrightarrow |a-b| < |1-\bar{a}b| \Leftrightarrow |a-b|^2 < |1-\bar{a}b|^2 \Leftrightarrow |a|^2+|b|^2-2\Re(a\bar{b}) < 1+|\bar{a}b|^2-2\Re(\bar{a}b)$$

(here we notice that $\Re(\bar{a}b) = \Re(a\bar{b})$ as they are conjugate numbers)

$$\Leftrightarrow |a|^2+|b|^2 < 1+|a|^2|b|^2 \Leftrightarrow 0 < 1+|a|^2|b|^2-|a|^2-|b|^2 = (1-|a|^2)(1-|b|^2).$$

The assumption $|a|<1$ and $|b|<1$ gives the result.

6. Prove that it is impossible to define a total ordering on $\mathbb{C}$. In other words, one cannot find a relation $\gg$ between complex numbers so that:

   (i) For any two complex numbers z and w one and only one of the following is true: $z \gg w$, $w \gg z$ or $z = w$.

   (ii) For all $z_1, z_2, z_3 \in \mathbb{C}$ the relation $z_1 \gg z_2$ implies $z_1 + z_3 \gg z_2 + z_3$.

   (iii) Moreover, for all $z_1, z_2, z_3 \in \mathbb{C}$ with $z_3 \gg 0$

   $$z_1 \gg z_2 \Longrightarrow z_1 z_3 \gg z_2 z_3.$$

   Hint: Is $i \gg 0$?

   Since $i \neq 0$, we have either $i \gg 0$ or $i \ll 0$.

   Step 1: If $i \gg 0$, then by (iii) we get

   $$i^2 \gg i \cdot 0 = 0 \Longrightarrow -1 \gg 0.$$

   Since $i \gg 0$ we also get by (iii) that $-i \gg 0$. We use (ii) to get

   $$0 = i + (-i) \gg 0 + (-i) = -i \gg 0.$$

   The transitive property gives a contradiction to (i).

   Step 2: Assume $i \ll 0$. Then $-i \gg 0$, since otherwise $-i \ll 0$ and (ii) implies $0 = i + (-i) \ll 0$ contradicting (i). As in Step 1, $-i \gg 0$ implies $-1 \gg 0$ and then $i \gg 0$, which is a contradiction.

7. Prove that the points z_1, z_2, z_3 are vertices of an equilateral triangle if $z_1 + z_2 + z_3 = 0$ and $|z_1| = |z_2| = |z_3|$.

   Rotating the complex numbers z_1, z_2, z_3 by the same angle keeps their lengths equal and keeps the relation $z_1 + z_2 + z_3 = 0$ valid. So we can assume that $z_1 = r$ is real. Then $r = |z_2| = |z_3|$. Setting $z_2 = r(\cos\theta + i\sin\theta)$, $z_3 = r(\cos\phi + i\sin\phi)$ in polar coordinates we get

   $$z_1 + z_2 + z_3 = 0 \Longrightarrow r(1 + \cos\theta + \cos\phi + i(\sin\theta + \sin\phi)) = 0.$$

   From the imaginary parts we get

   $$\sin\theta = -\sin\phi \Longrightarrow \sin^2\theta = \sin^2\phi \Longrightarrow \cos^2\theta = \cos^2\phi,$$

   by the basic trigonometric identity. This implies that the cosines are either equal or opposite. They cannot be opposite, since the real part of the equation gives

   $$1 + \cos\theta + \cos\phi = 0.$$

   So

   $$1 + 2\cos\theta = 0 \Longrightarrow \cos\theta = -1/2 \Longrightarrow \theta = 2\pi/3, 4\pi/3.$$

The equation $\sin\theta = -\sin\phi$ now gives respectively $\phi = 4\pi/3, 2\pi/3$ (notice that z_3 should be in the left-hand plane to have the same cos as z_2). The points are vertices of an equilateral triangle. If you want, you can check that

$$|1-(\cos 2\pi/3 + i\sin 2\pi/3)| = |1-(\cos 4\pi/3 + i\sin 4\pi/3)|$$

$$= |\cos 2\pi/3 + i\sin 2\pi/3 - (\cos 4\pi/3 + i\sin 4\pi/3)|.$$

8. Verify the Cauchy-Riemann equations for z^2 and z^3. We have

$$z^2 = (x^2-y^2) + i2xy, \quad z^3 = x^3 - 3xy^2 + i(3x^2y - y^3)$$

The first gives for $f(z) = z^2$, $u = x^2 - y^2$, $v = 2xy$ and

$$\frac{\partial u}{\partial x} = 2x = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -2y = -\frac{\partial v}{\partial x}.$$

For the function $f(z) = z^3$ we have $u = x^3 - 3xy^2$ and $v = 3x^2y - y^3$ and

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -6xy = -\frac{\partial v}{\partial x}.$$

9. Express $\cos(3\phi)$ and $\cos(4\phi)$ in terms of $\cos(\phi)$. Express $\sin(3\phi)$ in terms of $\sin(\phi)$.

By de Moivre's formula

$$(\cos\phi + i\sin\phi)^n = \cos(n\phi) + i\sin(n\phi)$$

for $n = 3, 4$ we get using the previous exercise:

$$\cos(3\phi) = \Re((\cos\phi + i\sin\phi)^3) = (\cos\phi)^3 - 3\cos\phi\sin^2\phi = \cos^3\phi - 3\cos\phi(1-\cos^2\phi)$$

$$= 4\cos^3\phi - 3\cos\phi,$$

$$\sin(3\phi) = \Im((\cos\phi + i\sin\phi)^3) = 3\cos^2\phi\sin\phi - \sin^3\phi = 3(1-\sin^2\phi)\sin\phi - \sin^3\phi$$

$$= 3\sin\phi - 4\sin^3\phi.$$

For $\cos(4\phi)$ we have

$$\cos(4\phi) = \Re((\cos\phi + i\sin\phi)^4) = \Re(\cos^2\phi - \sin^2\phi + i2\sin\phi\cos\phi)^2 = (\cos^2\phi - \sin^2\phi)^2$$

$$-(2\sin\phi\cos\phi)^2 = (2\cos^2\phi - 1)^2 - 4\sin^2\phi\cos^2\phi = 4\cos^4\phi - 4\cos^2\phi + 1 - 4(1-\cos^2\phi)\cos^2\phi$$

$$= 8\cos^4\phi - 8\cos^2\phi + 1.$$

10. Simplify $1+\cos(\phi)+\cos(2\phi)+\cdots+\cos(n\phi)$ and $\sin(\phi)+\sin(2\phi)+\cdots+\sin(n\phi)$.

Set $z=\cos\phi+i\sin\phi$, so that $z^j=\cos(j\phi)+i\sin(j\phi)$. Then

$$(1+\cos(\phi)+\cos(2\phi)+\cdots+\cos(n\phi))+i(\sin(\phi)+\sin(2\phi)+\cdots+\sin(n\phi))$$

$$=\sum_{j=0}^{n}(\cos(j\phi)+i\sin(j\phi))=z^0+z^1+\cdots+z^n=\frac{1-z^{n+1}}{1-z}$$

We substitute $z^{n+1}=\cos(n+1)\phi+i\sin(n+1)\phi$ is the last and multiply with $1-\bar{z}$ to get

$$=\frac{(1-z^{n+1})(1-\bar{z})}{|1-z|^2}=\frac{(1-\cos(n+1)\phi-i\sin(n+1)\phi)(1-\cos\phi+i\sin\phi)}{(1-\cos\phi)^2+\sin^2\phi}$$

We take real and imaginary parts to get

$$1+\cos(\phi)+\cos(2\phi)+\cdots+\cos(n\phi)=\frac{(1-\cos(n+1)\phi)(1-\cos\phi)+\sin(n+1)\phi\sin\phi}{2-2\cos\phi}$$

$$=\frac{1-\cos\phi}{2-2\cos\phi}+\frac{-\cos((n+1)\phi)(1-\cos\phi)+\sin((n+1)\phi\sin\phi)}{2-2\cos\phi}$$

$$=\frac{1}{2}+\frac{-\cos((n+1)\phi)2\sin^2\phi/2+\sin((n+1)\phi)2\sin(\phi/2)\cos(\phi/2)}{4\sin^2\phi/2}$$

$$=\frac{1}{2}+\frac{-\cos((n+1)\phi)\sin\phi/2+\sin((n+1)\phi)\cos\phi/2}{2\sin\phi/2}$$

$$=\frac{1}{2}+\frac{\sin((n+1_\phi-\phi/2)}{\sin\phi/2}=\frac{1}{2}+\frac{\sin((n+1/2)\phi)}{2\sin\phi/2}$$

using the identities $\sin^2(x/2)=(1-\cos x)/2$, $\sin x=2\sin(x/2)\cos(x/2)$ and the addition theorem for sin. The calculation can be simplified with the use of $e^{i\theta}$.

$$\sin\phi+\sin(2\phi)+\cdots+\sin(n\phi)=\frac{\Im(1-z^{n+1})(1-\bar{z})}{|1-z|^2}$$

$$=\frac{(1-\cos((n+1)\phi))\sin\phi-\sin((n+1)\phi)(1-\cos\phi)}{2-2\cos\phi}$$

$$=\frac{\sin\phi-\cos((n+1)\phi)\sin\phi-\sin((n+1)\phi)(1-\cos\phi)}{4\sin^2\phi/2}$$

$$=\frac{2\sin(\phi/2)\cos(\phi/2)-\cos((n+1)\phi)2\sin(\phi/2)\cos(\phi/2)-\sin((n+1)\phi)2\sin^2\phi/2}{4\sin^2\phi/2}$$

$$=\frac{1}{2}\cot\frac{\phi}{2}-\frac{\cos((n+1)\phi)\cos\phi/2+\sin((n+1)\phi)\sin\phi/2}{2\sin\phi/2}=\frac{1}{2}\cot\frac{\phi}{2}-\frac{\cos(n+1/2)\phi}{2\sin\phi/2}.$$

11. Prove that the diagonals of a parallelogram bisect each other and that the diagonals of a rhombus are orthogonal.

If the two sides of the parallelogram are represented by the geometric image of the complex numbers z_1 and z_2, then the diagonal they enclose is represented by $z_1 + z_2$. The midpoint of it is $(z_1 + z_2)/2$. The second diagonal corresponds to the difference $z_1 - z_2$. The midpoint of it corresponds to $(z_1 - z_2)/2$. However, the second diagonal is not a vector starting at the origin but at z_2. So the midpoint of the second diagonal is given by the geometric image of the complex number $(z_1 - z_2)/2 + z_2 = (z_1 + z_2)/2$. The two answers are equal, so the two diagonals meet at their midpoint.

We need to show that if $|z_1| = |z_2|$ then the vectors represented by $z_1 + z_2$ and $z_1 - z_2$ are perpendicular, or, equivalently, that the arguments of these two complex numbers differ by $\pi/2$. The difference of the arguments shows up in the quotient, so we need to prove that the complex numbers $z_1 + z_2$ and $z_1 - z_2$ representing the two diagonals have purely imaginary quotient. But this is true iff

$$\frac{\bar{z}_1 + \bar{z}_2}{\bar{z}_1 - \bar{z}_2} = -\frac{z_1 + z_2}{z_1 - z_2} \Leftrightarrow (\bar{z}_1 + \bar{z}_2)(z_1 - z_2) = -(z_1 + z_2)(\bar{z}_1 - \bar{z}_2) \Leftrightarrow |z_1|^2 - |z_2|^2 - \bar{z}_1 z_2 + \bar{z}_2 z_1$$

$$= -(|z_1|^2 - |z_2|^2 - z_1\bar{z}_2 + z_2\bar{z}_1) \Leftrightarrow |z_1|^2 - |z_2|^2 = -|z_1|^2 + |z_2|^2 \Leftrightarrow |z_1|^2 = |z_2|^2.$$

12. Prove rigorously that the functions $f(z)$ and $\overline{f(\bar{z})}$ are simultaneously holomorphic.

If $f(z) = u(x, y) + iv(x, y)$, then $\overline{f(\bar{z})} = u(x, -y) - iv(x, -y)$. We calculate the partial derivatives with respect to x and y for $\overline{f(\bar{z})}$ using the chain rule from multivariable calculus:

$$\frac{\partial u(x, -y)}{\partial x} = \frac{\partial u}{\partial x}(x, -y)\frac{\partial x}{\partial x} + \frac{\partial u}{\partial y}(x, -y)\frac{\partial(-y)}{\partial x} = \frac{\partial u}{\partial x}(x, -y).$$

$$\frac{\partial u(x, -y)}{\partial y} = \frac{\partial u}{\partial x}(x, -y)\frac{\partial x}{\partial y} + \frac{\partial u}{\partial y}(x, -y)\frac{\partial(-y)}{\partial y} = -\frac{\partial u}{\partial y}(x, -y).$$

$$\frac{\partial(-v(x, -y))}{\partial x} = \frac{\partial(-v)}{\partial x}(x, -y)\frac{\partial x}{\partial x} + \frac{\partial(-v)}{\partial y}(x, -y)\frac{\partial(-y)}{\partial x} = -\frac{\partial v}{\partial x}(x, -y).$$

$$\frac{\partial(-v(x, -y))}{\partial y} = \frac{\partial(-v)}{\partial x}(x, -y)\frac{\partial x}{\partial y} + \frac{\partial(-v)}{\partial y}(x, -y)\frac{\partial(-y)}{\partial y} = -\frac{\partial(-v)}{\partial y}(x, -y) = \frac{\partial v}{\partial y}(x, -y).$$

This gives:

$$\frac{\partial u(x, y)}{\partial x} = \frac{\partial v(x, y)}{\partial y} \Leftrightarrow \frac{\partial u(x, -y)}{\partial x} = \frac{\partial(-v(x, -y))}{\partial y}$$

and

$$\frac{\partial u(x, y)}{\partial y} = -\frac{\partial v(x, y)}{\partial x} \Leftrightarrow \frac{\partial u(x, -y)}{\partial y} = -\frac{\partial(-v(x, -y))}{\partial x}.$$

This proves the equivalence of the Cauchy-Riemann equations.

13. Suppose that U and V are open sets in the complex plane. Prove that if $f : U \to V$ and $g : V \to \mathbb{C}$ are two functions that are differentiable in the real sense (in x and y) and $h = g \circ f$, then the complex version of the chain rule is

$$\frac{\partial h}{\partial z} = \frac{\partial g}{\partial z}\frac{\partial f}{\partial z} + \frac{\partial g}{\partial \bar{z}}\frac{\partial \bar{f}}{\partial z}, \quad \frac{\partial h}{\partial \bar{z}} = \frac{\partial g}{\partial z}\frac{\partial f}{\partial \bar{z}} + \frac{\partial g}{\partial \bar{z}}\frac{\partial \bar{f}}{\partial \bar{z}}.$$

Set $f(x,y) = u(x,y) + iv(x,y)$, i.e., $(x,y) \to (u(x,y), v(x,y)) \to h(u,v)$. By the standard chain rule for functions of two variables we have

$$\begin{aligned} \frac{\partial h}{\partial x} &= \frac{\partial g}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial g}{\partial y}\frac{\partial v}{\partial x} \\ \frac{\partial h}{\partial y} &= \frac{\partial g}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial g}{\partial y}\frac{\partial v}{\partial y}. \end{aligned} \tag{1}$$

Moreover,

$$\begin{aligned} \frac{\partial f}{\partial z} &= \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) = \frac{1}{2}(u_x + iv_x - i(u_y + iv_y)) \\ \frac{\partial \bar{f}}{\partial z} &= \frac{1}{2}\left(\frac{\partial \bar{f}}{\partial x} - i\frac{\partial \bar{f}}{\partial y}\right) = \frac{1}{2}(u_x - iv_x - i(u_y - iv_y)). \end{aligned} \tag{2}$$

By the definition of $\partial/\partial z$ and $\partial/\partial \bar{z}$ we have

$$\begin{aligned} \frac{\partial g}{\partial z} &= \frac{1}{2}\left(\frac{\partial g}{\partial x} - i\frac{\partial g}{\partial y}\right) \\ \frac{\partial g}{\partial \bar{z}} &= \frac{1}{2}\left(\frac{\partial g}{\partial x} + i\frac{\partial g}{\partial y}\right). \end{aligned}$$

We add and subtract the last two equations to get

$$\frac{\partial g}{\partial x} = \frac{\partial g}{\partial z} + \frac{\partial g}{\partial \bar{z}}, \quad \frac{\partial g}{\partial y} = \frac{1}{i}\left(\frac{\partial g}{\partial \bar{z}} - \frac{\partial g}{\partial z}\right).$$

We substitute the last equations to (1), multiply the second equation in (1) by i, subtract them to get

$$\begin{aligned} \frac{\partial h}{\partial z} &= \frac{\partial g}{\partial x}\left(\frac{1}{2}\frac{\partial u}{\partial x} - \frac{1}{2}i\frac{\partial u}{\partial y}\right) + \frac{\partial g}{\partial y}\left(\frac{1}{2}\frac{\partial v}{\partial x} - \frac{1}{2}i\frac{\partial v}{\partial y}\right) \\ &= (g_z + g_{\bar{z}})\frac{1}{2}(u_x - iu_y) + \frac{1}{i}(g_{\bar{z}} - g_z)\frac{1}{2}(v_x - iv_y) \\ &= g_z\frac{1}{2}(u_x - iu_y + iv_x + v_y) + g_{\bar{z}}\frac{1}{2}(u_x - iu_y - iv_x - v_y) \\ &= g_z f_z + g_{\bar{z}}\bar{f}_z \end{aligned} \tag{3}$$

using equations (2). Similarly we get

$$\begin{aligned} \frac{\partial h}{\partial \bar{z}} &= \frac{\partial g}{\partial x}\left(\frac{1}{2}\frac{\partial u}{\partial x} + \frac{1}{2}i\frac{\partial u}{\partial y}\right) + \frac{\partial g}{\partial y}\left(\frac{1}{2}\frac{\partial v}{\partial x} + \frac{1}{2}i\frac{\partial v}{\partial y}\right) \\ &= (g_z + g_{\bar{z}})\frac{1}{2}(u_x + iu_y) + \frac{1}{i}(g_{\bar{z}} - g_z)\frac{1}{2}(v_x + iv_y) \\ &= g_z\frac{1}{2}(u_x + iu_y + iv_x - v_y) + g_{\bar{z}}\frac{1}{2}(u_x + iu_y - iv_x + v_y) \\ &= g_z f_{\bar{z}} + g_{\bar{z}}\bar{f}_{\bar{z}}, \end{aligned} \tag{4}$$

since

$$\begin{aligned} \frac{\partial f}{\partial \bar{z}} &= \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) = \frac{1}{2}(u_x + iv_x + i(u_y + iv_y)) \\ \frac{\partial \bar{f}}{\partial \bar{z}} &= \frac{1}{2}\left(\frac{\partial \bar{f}}{\partial x} + i\frac{\partial \bar{f}}{\partial y}\right) = \frac{1}{2}(u_x - iv_x + i(u_y - iv_y)). \end{aligned}$$

14. Show that in polar coordinates the Cauchy-Riemann equations take the form

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \quad \frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r}.$$

Use these equations to show that the logarithm function defined by

$$\log z = \log r + i\theta, \quad z = r(\cos\theta + i\sin\theta), \quad -\pi < \theta < \pi$$

is holomorphic in the region $r > 0$ and $-\pi < \theta < \pi$.

In polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$. By the chain rule for functions in two variables we have

$$\begin{aligned} \frac{\partial u}{\partial r} &= \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta \\ \frac{\partial u}{\partial \theta} &= \frac{\partial u}{\partial x}(-r)\sin\theta + \frac{\partial u}{\partial y}r\cos\theta \\ \frac{\partial v}{\partial r} &= \frac{\partial v}{\partial x}\cos\theta + \frac{\partial v}{\partial y}\sin\theta \\ \frac{\partial v}{\partial \theta} &= \frac{\partial v}{\partial x}(-r)\sin\theta + \frac{\partial v}{\partial y}r\cos\theta. \end{aligned} \tag{5}$$

Assume the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \tag{6}$$

Then we get

$$\begin{aligned} \frac{\partial u}{\partial r} &= \frac{\partial v}{\partial y}\cos\theta - \frac{\partial v}{\partial x}\sin\theta = \frac{1}{r}\frac{\partial v}{\partial \theta}, \\ \frac{\partial u}{\partial \theta} &= \frac{\partial v}{\partial y}(-r\sin\theta) - \frac{\partial v}{\partial x} r\cos\theta = -r\frac{\partial v}{\partial r}. \end{aligned} \tag{7}$$

Conversely assume the Cauchy-Riemann equations in polar form (7). By (5) we get

$$\begin{aligned} \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta &= -\frac{\partial v}{\partial x}\sin\theta + \frac{\partial v}{\partial y}\cos\theta \\ \frac{\partial u}{\partial x}(-r)\sin\theta + \frac{\partial u}{\partial y} r\cos\theta &= -\frac{\partial v}{\partial x} r\cos\theta + \frac{\partial v}{\partial y}(-r)\sin\theta \end{aligned}$$

This is a system of linear equations in the unknown functions $\partial u/\partial x, \partial u/\partial y$. Cramer's rule for solving the system gives

$$\frac{\partial u}{\partial x} = \frac{\begin{vmatrix} -v_x\sin\theta + v_y\cos\theta & \sin\theta \\ -rv_x\cos\theta - rv_y\sin\theta & r\cos\theta \end{vmatrix}}{\begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix}} = \frac{rv_y}{r} = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = \frac{\begin{vmatrix} \cos\theta & -v_x\sin\theta + v_y\cos\theta \\ -r\sin\theta & -rv_x\cos\theta - rv_y\sin\theta \end{vmatrix}}{\begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix}} = \frac{-rv_x}{r} = \frac{\partial v}{\partial x}.$$

So we get back the Cauchy-Riemann equations.

For the logarithm functions we have $u = \log r$ and $v = \theta$. This gives

$$\frac{\partial u}{\partial r} = \frac{1}{r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \quad \frac{\partial u}{\partial \theta} = 0 = -\frac{\partial v}{\partial r}.$$

15. Consider the function defined by

$$f(x+iy) = \sqrt{|x||y|}, \quad x, y \in \mathbb{R}.$$

Show that f satisfies the Cauchy-Riemann equations at the origin, yet f is not holomorphic at 0.

We have $f(x+i0) = \sqrt{|x||0|} = 0$, $f(0+iy) = \sqrt{|0||y|} = 0$, $f(0+i0) = 0$. Also, if $f = u + iv$, $u = f$ and $v = 0$. These give at the origin

$$\frac{\partial u}{\partial x} = \lim_{x\to 0}\frac{f(x+i0) - f(0+i0)}{x} = \lim_{x\to 0}\frac{0-0}{x} = 0,$$

$$\frac{\partial u}{\partial y} = \lim_{y \to 0} \frac{f(0+iy) - f(0+i0)}{x} = \lim_{y \to 0} \frac{0-0}{y} = 0.$$

Obviously $\partial v/\partial x = 0$ and $\partial v/\partial y = 0$. So the Cauchy-Riemann equations are satisfied. However, the function is not holomophic at 0. If it were, then $f'(0) = 0$. We set $x = y > 0$ and let $z = x + ix \to 0$.

$$\lim_{z \to 0} \frac{f(x+iy) - f(0+i0)}{z} = \lim_{z \to 0} \frac{\sqrt{|x||y|}}{z} = \lim_{x \to 0} \frac{\sqrt{|x|^2}}{x+ix} = \lim_{x \to 0} \frac{|x|}{x+ix} = \frac{1}{1+i}.$$