Accumulation constants of iterated function systems with Bloch target domains

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1 Introduction

Suppose that we are given a random sequence of holomorphic maps $f_1, f_2, f_3, \ldots$ of the unit disk Δ onto a subdomain $X \subset \Delta$. We consider the compositions

$$F_n = f_1 \circ f_2 \circ \ldots f_{n-1} \circ f_n.$$

The sequence $\{F_n\}$ is called the *iterated function system* coming from the sequence $f_1, f_2, f_3, \ldots$; we abbreviate this to *IFS*. By Montel's theorem (see for example [3]), the sequence F_n is a normal family, and every convergent subsequence converges uniformly on compact subsets of Δ to a holomorphic function F. The limit functions F are called accumulation points. Therefore every accumulation point is either an open self map of Δ or a constant map. The constant accumulation points may be located either inside X or on its boundary.

Note that for the iterated systems we consider here, the compositions are taken in the reverse of the usual order; that is, backwards. There is a theory for forward iterated function systems that is somewhat simpler and is dealt with in [5]. For example, for forward iterated function systems, by using constant functions, it is easy to construct systems with non-unique limits.

The first results for (backward) iterated function systems were found by Lorentzen and Gill ([8], [4]) who, independently proved that if X is relatively compact in Δ, the limit functions are always constant and each IFS has a unique limit.

In [2] the authors considered iterated function systems for which the target domain is non-relatively compact. Using techniques from hyperbolic geometry, they defined a hyperbolic Bloch condition for the target domain and proved that any X satisfying this condition has only constant limit functions. In [6] we proved that this Bloch condition is also necessary.

In [7] we turned to non-Bloch target domains. Using Blaschke products, we proved that any holomorphic map from Δ to X can be realized as the limit

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function of some IFS. We also proved that many sets of open maps and constants in $\bar{X}$ can be realized as limit functions of an IFS.

In this paper we turn our attention to the possible limit constants for Bloch target domains. We ask what points in X or ∂X can occur as limits. Our main result is that for a non-relatively compact Bloch domain X, any finite set of distinct points in X can be realized as the full set of limits of an IFS.

We also give a new proof of the Lorentzen Gill theorem based on reasoning in [2] and [6]. This theorem shows that if the target domain is relatively compact, the limit function must always lie inside X and not on its boundary. For non-Bloch domains, we saw in [7] that boundary points may be limit points. It is an open question whether boundary points can be limit points for arbitrary non-relatively compact Bloch domains. Here we give two examples of special classes of non-relatively compact Bloch domains for which any boundary point may be a limit point.

The paper is organized as follows. In section 2 we provide a proof of the Lorentz-Gill theorem and show that the limit point must be in X. In section 3 we prove the main result that for a non-relatively compact Bloch domain any n distinct points can be the limit set of an IFS. Finally, in section 4 we find two classes of Bloch domains for which any boundary point can be a limit point.

2 Relatively Compact Subdomains

In this section we consider iterated function systems where the target domain is relatively compact. We remark that if the function f_1 of any IFS is a constant map, then $f_1 \circ \ldots f_n$ is the same constant map and this constant is the unique accumulation point. Similarly, if $f_k(z) \equiv c$, c constant, then the unique accumulation point of the IFS is the constant $f_1 \circ \ldots f_{k-1}(c)$.

We now make the tacit assumption that the functions in our IFS are non-constant. We begin by recalling a theorem of Lorentz and Gill.

Theorem 1 (Lorentz-Gill) *If X is a relatively compact subset of the unit disk, then every IFS has a unique constant limit inside X. Moreover, every constant in X is the limit of some IFS.*

Proof. Set $\mu(X) = \sup_{z \in X} \frac{\rho_\Delta(z)}{\rho_X(z)}$. Because $X \subset \Delta$, for all $z \in X$, $\rho_\Delta(z) < \rho_X(z)$. Moreover, since X is relatively compact in Δ, $\rho_\Delta(z)$ is bounded, whereas $\rho_X(z)$ is not. It follows that there is some $k < 1$ such that $\mu(X) \leq k < 1$.

What follows is an infinitesimal version of the proof in [2] that says if $\mu(X) < 1$ then all limit functions are constants. Note that domains X with $\mu(X) < 1$ are Bloch domains (see [2] for more details).

For any holomorphic $f : \Delta \to X$, the Schwarz-Pick lemma, [1], implies that for any $z \in X$, $\rho_X(f(z))|f'(z)| \leq \rho_\Delta(z)$. Combining this with the above we obtain the infinitesimal strong contraction property,

$$\rho_\Delta(f(z))|f'(z)| \leq k\rho_X(f(z))|f'(z)| \leq k\rho_\Delta(z) \qquad (1)$$

Integrating the left and right sides of this inequality along a path $\gamma \subset X$ joining points $z, w \in X$ that closely approximates the infimum and taking a limit gives us the strong contraction property,

$$\rho_\Delta(f(z), f(w)) = \inf_\gamma \int_\gamma \rho_\Delta(f(z))|f'(z)||dz| \leq k \inf_\gamma \int_\gamma \rho_\Delta(z)|dz| = k\rho_\Delta(z,w) \tag{2}$$

Now suppose that some subsequence F_{n_j} of the IFS converges to a holomorphic map g. The derivatives $F'_{n_j}(z)$ converge to $g'(z)$. Showing that g is a constant map is equivalent to showing that $F'_{n_j}(z)$ converges to zero. By the infinitesimal strong contraction property, (1), we have

$$|F'_{n_j}(z)| = |f'_1(f_2 f_3 \dots f_{n_j} z) f'_2(f_3 f_4 \dots f_{n_j} z) \dots f'_{n_{j-1}}(f_{n_j} z) f'_{n_j}(z)| \leq$$
$$\frac{k\rho_\Delta(f_2 f_3 \dots f_{n_j} z)}{\rho_\Delta(f_1 f_2 f_3 \dots f_{n_j} z)} \frac{k\rho_\Delta(f_3 f_4 \dots f_{n_j} z)}{\rho_\Delta(f_2 f_3 f_4 \dots f_{n_j} z)} \dots \frac{k\rho_\Delta(z)}{\rho_\Delta(f_{n_j} z)} =$$
$$k^{n_j} \frac{\rho_\Delta(z)}{\rho_\Delta(F_{n_j} z)}.$$

On the one hand, k^{n_j} tends to zero, and on the other, $\frac{\rho_\Delta(z)}{\rho_\Delta(F_{n_j}(z))}$ converges to $\frac{\rho_\Delta(z)}{\rho_\Delta(g(z))} < \infty$. Therefore $g'(z) = 0$ for all z in Δ, and every limit function is constant.

To show the constant is unique note that we only need to look at $F_n(0)$. Since X is relatively compact, there is some positive B such that $\sup_{z,w \in X} \rho_\Delta(z,w) < B$. For any $m > n$, we apply the strong contraction property (2) $m - n$ times to get,

$$\begin{aligned}\rho_\Delta(F_m(0), F_n(0)) &= \rho_\Delta(f_1 \circ \dots f_m(0), f_1 \circ \dots f_n(0)) \\ &\leq k^{m-n} \rho_\Delta(f_{n+1} \circ \dots f_m(0), 0) < k^{m-n} B\end{aligned}$$

Thus, $F_n(0)$ is a Cauchy sequence and so has a unique limit.

To see that the limit cannot be in ∂X, look at the IFS $G_n = f_2 \circ f_3 \circ \dots f_n$; then, by the above, G_n converges to a point a in $\bar{X}$. Since $F_n = f_1 \circ G_n$, however, F_n converges to $f(a)$ and $f(a$ is in X.

Finally, to show that any point a in X is the limit of some IFS, set $f_1 \equiv a$.

Note that these arguments use the compactness and don't work for non-relatively compact domains. They do still work, however, if we replace the unit disk Δ with an arbitrary source domain. □

3 Non-relatively compact subdomains

We turn now to the question of subdomains X that are not relatively compact in Δ. Although most of the arguments work for non-Bloch domains, we are interested in the case when they are Bloch.

In [5], in addition to our discussion of forward iterated systems, we showed that if X is any non relatively compact subset of Δ, we could find a (backward) IFS that had two limit functions. Here we generalize this construction to show that for every integer n, we can find iterated function systems with any given set of n distinct points as the full set of accumulation points. A key to the construction is

Lemma 3.1 *Let X be any non relatively compact subset of Δ, and for any fixed n, let $a_1, \ldots, a_n$ be any distinct points in $\Delta \setminus \{0\}$. Then there exists a function $f : \Delta \to \Delta$ and points $x_1, \ldots, x_n \in X$ such that for all $i = 1, \ldots, n$, $f(x_i) = a_i/x_i$.*

Proof. We use the notation:

$$A(a, z) = \frac{z - a}{1 - \bar{a}z}$$

and note that $A(a, A(-a, z)) = z$.

Step 1: Since X is not relatively compact we choose an $x_1 \in X$ such that $|x_1| > |a_1|$. Let $g_1(z)$ be a self map of the unit disk to be determined. Define

$$f(z) = \frac{A(x_1, z)g_1(A(x_1, z)) + \frac{a_1}{x_1}}{1 + \frac{\bar{a}_1}{\bar{x}_1}A(x_1, z)g_1(A(x_1, z))}$$

It follows that $f(x_1) = a_1/x_1$ as required. Because we want to work inductively we rewrite this definition implicitly as follows

$$A(x_1, z)g_1(A(x_1, z)) = A(\frac{a_1}{x_1}, f(z)) \tag{3}$$

If $n = 1$ we set $g_1(z) \equiv 0$ and we are done. From now on we assume that $n > 1$ and that we have chosen x_1.

Step 2: Before we proceed, we set up some further notation:
For $1 \leq j \leq k \leq n$ set $a_{jk} = A(x_j, x_k)$. Next, for $k = 2, \ldots n$ set

$$b_{1k} = A(\frac{a_1}{x_1}, \frac{a_k}{x_k}) \tag{4}$$

For $j = 2, \ldots n - 1$ and $k = j, j + 1, \ldots n$ set

$$b_{jk} = A(\frac{b_{(j-1)j}}{a_{(j-1)j}}, \frac{b_{(j-1)k}}{a_{(j-1)k}}) \tag{5}$$

In order that our construction work we need to choose the x_i so that the following inequalities hold:

$$|\frac{a_i}{x_i}| < 1, \quad i = 1, \ldots n \tag{6}$$

In step 1 we chose x_1 so this holds for $i = 1$.

For all j, k such that $j < k$ we also need to have

$$|\frac{b_{jk}}{a_{jk}}| < 1 \tag{7}$$

To see that we can satisfy these inequalities note first that for fixed j, and all $k > j$, $|x_k| \to 1$ implies $|a_{jk}| \to 1$.

Next

$$|b_{1j}| \leq |A(\frac{a_1}{x_1}, a_j e^{\theta_j})| = B_{1j} < 1$$

where θ_j is chosen so that $\arg a_j e^{\theta_j} = \arg \frac{a_1}{x_1} + \pi$ and B_{1j} is maximal.

Since X is not relatively compact we get conditions on x_i, $i = 2, \ldots, n$ so that all the inequalities (6) and all the inequalities (7) with $j = 1$ hold.

Now fix x_2 so that (6) and (7) with $j = 1$ hold, assuming the remaining $|x_i|$ are close enough to 1.

We now find bounds

$$|b_{2j}| \leq |A(\frac{b_{12}}{a_{12}}, B_{1j} e^{\theta_j})| = B_{2j} < 1$$

where again θ_j is chosen to maximize.

We repeat this process, choosing $x_3, \ldots x_{n-1}, x_n$, in turn so that all the inequalities above hold.

Step 3:

Define the functions $g_k(z) : \Delta \to \Delta$, $k = 2, \ldots n$ recursively by

$$A(x_k, z) g_k(A(x_k, z)) = A(\frac{b_{(k-1)k}}{a_{(k-1)k}}, g_{(k-1)}(A(x_{(k-1)}, z))) \tag{8}$$

Now take $g_n(z)$ to be any holomorphic function of the disk to itself; in particular, we can take the function $g_n(z) \equiv 0$. Then work back through equations (8) to obtain the functions g_1 and f.

We check that $f(x_i) = \frac{a_i}{x_i}$ for $i = 1, \ldots, n$, so that we have the required points x_i and the function f. □

Now we show that we may construct an iterated function system that has n arbitrarily chosen distinct accumulation points for any integer $n > 1$. We construct it inductively.

Theorem 2 *Let X be any subdomain of Δ that is not relatively compact and let $n > 1$ be a given integer. There is an IFS that has at least n distinct accumulation points. If X is Bloch these accumulation points are constant and the IFS has no other accumulation points.*

Proof. With no loss of generality we may assume that $0 \in X$. The idea of the proof is to construct functions f_k such that the set $S = \{c_0 = 0, c_1 = f_1(0), c_2 = f_1 \circ f_2(0), \ldots c_{n-1} = f_1 \circ f_2 \circ \ldots f_{n-1}(0)\}$ consists of distinct points and such that the *cycle relation*

$$f_i \circ f_{i+1} \circ \ldots \circ f_{i+n-1}(0) = 0 \tag{9}$$

holds for all integers i.

Suppose we have such a system and we consider any subsequence $F_{n_k} = f_1 \circ f_2 \circ \ldots \circ f_{n_k}$. By the cycle relation we see that $F_{n_k}(0) \in S$ for all k. It follows that any limit function must map 0 to a point in S. Choosing subsequences appropriately, we can find n distinct limit functions G_i such that $G_i(0) = c_i$, $i = 0 \ldots n-1$.

If X is Bloch, these limit functions must be constant so there are at most n such functions and hence exactly n of them.

Suppose first that $n = 2$ and we are given two distinct points c_0 and c_1 in X. In this construction, all maps f_i will be different universal covering maps from Δ onto X. We may assume without loss of generality that $c_0 = 0$. We can find a covering map f_1 such that $f_1(0) = c_1$. Then because f_1 is defined up to a rotation about 0 and X is not relatively compact we can find $x_1 \in X$ with $f_1(x_1) = 0$.

By the same reasoning we let f_2 be a covering map from Δ onto X such that $f_2(0) = x_1$ and such that there is an $x_2 \in X$ with $f_2(x_2) = 0$. Again there is such an x_2 because X is not relatively compact in Δ. Continuing this process we obtain a sequence of covering maps f_k and a sequence of points x_k in X such that

$$f_k(0) = x_{k-1} \text{ and } f_k(x_k) = 0 \tag{10}$$

for all k. Choosing odd or even subsequences we obtain two distinct limit functions G_1, G_2 such that $G_1(0) = c_1$ and $G_2(0) = 0$.

For $n > 2$, the maps f_i are not covering maps. We need to apply lemma 3.1 repeatedly. This part of the construction comes in two parts. First we construct the maps $f_1, \ldots f_{n-1}$ and then construct the rest of the maps, f_{n+j}, $j = 0, 1, \ldots$. We obtain two collections of points: those are labeled x_* and belong to the cycles $\{f_{i+n-1}(0), f_{i+n-2} \circ f_{i+n-1}(0), f_{i+n-3} \circ f_{i+n-2} \circ f_{i+n-1}(0), \ldots f_{i+1} \circ \ldots f_{i+n-2} \circ f_{i+n-1}(0), 0\}$ and and those that are labeled b_* and don't belong to the cycles.

We assume we are given the n distinct points $c_0 = 0, c_1, \ldots c_{n-1} \in X$. We apply lemma 3.1 to obtain $n-1$ new distinct points

$$x_1, b_2, b_{23}, \ldots b_{2\ldots(n-1)} \in X$$

and a function f_1 such that $f_1(x_1) = 0$ and

$$f_1(0) = c_1, f_1(b_2) = c_2 \ldots f_1(b_{2\ldots(n-1)}) = c_{n-1}$$

Recall that in the construction of lemma 3.1, we obtain a function f such that for the given point a_i we have a new point x_i with $x_i f(x_i) = a_i$. Therefore to obtain f_1 we first apply a covering map $\pi : \Delta \to X$ with $\pi(0) = c_1$. We use the lemma to find a map f and points in X. We set $f_1(z) = \pi(z f(z))$. The new points $x_1, b_2, b_{23}, \ldots b_{2\ldots(n-1)}$ are the preimages of the points we get from the lemma. Because X is not compact, we can take these preimages in X.

We repeat this process for the n new points $x_1, b_2, \ldots b_{2\ldots(n-1)}$ and obtain a second set of $n-1$ distinct points x_2, x_{21} and $b_3, b_{3\ldots(n-1)}$ and a function f_2 such that $f_2(x_2) = 0, f_2(x_{21}) = x_1$ and

$$f_2(0) = b_2, \ldots f_2(b_3) = b_{23}, \ldots, f_2(b_{3\ldots(n-1)}) = b_{2\ldots(n-1)}$$

We continue in this way. For $i = 3, \ldots, n-1$ start with the $n-1$ points

$$x_{i-1}, x_{(i-1)(i-2)}, \ldots x_{(i-1)\ldots 1}, b_i, b_{i(i+1)}, b_{i\ldots(n-1)}$$

and obtain a function f_i and $n-1$ new points

$$x_i, x_{i(i-1)}, \ldots, x_{i(i-1)\ldots 1}, b_{i+1}, b_{(i+1)(i+2)}, \ldots b_{(i+1)\ldots(n-1)}$$

such that

$$f_i(x_i) = 0, f_i(x_{i(i-1)}) = x_{i-1}, \ldots, f_i(x_{i(i-1)\ldots 1}) = x_{(i-1)\ldots 1}$$

and

$$f_i(0) = b_i, f_i(b_{i+1}) = b_{i(i+1)}, \ldots, f_i(b_{(i+1)\ldots(n-1)}) = b_{i\ldots(n-1)}$$

We thus obtain the first $n-1$ maps and check that they satisfy $f_1(0) = c_1$, $f_1 \circ f_2(0) = c_2$, $\ldots f_1 \circ \ldots \circ f_{n-1}(0) = c_{n-1}$. Moreover, we have the points of the cycles such that

- $x_1, \ldots, x_{n-1} \in X$ satisfying $f_i(x_i) = 0$, $i = 1, \ldots, n-1$.
- $x_{21}, x_{32}, \ldots x_{(n-1)(n-2)} \in X$ satisfying $f_i(x_{i(i-1)}) = x_{i-1}$, $i = 2, \ldots, n-1$.
- $x_{321}, x_{432}, \ldots x_{n(n-1)(n-2)} \in X$ satisfying $f_i(x_{i(i-1)(i-2)}) = x_{(i-1)(i-2)}$, $i = 2, \ldots, n-1$.
- $\ldots$
- $x_{(n-1)\ldots 21} \in X$ satisfying $f_{n-1}(x_{(n-1)\ldots 21}) = x_{(n-2)\ldots 21}$

We now have $n-1$ points of the first cycle, $n-2$ points of the second and so forth. The next step is the general step; we need to complete the cycles.

We construct a holomorphic map f_n from Δ to X to complete the first cycle; that is, so that $f_n(0) = x_{(n-1)\ldots 21}$ and $f_1 \circ \ldots \circ f_n(0) = 0$. We also obtain new points $x_{n(n-1)}, \ldots, x_{n(n-1)\ldots 32}$ in $X \setminus \{0\}$ in each of the second through $n-1$-st cycles and start a new cycle with a new point x_n. That is,

$$f_n(0) = x_{(n-1)\ldots 21} \tag{11}$$

$$f_n(x_{n(n-1)\ldots 32}) = x_{(n-1)(n-2)\ldots 32} \tag{12}$$

$$f_n(x_{n(n-1)\ldots 43}) = x_{(n-1)(n-2)\ldots 43} \tag{13}$$

$\ldots$

$$f_n(x_{n(n-1)}) = x_{(n-1)} \tag{14}$$

and

$$f_n(x_n) = 0 \tag{15}$$

For the construction of f_n, we again begin with a covering map. Let π_1 be a holomorphic covering map from Δ onto X such that $\pi_1(0) = x_{(n-1)\dots 21}$. We now choose any $n-1$ points in Δ that are preimages under π_1 of the dangling points of the cycles we are constructing as follows:

$$y_{(n-1)\dots 2} \text{ such that } \pi_1(y_{(n-1)\dots 2}) = x_{(n-1)\dots 2}$$

$$y_{(n-1)\dots 3} \text{ such that } \pi_1(y_{(n-1)\dots 3}) = x_{(n-1)\dots 3}$$

...

$$y_{n-1} \text{ such that } \pi_1(y_{(n-1)}) = x_{(n-1)}$$

$$y_n \text{ such that } \pi_1(y_n) = 0$$

These $n-1$ points together with 0 form a set of n distinct points in Δ. Using lemma 3.1 we can find $n-1$ points $x_n, x_{n(n-1)}, \dots, x_{n(n-1)\dots 32}$ in $X \setminus \{0\}$ and a function g such that

$$g(x_{n(n-1)\dots 2}) = \frac{y_{(n-1)\dots 2}}{x_{n(n-1)\dots 2}}$$

$$g(x_{n(n-1)\dots 3}) = \frac{y_{(n-1)\dots 3}}{x_{n(n-1)\dots 3}}$$

...

$$g(x_{n(n-1)}) = \frac{y_{n-1}}{x_{n(n-1)}}$$

and

$$g(x_n) = \frac{y_n}{x_n}$$

Finally, let $f_n(z) = \pi_1(zg(z))$. We have completed the first cycle so that the composition $f_1 \circ f_2 \circ \dots \circ f_n$ fixes zero. We now repeat this construction ad infinitum to obtain $f_{n+1}, f_{n+2}, \dots$. At each stage we complete one cycle and add points to the next $n-1$ cycles. Thus, the cycle relation, (9), holds for each i. Let $F_k = f_1 \circ \dots f_k$. Then, $F_k(0) = c_r$ where $r = k \bmod n$. The accumulation points are limits of subsequences $\{F_{n_k}\}$. For any such limit F, $F(0) = c_r$ for some $r = 0, \dots, n-1$. Because the c_r are distinct, we have at least n distinct accumulation points.

If X is Bloch, all the limit functions of this IFS are constant. Since $F(0) = c_k$ for some k for every limit function there are exactly n possible constant functions.
□

4 Boundary points as limiting values

As we saw in section 2, if X is relatively compact, all limit functions lie inside X. As we mentioned in the introduction, in [7] we proved that if X is non-Bloch, we can find an IFS whose limit functions take on any or all boundary points. If X is Bloch and not relatively compact, however, it is not known whether we can obtain limits that are boundary points of X.

In this section we exhibit two special classes of subdomains that do admit an IFS whose limit point does lie on the boundary. This gives an affirmative answer to our question for those non-relatively compact Bloch domains in these classes.

Theorem 3 *Let X be a subdomain of Δ formed by removing an infinite collection of isolated points from Δ. For any boundary point $b \in \partial X$, there is an IFS with a limit function that takes the value b.*

PROOF. Choose some $b \in \partial X$; either b is one of the isolated boundary points of X or $b \in \partial\Delta$. Let $c_1, c_2, \ldots$ be a sequence of points in X that tend to b. Assume, without loss of generality that the origin belongs to X. The idea of the proof is similar to the one above, and works because, although the arguments in the proof of lemma 3.1 do not extend to an infinte number of points, we can use the special nature of X to obtain an infinite point version of lemma 3.1.

Let $g_1 : \Delta \to X$ be a covering map such that $g_1(0) = c_1$. It is uniquely determined up to pre-composition by a rotation about the origin. Since X is not simply connected, we may pick points $a_2, a_3, \ldots$ in Δ such that $g_1(a_j) = c_j$ and $|a_j| < |a_{j+1}|$. The sets

$$A_\theta = \{e^{-i\theta} a_j \; : \; j = 2, 3, \ldots\}$$

are disjoint for $0 < \theta < 2\pi$. Since $\Delta \setminus X$ is countable, there exists θ such that $A_\theta \subset X$. Let $c_{1j} = e^{-i\theta} a_j$ and let $f_1(z) = g_1(e^{i\theta} z)$. Then $f_1(0) = c_1$, and $f_1(c_{1j}) = c_j$ for $j > 1$.

We next construct f_2 in the same way. We choose a covering map f_2 so that $f_2(0) = c_{12}$; then $f_1 \circ f_2(0) = c_2$. We choose preimages c_{2j}, $j = 3, 4, \ldots$ such that $f_2(c_{2j}) = c_{1j}$. We use the same argument as above to adjust f_2 so that all these preimages lie in X.

We repeat the construction for each n. We take f_n as a covering map such that $f_n(0) = c_{(n-1)n}$ and adjust so that we can find points c_{nj}, $j = n+1, n+2, \ldots, \in X$ with $f_n(c_{nj}) = c_{(n-1)j}$. Then $f_1 \circ \ldots f_n(0) = c_n$.

Set $F_n(z) = f_1 \circ \ldots f_n(z)$. Since $c_n \to b$, if G is a limit function of F_n, then $G(0) = b$.

Note that if X is Bloch, then G must be constant, $G(z) \equiv b$. □

Theorem 4 *Suppose Y is non relatively compact subdomain of Δ with locally connected boundary. Then, for any boundary point $c \in \partial Y$, there is an IFS with a limit function that takes the value c.*

PROOF. Let $c \in \partial\Delta \bigcap \partial Y$. We will construct an IFS whose accumulation point is c. All our maps f_i will map the unit disk conformally onto Y. Let f be a Riemann map from the unit disk onto Y. By Caratheodory's theorem f extends continuously to the boundary of the unit disk (see Theorem 2.1 in [3]). The preimage of c under this extension is a point on the unit circle, and precomposing by a Mobius map if necessary, we may assume that the continuous extension of f, which we will still call f, fixes c. Take a sequence z_n of points in Y such that z_n converges to c. Then $f(z_n)$ converges to c. Therefore there exists a point z_{n_1} such that $|f(z_{n_1}) - c| < \frac{1}{2}$. Let A_1 be a hyperbolic isometry of the unit disk such that $A_1(c) = c$ and $A_1(0) = z_{n_1}$. Let $f_1 = f \circ A_1$. Then

$$|f_1(0) - c| < \frac{1}{2}$$

and

$$\lim_{z\to c} f_1(z) = c.$$

Therefore $f_1(f(z_n))$ converges to c, and we may choose z_{n_2} such that $|f_1 f(z_{n_2}) - c| < \frac{1}{4}$. Now we take a hyperbolic isometry A_2 of the unit disk such that $A_2(c) = c$ and $A_2(0) = z_{n_2}$. Let $f_2 = f \circ A_2$. Then

$$|f_1 f_2(0) - c| < \frac{1}{4}$$

and

$$\lim_{z\to c} f_2(z) = c.$$

In this way, we obtain a sequence of maps f_n from Δ onto $Y \subset X$ such that $|f_1 f_2 \dots f_n(0) - c| \leq \frac{1}{2^n}$. Therefore c is the accumulation point of the IFS $f_1 f_2 \dots f_n$. Suppose now that c is any point on the boundary of Y and let f be a Riemann map from the unit disk onto Y. Then there exists a point c_0 on the unit circle such that $f(c_0) = c$. Precomposing f by a rotation if necessary, we may assume that $c_0 \in \partial\Delta \bigcap \partial Y$. By the above, there exists an IFS F_n whose accumulation functions all map 0 to c_0. Then every accumulation function of the IFS $G_n = f \circ F_n$ maps 0 to c. $\square$

Examples of domains satisfying conditions in Theorem 4 are a those that meet the boundary in a Stolz angle and polygons with ideal boundary.

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