How to Bend Pairs of Punctured Tori

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1. Introduction

This paper is about the possible pairs of bending laminations for a quasifuchsian punctured torus group.

A *quasifuchsian punctured torus group* G is a discrete subgroup of $PSL(2, \mathbb{C})$ whose set of discontinuity $\Omega(G)$ consists of exactly two simply connected invariant components $\Omega^{\pm}(G)$ such that the quotients $\Omega^{\pm}(G)/G$ are both homeomorphic to a punctured torus $\mathcal{T}_1$. Let $\mathcal{C}$ be the convex hull in hyperbolic three space $\mathbb{H}^3$ of the limit set $\Lambda(G) \subset \hat{\mathbb{C}}$. Corresponding to each of the two components $\Omega^{\pm}(G)$, the boundary $\partial\mathcal{C}$ of $\mathcal{C}$ has a component $\partial\mathcal{C}^{\pm}$, whose quotient $\partial\mathcal{C}^{\pm}/G$ is also homeomorphic to $\mathcal{T}_1$. The components $\partial\mathcal{C}^{\pm}$ and their quotients $\partial\mathcal{C}^{\pm}/G$ are *pleated surfaces* in the sense of Thurston. The bending lines or pleating lines of $\partial\mathcal{C}^{\pm}/G$ form geodesic laminations which support natural transverse measures called the *bending measures*, see [**7**]. We call these measured laminations $pl^{\pm}(G)$ the *bending laminations* of G and denote the corresponding projective measured laminations by $[pl^{\pm}(G)]$.

In this paper we prove that every pair of distinct projective measured geodesic laminations on $\mathcal{T}_1$ occurs as $[pl^{\pm}(G)]$ for some quasifuchsian punctured torus group G.

A geodesic lamination is said to be *rational* if it consists solely of simple closed geodesics. In [**18**], Otal has proved a result closely related to ours: if $\mathbb{H}^3/G$ is a handlebody, then, subject to certain simple and necessary restrictions, every possible rational measured lamination occurs exactly once in the appropriate deformation space. His result has been extended to the case of arbitrary laminations by Bonahon [**5**]. Their results, in particular the occurrence of each projective measure class as the pleating locus for some group G, depend crucially on very heavy machinery: the Hodgson-Kerckhoff theory of deformations of cone structures and the theory of Thurston-Morgan-Shalen on degeneration of representations in $PSL(2, \mathbb{C})$, among others.

The methods of this paper are, in contrast, relatively elementary and completely independent of the above machinery. We hope that they will lend new insight by revealing the exact mechanism which controls the creation of different pleating loci as we deform away from Fuchsian space. Using extensions of these methods combined with ideas from our theory of pleating coordinates, see [**11, 12, 10**], it

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is also possible to prove the analogue of Otal's result, that in the rational case all actual bending measures occur. We shall not, however, discuss this question here.

We concentrate on the case in which the pleating loci of both surfaces $\partial\mathcal{C}^{\pm}/G$ are rational. Since $\partial\mathcal{C}^{\pm}/G$ are punctured tori, and since the pleating locus always supports a transverse measure, any such lamination consists of exactly one bending line $\gamma^{\pm}$. In this case the pleated surfaces $\partial\mathcal{C}^{\pm}$ are made up of planar pieces meeting along the lifts to $\mathbb{H}^3$ of $\gamma^{\pm}$, at angles $\theta^{\pm}$ which we call the *bending angles*.

In this situation, the twist parameter of the complex Fenchel-Nielsen coordinates introduced in [**15**] and [**22**] has a natural interpretation in terms of the quakebend deformation introduced by Thurston as a generalization of the complex Fenchel-Nielsen twist.

Let G_0 be a Fuchsian group acting on the hyperbolic plane $\mathbb{D}$, with quotient $\mathbb{D}/G_0$ homeomorphic to $\mathcal{T}_1$, and let γ be a simple closed geodesic on $\mathbb{D}/G_0$. For $\tau = t + i\theta \in \mathbb{C}$, the quakebend $Q_\tau = Q_\tau(\gamma)$ is the composition of an earthquake (equivalently a Fenchel-Nielsen twist) along γ through a distance t with a bend through an angle θ. The quakebend induces a deformed group G_τ and a pleated surface $\phi_\tau : \mathbb{D} \to \mathbb{H}^3$, that intertwines the actions of G_0 on $\mathbb{D}$ and G_τ on $\mathbb{D}_\tau = \phi_\tau(\mathbb{D})$. The image $\mathbb{D}_\tau$ is a bent surface made up of planar pieces joined, at an angle $\theta = \Im\tau$, along the images in $\mathbb{H}^3$ of the axes in $\mathbb{D}$ which project to γ (see section 5.3).

The first part of our work is a careful proof of the intuitively obvious result, interesting in its own right, that if in the above situation θ is sufficiently small, then $\mathbb{D}_\tau$ is embedded and G_τ is quasifuchsian, see theorem 6.1 and the remark following it. It follows by convexity that $\mathbb{D}_\tau$ is one of the boundary components of $\mathcal{C}$. Clearly $\mathbb{D}_\tau$ has bending lamination γ and bending angle $\pm\Im\tau$. This allows us to construct groups with any prescribed rational pleating locus on one component of $\partial\mathcal{C}/G$.

To simultaneously control the pleating locus on the other component, we use the general observation made in [**11**] that a necessary condition for an axis corresponding to a simple geodesic to lie in the pleating locus of the convex hull boundary is that the trace of the corresponding group element be real. In general, this condition is not sufficient (see e.g. fig. 3 of [**11**]); we prove however, with a simple convexity argument that, in the special case of the punctured torus with small bending angle, it is. (We remark that the convexity argument does not immediately work on a higher genus surface even for small bending angles: if a rational lamination contains more than one geodesic whose corresponding group elements have real trace, then on the corresponding bent surface there may be two or more bending angles with opposite signs. As above, small bending angle forces a *convex* surface $\mathbb{D}_\tau$ to be embedded.)

Let γ, γ' be distinct simple closed geodesics on $\mathcal{T}_1$. By the above, it suffices to show that we can quakebend along γ as above, in such a way that the trace of a group element corresponding to γ' also stays real. To do this we make use of Kerckhoff's theorem [**14**], that the length $l_\beta(t)$ of a closed geodesic $\beta(t)$ transverse to a geodesic lamination Λ on $\mathbb{D}/G_0$ is a strictly convex and analytic function of the earthquake parameter t. Moreover, by a result of Wolpert [**24**], the second derivative at the unique minimum point $t = t_\beta$ is strictly positive. We fix the length of the geodesic γ and we look for a path of quakebends whose initial point lies in Fuchsian space but which then leaves Fuchsian space in such a way that $\mathrm{Tr}\,\beta$ stays real. Our fundamental observation is that there is a unique pair of paths, both with initial point $t = t_\beta$, and both leaving Fuchsian space in the direction of a pure bend, that is tangent to the directions $\pm\tau \in i\mathbb{R}$. This, combined with the results about real trace, yields a proof of our main existence theorem in the rational case, theorem 3.1.

We generalize our existence theorem to arbitrary laminations, first by extending the estimates that allow us to show that the convex pleated surface $\mathbb{D}_\tau$ is embedded for small $|\Im\tau|$ to arbitrary bending laminations and then by using the continuity theorem of [**13**].

We plan to use these ideas in a fuller discussion of quasifuchsian space for punctured tori elsewhere.

The plan of the paper is as follows:

In section 2 we set notation and discuss quasifuchsian groups. In section 3 we discuss pleated surfaces and the convex hull boundary. In section 4 we review real and then complex Fenchel-Nielsen coordinates and state our main result, the existence theorem, theorem 3.1. In section 5 we discuss the relationship between rational quakebends, complex Fenchel-Nielsen twists and the pleated surfaces $\mathbb{D}_\tau$. In section 6 we prove two lemmas which control the geometry of $\mathbb{D}_\tau$ for small bends and obtain our first important result, theorem 6.1. The existence theorem 3.1 is proved for rational laminations in section 7. In the last section, we extend the results of section 6 to arbitrary laminations and prove theorem 3.1 in full generality.

2. Quasifuchsian Punctured Tori

A Kleinian group G is a discrete subgroup of $PSL(2,\mathbb{C})$. It acts on $\hat{\mathbb{C}}$ as a group of conformal automorphisms and on $\mathbb{H}^3$ as a group of hyperbolic isometries. Its set of discontinuity $\Omega(G)$ is the maximal invariant set in $\hat{\mathbb{C}}$ on which the group elements form a normal family; the limit set $\Lambda(G)$ is the complement of $\Omega(G)$ in $\hat{\mathbb{C}}$.

Let $\mathcal{T}_1$ be a punctured torus. Recall that any pair α,β of simple closed loops on $\mathcal{T}_1$ which intersect exactly once are free generators of $\pi_1(\mathcal{T}_1)$; their commutator $[\alpha,\beta]=\alpha\beta\alpha^{-1}\beta^{-1}$ represents a loop round the puncture. We fix once and for all an orientation on $\mathcal{T}_1$. We say that β is *positively oriented* with respect to α if, relative to this orientation on $\mathcal{T}_1$, the rotation from β to α is counterclockwise; in this case we say the generators (α,β) are an *oriented pair*.

Definitions:

- A *Quasifuchsian Punctured Torus Group* G is the image of a discrete faithful representation
$$\rho:\pi_1(\mathcal{T}_1)\to PSL(2,\mathbb{C})$$
such that, if ω represents a loop around the puncture then $\rho(\omega)$ is parabolic, and such that $\Omega(G)$ consists of exactly two simply connected invariant components $\Omega^\pm$, whose quotients $\Omega^\pm(G)/G$ are each homeomorphic to $\mathcal{T}_1$.

The quotient $M(G)=\mathbb{H}^3/G$ is homeomorphic to $\mathcal{T}_1\times(0,1)$; it has a hyperbolic structure induced from $\mathbb{H}^3$. Each quotient $\Omega^\pm(G)/G$ inherits a complex structure, and hence also an orientation, from $\hat{\mathbb{C}}$. We make the convention that the homeomorphisms from $\Omega^+(G)/G$ and $\Omega^-(G)/G$ to $\mathcal{T}_1$ are orientation preserving and reversing respectively.

Let (A,B) be an ordered pair of generators of the quasifuchsian punctured torus group G. Let α,β denote simple closed loops on $\Omega^+(G)/G$ homotopic in $\mathbb{H}^3/G$ to the projection of the axes of A,B in $\mathbb{H}^3/G$; clearly with the given identification of $\Omega^+(G)/G$ and $\mathcal{T}_1$, (α,β) are generators of $\pi_1(\mathcal{T}_1)$. If β is positively oriented with respect to α then we call the generators (A,B) of G an *oriented pair*.

- A *marked* quasifuchsian punctured torus group $(G; A, B)$ is a quasifuchsian punctured torus group G together with an ordered oriented pair of generators (A, B).
- Marked quasifuchsian groups, $(G_1; A_1, B_1)$ and $(G_2; A_2, B_2)$ are *equivalent* if they are conjugate in $PSL(2, \mathbb{C})$ by a conjugacy respecting the markings; in this case the conformal structures on $\Omega^+(G_i)/G_i$ and on $\Omega^-(G_i)/G_i$ are equivalent and the hyperbolic structures on $\mathbb{H}^3/G_i$ agree for $i = 1, 2$.
- *Quasifuchsian space* $\mathcal{QF} = \mathcal{QF}(\mathcal{T}_1)$ is the set of equivalence classes of representations such that $\rho(\pi_1(\mathcal{T}_1))$ is a marked quasifuchsian punctured torus group. Fuchsian space $\mathcal{F}$ is the subspace of $\mathcal{QF}$ consisting of equivalence classes of Fuchsian groups, that is, of groups which are conjugate to a subgroup of $PSL(2, \mathbb{R})$.

Let $\mathrm{Teich}(\mathcal{T}_1)$ be the Teichmüller space of the marked punctured torus $\mathcal{T}_1$; conformally $\mathrm{Teich}(\mathcal{T}_1)$ is the unit disk. By the Bers theory of simultaneous uniformization, see [**1**], we know that $\mathcal{QF} \cong \mathrm{Teich}(\mathcal{T}_1) \times \overline{\mathrm{Teich}(\mathcal{T}_1)}$ and thus can be embedded as a connected set in $\mathbb{C}^2$. We use the notation $\overline{\mathrm{Teich}}$ to record the fact that the basepoint Riemann surfaces of the factors have opposite orientation.

In the special case where $G = \rho(\pi_1(\mathcal{T}_1))$ is Fuchsian, it is conjugate to a group in $PSL(2, \mathbb{R})$. In this case $\Lambda(G)$ is a round circle and $\Omega^{\pm}(G)$ are round disks. Inversion in $\Lambda(G)$ defines a G-invariant antiholomorphic (orientation reversing) homeomorphism between $\Omega^+(G)/G$ and $\Omega^-(G)/G$. Fuchsian space $\mathcal{F}$ is embedded in $\mathcal{QF}$ as the image of $\mathrm{Teich}(\mathcal{T}_1)$ under the map $\rho \mapsto (\rho, \bar{\rho})$ where $\bar{\rho}$ is the representation defined by $\bar{\rho}(\alpha) = \rho(\alpha)^{-1}, \bar{\rho}(\beta) = \rho(\beta)$ for an oriented pair of generators α, β of $\pi_1(\mathcal{T}_1)$.

We shall also need to consider representations of $\pi_1(\mathcal{T}_1)$ into $PSL(2, \mathbb{C})$ whose images are not necessarily quasifuchsian.

Let $\mathcal{R}(\mathcal{T}_1)$ be the set of marked two generator subgroups $G = \langle A, B \rangle \subset PSL(2, \mathbb{C})$ such that A, B are loxodromic and $[A, B]$ is parabolic but where we do not require that the groups in $\mathcal{R}(\mathcal{T}_1)$ be either discrete or free.

Any representation of $\pi_1(\mathcal{T}_1) = \langle \alpha, \beta \rangle$ in $PSL(2, \mathbb{C})$ such that $\rho(\alpha), \rho(\beta)$ are loxodromic and $\rho([\alpha, \beta])$ is parabolic defines an element of $\mathcal{R}(\mathcal{T}_1)$, where we require that the orientations of α, β and the orientations given by translation along the axes of A, B should agree. The group defined by the representation $(\rho(\pi_1(\mathcal{T}_1); \rho(\alpha), \rho(\beta))$ is a marked quasifuchsian punctured torus group if it defines an element of $\mathcal{R}(\mathcal{T}_1)$ and if in addition, $\rho(\pi_1(\mathcal{T}_1))$ is discrete and free.

2.1. Changing the marking. Suppose that $G = (\rho(\pi_1(\mathcal{T}_1); \rho(\alpha), \rho(\beta))$ is a marked quasifuchsian punctured torus group and suppose that γ is an oriented simple closed curve on $\mathcal{T}_1$ with $\rho(\gamma) = V \in PSL(2, \mathbb{C})$. Let δ be any other simple closed curve on $\mathcal{T}_1$ intersecting γ exactly once and such that (γ, δ) is an oriented pair. If $\rho(\delta) = W$, then (V, W) is a marking for G and and $(G; V, W) \in \mathcal{R}(\mathcal{T}_1)$. We say that W is a *complementary generator* to V. Note that if δ' is any other simple geodesic with the same properties as δ, then since the mapping class group of $\mathcal{T}_1$ is generated by Dehn twists about γ and δ, it follows that δ' differs from δ by some power of the Dehn twist about γ and hence $W' = V^n W, n \in \mathbb{Z}$.

2.2. Remark. We note that if $A = \rho(\alpha)$ and $B = \rho(\beta)$ in $PSL(2, \mathbb{C})$, then the parabolicity condition always takes the form $\mathrm{Tr}[A, B] = -2$. This condition can be written as the *Markov equation*

$$\operatorname{Tr}^2 A + \operatorname{Tr}^2 B + \operatorname{Tr}^2 AB = \operatorname{Tr} A \operatorname{Tr} B \operatorname{Tr} AB. \tag{1}$$

where $\operatorname{Tr} A$ is the trace of A.

We see from this equation that $\operatorname{Tr} A$ and $\operatorname{Tr} B$ are almost but not quite global parameters for $\mathcal{QF}$, (see also remark 4.5).

3. Pleated Surfaces and the Convex Hull Boundary

In this section we briefly review the background we need about geodesic laminations, pleated surfaces and the convex hull boundary. All these concepts were introduced by Thurston: we refer to **[6, 7, 13, 20]** for more detailed discussion. We then state our main existence theorem, theorem 3.1.

3.1. Geodesic laminations. A *geodesic lamination* on a hyperbolic surface Σ is a closed set which is a union of pairwise disjoint simple geodesics on Σ called its *leaves*. We denote by $\mathcal{GL} = \mathcal{GL}(\Sigma)$ the set of all geodesic laminations on Σ. Note that $\mathcal{GL}(\Sigma)$ is independent of the hyperbolic structure on Σ, (see e.g. [**13**] section 3.7 and [**6**] section 4.1.4). A geodesic lamination on Σ is *rational* if it consists entirely of closed leaves.

A *measured lamination* λ on Σ is a geodesic lamination together with a transverse measure. We denote by $|\lambda|$ the underlying lamination and by $ML(\Sigma)$ the set of measured laminations on Σ. Two measured laminations λ and λ' are *projectively equivalent* if $|\lambda| = |\lambda'|$ and if there exists $t > 0$ such that for any arc σ transverse to the leaves of $|\lambda|$, $\lambda(\sigma) = t\lambda'(\sigma)$. In particular, if γ is a simple closed curve on Σ, then there is an associated projective class $[\gamma]$ of measured laminations $t\delta_\gamma, t > 0$ in $ML(\Sigma)$ whose underlying lamination is the unique geodesic in the homotopy class of γ, where δ_γ is the delta function on γ. We write $[\lambda]$ for the projective class of $\lambda \in ML(\Sigma)$ and we write $[\gamma]$ for $[\delta_\gamma]$ when there is no risk of confusion. We denote the set of projective equivalence classes by $PML(\Sigma)$; this is the set which will be of main interest to us here.

3.2. Pleated surfaces. A *pleated surface* in a hyperbolic 3-manifold M is a pair (ϕ, Σ) where Σ is a complete hyperbolic surface and $\phi : \Sigma \to M$ is an isometric map satisfying:

1. For each point P in Σ there is at least one geodesic segment through P that is mapped to a geodesic segment in M.
2. The induced map $\phi_* : \pi_1(\Sigma) \to \pi_1(M)$ is injective.

For simplicity we call the image $\phi(\Sigma)$ a pleated surface also.

The set of points through which there is only one geodesic segment whose image is a geodesic segment is a geodesic lamination on Σ called the *pleating locus* or *bending lamination* of $\phi(\Sigma)$.

The complement of the pleating locus in $\phi(\Sigma)$ consists of points through which there passes a geodesic in every direction; the images of the components of the complement are called the *flat pieces* of $\phi(\Sigma)$. We remark that if M is $\mathbb{H}^3$, the flat pieces are contained in hyperbolic planes. We refer to [**7**] for a fuller discussion of these ideas.

3.3. The Convex Hull Boundary. For any Kleinian group G, we can form the hyperbolic convex hull $\mathcal{C}$ of $\Lambda(G)$ in $\mathbb{H}^3 \cup \hat{\mathbb{C}}$. Clearly $\mathcal{C}$ is invariant under G. There is a natural retraction $r : \mathbb{H}^3 \cup \hat{\mathbb{C}} \to \mathcal{C}$ that induces a G-equivariant homeomorphism $r : \Omega \to \partial\mathcal{C}$ (see [**7**], section 1.3, [**13**], proposition 3.1). Hence, in the case of a quasifuchsian punctured torus group, $\partial\mathcal{C}$ consists of two connected components $\partial\mathcal{C}^{\pm}$, such that each quotient $\partial\mathcal{C}^{\pm}/G$ is homeomorphic to a punctured torus.

The components $\partial\mathcal{C}^{\pm}$ and the quotients $\partial\mathcal{C}^{\pm}/G$ are pleated surfaces. The bending laminations of these surfaces carry a natural transverse measure, the *bending measure*, see [**7, 13**]. We denote these measured laminations by $pl(G)^{\pm}$. In the special case in which G is a Fuchsian group acting invariantly on the hyperbolic plane $\mathbb{D} \subset \mathbb{H}^3$, then $\mathcal{C} = \mathbb{D}$ is degenerate. In this case we regard $\partial\mathcal{C}$ and $\partial\mathcal{C}/G$ as 2-sided surfaces, each side of which is a pleated surface with empty pleating locus (and zero measure).

3.4. Statement of the main theorem. It is natural to ask which pairs of measured laminations occur as $pl^{\pm}(G)$ for quasifuchsian punctured torus groups G. In this paper we shall concern ourselves only with the question of which pairs of projective classes occur. The question of which actual bending measures occur is more subtle, as is the extension of a suitably modified result to general quasifuchsian groups. While our methods still apply, the situation is more complicated and we shall not attempt to discuss either question here, (but see also the discussion of the results of Otal and Bonahon in the introduction).

The main result of this paper is:

THEOREM 3.1. *Let $\mu, \nu \in PML(\mathcal{T}_1)$, $\mu \neq \nu$. Then there exists $G \in \mathcal{QF}$ such that $[pl^+(G)] = \mu$ and $[pl^-(G)] = \nu$.*

In fact groups with bending measures in arbitrarily prescribed distinct projective classes occur in any neighborhood of Fuchsian space $\mathcal{F}$. Our proof reveals the mechanism that governs this phenomenon.

We shall prove theorem 3.1 first in the case where μ and ν are both rational; in this case the argument is simpler but contains the essentials of our proof. We remark that since any two distinct simple closed geodesics on $\mathcal{T}_1$ intersect, the rational laminations on $\mathcal{T}_1$ are in one to one correspondence with the simple closed geodesics. Thus we show:

THEOREM 3.2. *Let γ, γ' be distinct simple closed curves on $\mathcal{T}_1$. Then there is a group $G \in \mathcal{QF}$ such that $[pl^+(G)] = [\gamma]$ and $[pl^-(G)] = [\gamma']$.*

We generalize this theorem to arbitrary pairs of laminations in the last section.

The reason for the restriction $\mu \neq \nu$ is a consequence of proposition 3.3 and corollary 3.4 below. Proposition 3.3 applies to any quasifuchsian group, while corollary 3.4 is special to the punctured torus. In general the correct restriction is that the intersection number of $pl^+(G), pl^-(G)$ should not vanish.

PROPOSITION 3.3. *Let G be a quasifuchsian punctured torus group. If there is a geodesic $\gamma \subset \mathbb{H}^3$ contained in both $|pl^+(G)|$ and $|pl^-(G)|$, then G is Fuchsian.*

PROOF. Let $r : \mathbb{H}^3 \cup \hat{\mathbb{C}} \to \mathcal{C}$ be the retraction map as in 3.3. If $\gamma \subset \partial\mathcal{C}^+ \cap \partial\mathcal{C}^-$, then $r^{-1}(\gamma) \cap \hat{\mathbb{C}}$ intersects both components $\Omega^{\pm}(G)$ of $\Omega(G)$. We shall show, however, that $r^{-1}(\gamma) \cap \hat{\mathbb{C}}$ is connected unless G is Fuchsian. Let $\xi \in r^{-1}(\gamma) \cap \hat{\mathbb{C}}$ and let $\eta = r(\xi)$. There is a horoball B based at ξ which meets $\partial\mathcal{C}$ at η; the tangent plane Π to B at η is a support plane to $\mathcal{C}$ at η so that $\mathcal{C}$ lies entirely to one side of Π ([**7**] sec 1.3.2). Thus the plane Π cuts out an open disc Δ in $\hat{\mathbb{C}}$ which contains ξ and is contained in Ω ; clearly $\gamma \subset \Pi$ so that $\partial\Delta$ contains the fixed points of γ. If Π, Π' are any two such planes then the union $\Delta \cup \Delta'$ of the corresponding discs is connected and non-empty, unless possibly Π coincides with Π' and Δ, Δ' are the discs cut out on opposite sides of Π. In this case , $\mathcal{C} \cap \hat{\mathbb{C}} = \Lambda$ is contained in a round circle so that G is Fuchsian. Otherwise, $r^{-1}(\gamma)$ is connected as claimed. $\square$

COROLLARY 3.4. *Let $G \in \mathcal{QF}(\mathcal{T}_1)$. Then laminations $\mu, \nu \in PML(\mathcal{T}_1)$ contain a common leaf if and only if $\mu = \nu$.*

PROOF. Clearly if $\mu = \nu$ then the laminations contain common geodesics. Thus suppose that $\mu \neq \nu$ are laminations with a common leaf. Now any measured geodesic lamination on a hyperbolic surface decomposes into a finite number of sublaminations each carrying a transverse measure and in which every leaf is dense, ([**20**] corollary 1.7.3.) Also, by a well known result of Thurston, dim $PML(\mathcal{T}_1) = 1$, (see e.g. [**20**] theorem 3.1.10). Thus in a measured geodesic lamination on $\mathcal{T}_1$, every leaf is dense. Hence $|\mu| = |\nu|$. Further, measured laminations on $\mathcal{T}_1$ are uniquely ergodic; that is, a given lamination can support only one projective measure class. (This well known fact follows for example from the discussion in [**21**]; the return map on a suitable cross section to the lamination is an irrational rotation of the circle and as such, is uniquely ergodic). Hence $\mu = \nu$. $\square$

4. Complex Fenchel-Nielsen coordinates

In order to prove theorem 3.1, it will be helpful (although not strictly necessary) to think in terms of a specific global parametrization of $\mathcal{QF} \subset \mathbb{C}^2$, namely the complex Fenchel-Nielsen parameters introduced in [**15**] and [**22**]. These coordinates generalize the classical Fenchel-Nielsen coordinates that embed Fuchsian space $\mathcal{F}$ as $\mathbb{R}^+ \times \mathbb{R}$. As we shall see, they form a nice link between classical Teichmüller theory and the Thurston picture of quakebends and the convex hull boundary, and we therefore spend some time here explaining the construction in detail in the special case of the punctured torus.

If $M \in PSL(2,\mathbb{C})$ we write $\operatorname{Tr} M$ for the trace of M and $\operatorname{Ax} M$ for its axis. Although $\operatorname{Tr} M$ is only defined up to sign, this ambiguity will not matter in what follows, see e.g. [**19**].

4.1. Classical Fenchel-Nielsen coordinates. Suppose that $\rho : \pi_1(\mathcal{T}_1) \to PSL(2,\mathbb{C})$ is a representation of $\pi_1(\mathcal{T}_1)$ as a marked Fuchsian group G, and let (A, B) be an oriented pair of generators for G.

Classical Fenchel-Nielsen coordinates (λ_A, τ_A) specify, up to conjugation in $PSL(2,\mathbb{R})$, the position of the marked torus group $(\rho(\pi_1(\mathcal{T}_1)); A, B)$ in $\mathcal{F}$. The first coordinate λ_A is half the hyperbolic translation length of A, given by the formula $\operatorname{Tr} A = 2\cosh \lambda_A$, $\lambda_A > 0$. The second coordinate is the *twist parameter* τ_A, defined by cutting $\mathcal{T}_1$ along $\hat{\alpha}$, the projection on $\mathcal{T}_1$ of $\operatorname{Ax} A$, to obtain a punctured cylinder C, and measuring the twisting along $\hat{\alpha}$ when the boundary curves are reglued to form $\mathcal{T}_1$.

To specify precisely how to measure τ_A, lift to the hyperbolic plane $\mathbb{D}$ and let δ be the common perpendicular from $\operatorname{Ax} A$ to $\operatorname{Ax} B^{-1}AB$, meeting the axes at points X, Y respectively. The projection of δ to $\mathcal{T}_1$ is the unique common perpendicular to the two boundary curves of C. Clearly, $B(Y) \in \operatorname{Ax} A$. The twist parameter

τ_A is the *signed* distance from X to $B(Y)$ measured along $\mathrm{Ax}\, A$, counting distance positive in the direction of translation of A. In the special case where $\tau_A = 0$ and $B(Y) = X$, then $\mathbb{D}/G$ is *rectangular*. We note that τ_A depends on the generator B also; we suppress B in the notation for the reasons explained in 4.3 below.

Before beginning the converse construction of a group given its coordinates, recall that for each $L > 0$, there is up to congruence, a unique hyperbolic pentagon $\Pi(L)$ with four right angles and one zero angle, in which the two non-adjacent finite sides each have length L. The length $d = d(L)$ of the remaining finite side is given by the formula $\sinh \frac{d}{2} \sinh L = 1$, [**2**].

Now suppose we are given $(\lambda, \tau) \in \mathbb{R}^+ \times \mathbb{R}$; we wish to construct a marked group $(G; A, B)$ with $(\lambda_A, \tau_A) = (\lambda, \tau)$. Choose a hyperbolic isometry A acting in $\mathbb{D}$ with translation length 2λ, so that $\mathrm{Tr}\, A = 2\cosh\lambda$. Choose a point X on $\mathrm{Ax}\, A$. Two copies of pentagon $\Pi(\lambda)$ fit together along the finite side of length $d(\lambda)$ to form a six sided polygon P with two opposite finite sides s, s' each of length 2λ and the remaining four sides infinite. Let l, l' be the geodesics containing s, s' and place P so that l coincides with $\mathrm{Ax}\, A$ and so that X is the point on l where the two pentagons meet. Let Y be the point where the common boundary δ of the two pentagons meets l'. Let B be the isometry which carries l' to l and Y to a point at (signed) distance τ from X measured along l with distance measured positive in the direction of translation of A. This defines B uniquely and it is not too hard to show that $G =< A, B >\in \mathcal{F}$ with $(\lambda_A, \tau_A) = (\lambda, \tau)$.

This construction leads to the well-known result (see e.g. [**9, 24**]):

THEOREM 4.1. *Fenchel-Nielsen coordinates are global coordinates for $\mathcal{F}$ and the map $\mathcal{F} \to \mathbb{R}^+ \times \mathbb{R}, (G; A, B) \mapsto (\lambda_A, \tau_A)$ is an analytic bijection.*

REMARK 4.2. In the construction above, there is a fundamental domain P for G with six sides described as follows. Let t_1 and t_2 be the orthogonals from the fixed point p of the parabolic $AB^{-1}A^{-1}B$ to $\mathrm{Ax}\, A$ and $\mathrm{Ax}\, B^{-1}AB$ respectively. The lines $t_1' = A^{-1}(t_1)$ and $t_2' = B^{-1}A^{-1}B(t_2)$ are also orthogonal to $\mathrm{Ax}\, A$ and $\mathrm{Ax}\, B^{-1}AB$ respectively and end at the fixed point q of $B^{-1}A^{-1}BA$. Let s be the segment of $\mathrm{Ax}\, A$ between t_1 and t_1' and let s' be the segment of $\mathrm{Ax}\, B^{-1}AB$ between t_2 and t_2'. Then P is the polygon with sides $t_1, s, t_1', t_2', s', t_2$. Note that B does *not* identify s' with s; instead P is symmetric under the reflection in the common perpendicular δ to the two sides s, s' so that δ divides P into two congruent pentagons $\Pi(\lambda_A)$. Thus the hyperbolic length d_A of δ is given by $\sinh \frac{d_A}{2} \sinh \lambda_A = 1$. Note also that d_A is the minimum distance between lifts to $\mathbb{D}$ of $\hat{\alpha}$. Equivalently, d_A is the distance between the two boundary components of the cylinder $\mathcal{T}_1 - \hat{\alpha}$. This distance depends only on the conjugacy class of A and we shall therefore denote it by either d_A or $d_{\hat{\alpha}}$.

4.2. Complex Fenchel-Nielsen coordinates. Complex Fenchel-Nielsen coordinates are obtained by allowing (λ_A, τ_A) in the above construction to become complex. They give convenient holomorphic parameters for $\mathcal{QF}$ and extend to a set of holomorphic parameters for any marked two generator group $(G; A, B) \in \mathcal{R}(\mathcal{T}_1)$.

We refer to [**8, 15, 19**] for fuller discussions of three dimensional hyperbolic geometry, in particular skew hexagons and complex length.

Suppose $\rho : \pi_1(\mathcal{T}_1) \to G$ is any representation of the marked torus $(\mathcal{T}_1; \alpha, \beta)$ into $PSL(2, \mathbb{C})$ such that $(\rho(\pi_1(\mathcal{T}_1); \rho(\alpha), \rho(\beta)) \in \mathcal{R}(\mathcal{T}_1)$. The image $\rho(\alpha) = A$ considered as an isometry of $\mathbb{H}^3$ has a *complex translation length* $2\lambda_A$ given by

$\operatorname{Tr} A = 2\cosh\lambda_A$. We always assume that $\lambda_A \in \mathbb{C}^+ = \{z = x + iy \in \mathbb{C}, x > 0\}$. This defines the first parameter λ_A. The twist $\tau_A \in \mathbb{C}$ is now defined as the *complex distance* between the oriented lines δ and $B(\delta)$, where δ is again the common perpendicular to the axes $\operatorname{Ax} A$ and $\operatorname{Ax} B^{-1}AB$. More precisely, $\Re\tau_A$ is the *signed* distance from X to $B(Y)$, where X, Y and the convention on measuring signed real distance are chosen as before. The imaginary part $\Im\tau_A$ is an angle, measured as follows. Let $\bar{v}, \bar{w}$ be vectors at Y, X respectively, both pointing along δ in the direction from Y to X. Then $\Im\tau_A$ is the angle between $B(\bar{v})$ and $\bar{w}$, where $\bar{w}$ is parallel translated along $\operatorname{Ax} A$ from X to $B(Y)$ and the angle is measured facing along $\operatorname{Ax} A$ in the direction of translation of A.

Conversely, given $(\lambda, \tau) \in \mathbb{C}^+ \times \mathbb{C}$, we can construct a marked group $(G_{\lambda,\tau}; A, B)$ in $\mathcal{R}(\mathcal{T}_1)$ with $(\lambda_A, \tau_A) = (\lambda, \tau)$ as follows. (Note that we do not claim that $G_{\lambda,\tau}$ belongs to $\mathcal{QF}$.)

Let $A \in PSL(2, \mathbb{C})$ be loxodromic with complex translation length 2λ, so that $\operatorname{Tr} A = 2\cosh\lambda$. The Fuchsian construction depended on the existence, for any $L > 0$, of the hyperbolic pentagons $\Pi(L)$ with four right angles, one zero angle and two sides of length L. We now wish to construct an object in $\mathbb{H}^3$ analogous to $\Pi(L)$, where L is complex. In $\mathbb{H}^3$ a skew right angled hexagon is determined by the lengths of three alternate sides; it lies in a plane if and only if the three complex lengths are in $\mathbb{R}^+ \cup \{0, \infty\}$. Given $\lambda \in \mathbb{C}^+$, we construct a degenerate skew hexagon (that is, a hexagon with one side of zero length) $\Pi(\lambda)$ with alternate sides of complex length λ, λ and zero. $\Pi(\lambda)$ is equally described as a pentagon with four right angles and one zero angle. The sides meeting the zero length side have infinite length and the remaining finite side has length $d(\lambda)$, given by the same formula as before.

Once again, glue two copies of $\Pi(\lambda)$ together along the sides of complex length $d(\lambda)$ to obtain a six sided skew polygon P with two opposite finite sides s, s' each of complex length 2λ and the remaining four sides infinite. (Note that for such a procedure to work, the *complex* lengths of the two sides to be glued must agree.)

The construction of the group now proceeds as before: place P so that the side s lies in the geodesic l that is the axis of A and so that it is symmetric about a chosen point $X \in l$; then define B to be the isometry which carries l' to l and Y to a point at (signed) distance τ from X measured along l with distance measured positive in the direction of translation of A, and which, in addition, rotates the vector $\bar{v}$ at Y pointing along δ in the direction from Y to X through an angle $\Im\tau$ relative to the vector $\bar{w}$ which points along δ in the same direction at X. This clearly defines B uniquely, and determines a representation $\rho : \pi_1(\mathcal{T}_1) \to (G_{\lambda,\tau}; A, B) \in PSL(2, \mathbb{C})$ with $(\lambda_A, \tau_A) = (\lambda, \tau)$.

The above discussion is an outline of the proof of the following special case of Theorem A of [**15**] and Theorem 1 of [**22**]:

THEOREM 4.3. *Complex Fenchel-Nielsen coordinates are holomorphic coordinates for $\mathcal{R}(\mathcal{T}_1)$. They restrict to holomorphic coordinates for $\mathcal{QF}$; that is, the map*

$$\mathcal{QF} \to \mathbb{C}^+ \times \mathbb{C}/2\pi i, (G; A, B) \mapsto (\lambda_A, \tau_A)$$

is a holomorphic embedding.

REMARK 4.4. *In the Fuchsian case, the map $\mathcal{F} \to \mathbb{R}^+ \times \mathbb{R}$ is surjective. By contrast, in theorem 4.3 the mapping of $\mathcal{QF}$ into $\mathbb{C}^+ \times \mathbb{C}/2\pi i$ is certainly not surjective, although, it follows from the Bers simultaneous uniformization theory,* *(see also theorem 6.1 below), that the image of $\mathcal{QF}$ contains a neighborhood of* $\mathbb{R}^+ \times \mathbb{R}$.

REMARK 4.5. *The twist τ can be related to the translation lengths of the generators $2\lambda_A$ and $2\lambda_B$ by the formula*

$$\cosh\left(\frac{\tau_A}{2}\right) = \cosh\lambda_B \tanh\lambda_A, \tag{2}$$

see [**19, 25**]. *Applying a Dehn twist about the projection on $\mathcal{T}_1$ of* $\operatorname{Ax} A$, *we also find*

$$\cosh\left(\frac{\tau_A}{2} + \lambda_A\right) = \cosh\lambda_{AB} \tanh\lambda_A. \tag{3}$$

From these formulas we see directly that (λ_A, τ_A) determine and are determined by the ordered triple $(\cosh A, \cosh B, \cosh AB)$, *(see remark 2.2).*

4.3. Change of generators. We may define complex Fenchel-Nielsen coordinates relative to any marking on $\mathcal{T}_1$. Thus, if γ is a simple closed curve on $\mathcal{T}_1$, represented by an element $V \in G_0$, and if W is a complementary generator as in section 2.1, we denote by (λ_V, τ_V) the complex Fenchel-Nielsen coordinates determined as above with (V, W) in place of (A, B). We write $\tau_V = \tau_{(V,W)}$ when we want to indicate the dependence on W. Note that by theorem 4.3, (λ_V, τ_V) are holomorphic functions of (λ_A, τ_A).

If we replace W by any other complementary generator W', then since $W' = V^n W$ for some $n \in \mathbb{Z}$, we have $\tau_{(V,W)} = \tau_{(V,W')} + n\lambda_V$. Thus the only difference between complex Fenchel-Nielsen coordinates relative to the pairs (V, W) and (V, W') is the base point from which we measure τ_V. Usually in what follows, the choice of base point is unimportant and it will not matter that we have suppressed the dependence on the second generator in our notation.

5. Complex Fenchel-Nielsen twists and Rational Quakebends

5.1. Complex Fenchel-Nielsen twists. The classical Fenchel-Nielsen twist is a deformation of Fuchsian space defined by varying the twist parameter along a simple closed geodesic γ on $\mathcal{T}_1$ ([**24**]). More precisely, let $G_0 \in \mathcal{F}$ and let $V \in G_0$ be an element corresponding to γ. Let W be a complementary generator to V and let (λ_V, τ_V) be Fenchel-Nielsen coordinates for $\mathcal{R}(\mathcal{T}_1)$ relative to V, W. The Fenchel-Nielsen twist $Q_t = Q_t(V) : \mathcal{F} \to \mathcal{F}$ is defined by:

$$Q_t(V) : G(\lambda_V, \tau_V) \mapsto G(\lambda_V, \tau_V + t)$$

where $G(\lambda, \tau)$ denotes the group with complex Fenchel-Nielsen coordinates (λ, τ) relative to (V, W) as in section 4.3.

Note that if we replace W by another generating curve W' then as explained in section 4.3, the only difference in the definition of (λ_V, τ_V) is the base point from which we read off the twist coordinate. The expression for $Q_t = Q_t(V)$ is identical.

Complex Fenchel-Nielsen twists can be defined in exactly the same way, relative to complex Fenchel-Nielsen coordinates. Choose $(G_0; V, W) \in \mathcal{R}(\mathcal{T}_1)$. The complex Fenchel-Nielsen twist $Q_\tau = Q_\tau(V), \tau \in \mathbb{C}$ is the deformation $Q_\tau : \mathcal{R}(\mathcal{T}_1) \to \mathcal{R}(\mathcal{T}_1)$, defined by $G(\lambda_V, \tau_V) \mapsto G(\lambda_V, \tau_V + \tau)$. Clearly, Q_τ is a bijection for each τ and $Q_\tau Q_{\tau'} = Q_{\tau+\tau'}$. Again, the same remarks about the suppression of W in the notation apply. Notice that it is not necessary in this definition that the initial point G_0 be a point in Fuchsian space; our definition reduces to the standard one if it is.

Just as in the real case, the real part of τ should be thought of as representing a twist along the geodesic γ; a deformation with τ real keeps us within $\mathcal{F}$. The imaginary part of τ should be thought of as a bend which moves out of Fuchsian space; if τ is purely imaginary, we call Q_τ a *pure bend*.

Let $(G_0; A, B) \in \mathcal{R}(\mathcal{T}_1)$. Via its action on the generating pair (V, W), the twist $Q_\tau(V)$ induces a map $A \mapsto A_\tau, B \mapsto B_\tau$ on the generators (A, B) and more generally a map $U \mapsto Q_\tau(U)$ for each $U \in G_0$. By theorem 4.3 these maps are holomorphic. We write $G_\tau = G_\tau(V)$ for the marked group with generating pair (A_τ, B_τ).

5.2. Twists and quakebends. Classical Fenchel-Nielsen twists were generalized by Thurston to the deformations of Fuchsian groups called earthquakes, in which a closed geodesic is replaced by a measured geodesic lamination Λ, (see e.g. [**14**]). Likewise, complex Fenchel-Nielsen twists were generalized to quakebends, (see e.g.[**7**]). In this case the lamination Λ is called the *bending lamination.* Thus a (complex) Fenchel-Nielsen twist is nothing other than a quakebend whose bending lamination is rational; for this reason we shall call such a twist a *rational quakebend.* In the main part of our proof of theorem 3.1 we shall restrict to complex Fenchel-Nielsen twists. It will, however, frequently be helpful to think in terms of quakebends and to keep both sets of terminology in mind.

According to Thurston, ([**7**] Chap. 3) a quakebend $Q_\tau(\Lambda)$ along a measured geodesic lamination Λ on a hyperbolic surface represented by a Fuchsian group G_0 determines, via the quakebend cocycle, a pleated surface $\phi_\tau : \mathbb{D} \to \mathbb{D}_\tau \subset \mathbb{H}^3$, where $\mathbb{D}_\tau$ is invariant under a group G_τ. The map ϕ_τ intertwines the actions of G_0 on the hyperbolic disk $\mathbb{D}$ and G_τ in $\mathbb{H}^3$.

In section 5.3 below, we discuss this construction in detail when the quakebend is along a rational lamination, (see also the discussion in [**7**] Sec. 3.2.) We discuss the case of a quakebend along an arbitrary measured lamination in section 8.

5.3. Rational quakebends and pleated surfaces. In this section, we consider a rational quakebend Q_τ along a simple closed geodesic γ on $\mathcal{T}_1$, together with the construction of its associated pleated surface $\mathbb{D}_\tau$. For the moment we shall make the additional assumptions that the initial group G_0 is Fuchsian and that Q_τ is a pure bend so that $\tau = i\theta$.

To define the quakebend $Q_\tau = Q_\tau(\gamma)$ precisely, we proceed as follows. Write $V = V(\gamma) \in G_0$ for a fixed element corresponding to γ in G_0 and $\mathcal{V} = \mathcal{V}(\gamma)$ for the set of all lifts of γ to $\mathbb{D}$. Since γ is simple, $\mathcal{V}$ consists of a set of pairwise disjoint geodesics in $\mathbb{D}$, namely the axes of the element V and all of its conjugates. These axes in $\mathcal{V}$ partition $\mathbb{D}$ into pieces P_i. The map ϕ_τ is defined in such a way that $(\phi_\tau, \mathbb{D})$ is a pleated surface in $\mathbb{H}^3$ such that ϕ_τ is an isometry on each axis in $\mathcal{V}$ and on each (closed) piece P_i. If pieces P_i, P_j meet along an axis $\alpha \in \mathcal{V}$, then $\phi_\tau(P_i), \phi_\tau(P_j)$ lie in planes which meet along the geodesic $\phi_\tau(\alpha)$. With a suitable convention, the angle between these planes is the bending angle $\theta = \Im\tau$.

The convention for measuring the angle is the following. It is chosen so as to be in accord with the definition of complex Fenchel-Nielsen coordinates in section 4.2. Let $x, y \in \mathbb{D} - \mathcal{V}$ and let β be an oriented geodesic from x to y. Let $P_0, P_1, \dots, P_k$ be the pieces cut in order by β, that meet along axes $\alpha_1, \alpha_2, \dots, \alpha_k \in \mathcal{V}$. Orient α_i so that, in $\mathbb{D}$, P_{i-1} lies to the left of α_i and P_i to the right. Let $X_i = \beta \cap \alpha_i$ and let $\bar{v}_i, \bar{w}_i$ be tangent vectors to $\phi_\tau(P_{i-1} \cap \beta)$ and $\phi_\tau(P_i \cap \beta)$ at $\phi_\tau(X_i)$, oriented in the direction inherited from β, so that $\bar{v}_i$ points out of $\phi_\tau(P_{i-1})$ and $\bar{w}_i$ points into $\phi_\tau(P_i)$. Let $\bar{v}'_i, \bar{w}'_i$ be the projections of $\bar{v}_i, \bar{w}_i$ onto the directions orthogonal to the image of the bending axis at $\phi_\tau(X_i)$. Then θ is the angle from $\bar{v}_i$ to $\bar{w}_i$ measured facing along $\phi_\tau(\alpha_i)$. When $\phi_\tau(P_{i-1}), \phi_\tau(P_i)$ are coplanar, $\theta = 0$. (Note that with this convention, if θ is positive and if the group $G_{i\theta}$ obtained by deforming the Fuchsian group G_0 is quasifuchsian, then with the obvious conventions about "up" and "down", the pleated surface $(\phi_{i\theta}, \mathbb{D})$ is the "bottom" surface $\partial\mathcal{C}^-$ of $G_{i\theta}$.)

The above description completely specifies ϕ_τ up to the choice of an initial point and direction. This we make as follows. We always use the hyperbolic ball model $\mathbb{B}^3$ of $\mathbb{H}^3$ (where $\mathbb{B}^3$ here denotes the open ball). We embed $\mathbb{D}$ in $\mathbb{B}^3$ as the equatorial plane such that the origins in $\mathbb{D}$ and $\mathbb{B}^3$ coincide. We arrange that the axes of V and WVW^{-1} in G_0 lie in the boundary of a piece P_0 contained in $\mathbb{D}$. We then choose ϕ_τ to be the identity on P_0, thus $\phi_\tau(O) = (O)$. We write $\mathbb{D}_\tau = \phi_\tau(\mathbb{D})$ for the image of the pleated surface in $\mathbb{B}^3$.

We can also describe $\mathbb{D}_\tau$ in terms of the Fenchel-Nielsen construction in section 4.2. With $\mathbb{D}$ embedded in $\mathbb{B}^3$ as above, V, W become axes in $\mathbb{B}^3$. Let P be the fundamental domain for the group G_0 relative to the generator V and a complementary generator W and assume that $O \in P$. The quakebend Q_τ acts on G_0 without changing the length of V. We choose the tangent vector at the origin so that ϕ_τ is the identity on P. Then one sees that P is also a fundamental domain for G_τ acting on $\mathbb{D}_\tau$, and $\mathbb{D}_\tau$ is precisely the developing image of P under the action of G_τ in $\mathbb{B}^3$. The polygon P is contained in the planar region $P_0 = \mathbb{D} \cap \mathbb{D}_\tau$ invariant under the subgroups $\langle V, W^{-1}VW \rangle \subset G_0$ and $\langle V, W_\tau^{-1}VW_\tau \rangle \subset G_\tau$. The planar pieces $P_i \subset \mathbb{D}$ are developing images of P_0 under G_0; they are the lifts to $\mathbb{B}^3$ of the punctured cylinder obtained by cutting $\mathbb{D}/G_0$, (equivalently $\mathbb{D}_\tau/G_\tau$), along γ. Likewise the images $\phi_\tau(P_i)$ are developing images of P_0 under G_τ. From this discussion it is clear that the Fenchel-Nielsen twist $Q_\tau(V)$ and the quakebend $Q_\tau(\gamma)$ are the same.

As a consequence of this picture, we frequently refer to ϕ_τ as the developing map and to $\mathbb{D}_\tau$ as the developing image of $\mathbb{D}$.

In section 7.2 we shall want to perform the above construction when the initial group G_0 is not Fuchsian, but where nevertheless $\lambda_V \in \mathbb{R}$. Because $[V, W]$ is parabolic the elements $V, W^{-1}VW, V^{-1}W^{-1}VW$ all have real trace so that the skew polygon P of section 4.2 is planar. Thus one sees easily that the above construction still works and again defines a pleated surface $\mathbb{D}_\tau$.

6. Small deformations are quasifuchsian

In this section, we continue our study of rational quakebends. We prove:

THEOREM 6.1. *Let $G_0 \in \mathcal{F}$ and let γ be a simple closed geodesic on $\mathbb{D}/G_0$. Let $Q_\tau(\gamma)$ be the associated quakebend by τ along γ. Then there exists $\epsilon > 0$, depending only on the hyperbolic length of γ, such that if $|\Im\tau| < \epsilon$ then $\mathbb{D}_\tau$ is embedded, the induced map $G_0 \to G_\tau$ is a type preserving isomorphism and G_τ is quasifuchsian.*

We shall generalize this theorem to arbitrary bending laminations in the last section.

Remarks:

- That G_τ is quasifuchsian for small $\Im\tau$ follows directly from the analytic theory, and in particular from the Bers simultaneous uniformization theorem (see [**1, 3**]). The proof below, however, lends geometric insight and results easily from the methods we need for the rest of the paper.

  We thank the referee for pointing out that a direct proof that G_τ is quasifuchsian in the case of a pure bend, with a more precise statement on the size of the bending angle, has been given by Kourioniotis in Theorem 4.7 of [**16**]. His method is to write down a quasiconformal deformation that induces bending along one axis and to show that a suitable sequence of such homeomorphisms converge. His results, however, do not address the question of whether D_τ is embedded, nor is it immediately obvious how to extend them to the general case.
- As $|\Im\tau|$ increases, $\mathbb{D}_\tau$ remains embedded until G_τ reaches the boundary of $\mathcal{QF}$. We address the question of determining exactly the position of the boundary of $\mathcal{QF}$, using our theory of pleating coordinates [**11, 12**] in [**10**].
- Although theorem 6.1 is stated here for punctured tori, the same proof applies to any surface.

6.1. Geometry of $\mathbb{D}_\tau$. Let $G_0 \in \mathcal{F}$ and let γ be a simple closed geodesic on $\mathbb{D}/G_0$. In this section we prove two geometric lemmas that govern the behavior of the developed surface $\mathbb{D}_\tau$ for small τ when Q_τ is a pure bend.

As in section 5.3, we choose an element $V = V(\gamma)$ corresponding to γ in G_0 and let $\mathcal{V} = \mathcal{V}(\gamma) \subset \mathbb{D}$ be the set of lifts of γ to $\mathbb{D}$. Let $Q_\tau = Q_\tau(\gamma)$ be the associated quakebend by $\tau = i\theta$.

Suppose $\eta \subset \mathbb{D}$ is an oriented geodesic, transverse to (a subset of) the axes in $\mathcal{V}$, so that η is divided into segments by its intersections with these axes; we say η is transverse to $\mathcal{V}$. Under the developing map ϕ_τ, each of the segments of η maps to an oriented geodesic segment in $\mathbb{B}^3$, whose union is the developed image $\phi_\tau(\eta) = \eta_\tau$. We call the points where these segments meet, *bending points* of η_τ. If σ is a (closed) segment of η_τ and $X \in \sigma$, then we write $\bar{v}(X,\sigma)$ for the tangent vector to σ at the point X pointing in the direction inherited from the orientation of η. Let σ, σ' be two successive segments of η_τ meeting at a bending point X. The *bending angle* of η_τ at X is the angle between $\bar{v}(X,\sigma), \bar{v}(X,\sigma')$, measured facing along the bending line in $\mathcal{V}$ at X, in the direction induced by the translation direction of V along its axis in $\mathbb{B}^3$.

LEMMA 6.2 (Angle lemma). *Let G_0 be a Fuchsian group and let $Q_\tau, \tau = i\theta, \theta \in \mathbb{R}$ be a pure bend with rational bending lamination γ. Let $\eta \subset \mathbb{D}$ be an oriented geodesic transverse to $\mathcal{V}(\gamma)$ and let ϕ be a bending angle of η_τ. Then $|\phi| \le |\theta|$.*

PROOF. Let X be a bending point of η_τ, and let β be the image under ϕ_τ of the axis of $\mathcal{V}(\gamma)$ containing X.

Let σ, σ' be the oriented geodesic segments of η_τ meeting at X and let $\bar{v} = \bar{v}(X,\sigma), \bar{v}' = \bar{v}(X,\sigma')$. Let Y, Y' be points equidistant from X along the extensions of σ, σ' in the directions of $\bar{v}, \bar{v}'$. The perpendiculars from Y, Y' to β meet in a common point R and $\angle YRY' = |\theta|$. The bending angle of η_τ at X is, on the other hand, $\angle YXY'$. Let T be the midpoint of YY'. Comparing the triangles XYT and XRT, the result is clear. $\square$

Definitions:

- Let $X \in \mathbb{H}^3$ and let $\bar{v}$ be a vector in $\mathbb{H}^3$ at X. The *forward cone* $C_{\bar{v}}(\alpha, X)$ is the closed solid cone of angle α with vertex at X whose central axis is the $\mathbb{H}^3$ geodesic in the direction of $\bar{v}$.
- The *shadow* of a cone $C_{\bar{v}}(\alpha, X)$ on $\partial\,\mathbb{H}^3$ is the set of endpoints on $\partial\,\mathbb{H}^3$ of the semi-infinite geodesics emanating from X and contained in $C_{\bar{v}}(\alpha, X)$.

Let $\eta : [0, \infty) \to \mathbb{D}$ be an oriented geodesic and let $s \in [0, \infty)$. We write $\sigma(s)$ for any segment of η_τ containing $\phi_\tau(\eta(s))$: for most s there is a unique such segment but at the bending points of η_τ there are two such. We write $C_\eta(\alpha, s)$ to mean either of the cones $C_{\bar{v}(\sigma(s), \phi(\eta(s)))}(\alpha, \phi(\eta(s)))$.

We can now state the first of our results used to control the geometry of $\mathbb{D}_\tau$. As in remark 4.2, denote by d_γ the shortest distance between pairs of axes in $\mathcal{V}(\gamma)$. Note that $d_\gamma \to 0$ as $\lambda_V \to \infty$.

Lemma 6.3 (Cone lemma). *Let $G_0 \in \mathcal{F}$ and let $\tau = i\theta \in \mathbb{C}$. Let γ be a simple closed geodesic on $\mathbb{D}/G_0$ and let $\alpha \in (0, \frac{\pi}{2})$. Then there exists $\epsilon > 0$, depending and α and the hyperbolic length of γ, such that if $Q_{i\theta}$ is a pure bend along γ, and if $\eta : [0, \infty) \to \mathbb{D}$ is an oriented geodesic transverse to $\mathcal{V}(\gamma)$, then for $|\theta| < \epsilon$,*

$$\eta_\tau([s, \infty)) \subset C_\eta(\alpha, s), \ \ \forall s \geq 0$$

and whenever $t \geq 2d_\gamma$,

$$C_\eta(\alpha, s) \supset C_\eta(\alpha, s + t).$$

Remark 6.4. *When $\eta(s)$ is a bending point of η, there are two cones C_η; this possibility is included in our statement.*

Proof. Consider the hyperbolic triangle ABC with one vertex B at ∞, side AC of hyperbolic length b and angle $\angle BAC = \psi$. Let ψ' be the exterior angle at C. Then $\psi' = f(\psi, b)$ where f is a continuous function of ψ, b. Note that if ψ is fixed, ψ' increases as b increases and if b is fixed, ψ' decreases as ψ decreases. Note also that $f(\psi, b) > \psi$ for $\psi > 0$.

Choose α, $0 < \alpha < \pi/2$. Using the above, we can find $\epsilon > 0$ such that $f(\alpha - \epsilon, d_\gamma) > \alpha + \epsilon$.

Now let $\eta : [0, \infty) \to \mathbb{D}$ be a geodesic and let $\sigma_0, \sigma_1, \sigma_2$ be adjacent geodesic segments of η_τ, meeting at bending points X_0, X_1, and oriented in the direction of η.

Let $\bar{u}_0 = \bar{v}(\sigma_0, X_0), \bar{v}_0 = \bar{v}(\sigma_1, X_0)$, and let $\bar{u}_1 = \bar{v}(\sigma_1, X_1), \bar{v}_1 = \bar{v}(\sigma_2, X_1)$ be tangent vectors at X_i pointing in the direction of the segment σ_i, $i = 0, 1$.

By lemma 6.2, the angles between $\bar{u}_0, \bar{v}_0$ and $\bar{u}_1, \bar{v}_1$ are both bounded above in modulus by $|\theta|$.

(i) Since $d(X_0, X_1) \geq d_\gamma$, the vertex angle of the maximal cone with vertex at X_1 and central axis in the direction $\bar{u}_1$ and contained in $C_{\bar{v}_0}(\alpha, X_0)$ is greater than $f(\alpha, d_\gamma) > \alpha + \epsilon$. The angle between $\bar{v}_1$ and $\bar{u}_1$ is less than ϵ so $C_{\bar{v}_0}(\alpha, X_0) \supset C_{\bar{v}_1}(\alpha + \epsilon, X_1)$.

(ii) The minimum angle between $\bar{v}_0$ and the boundary of $C_{\bar{u}_0}(\alpha, X_0)$ is $\alpha - \epsilon$. Thus $C_{\bar{u}_0}(\alpha, X_0) \supset C_{\bar{v}_0}(\alpha - \epsilon, X_0)$. Now apply the reasoning of (i) with $\alpha - \epsilon$ in place of α: since $f(\alpha - \epsilon, d_\gamma) > \alpha + \epsilon$ we get $C_{\bar{v}_0}(\alpha - \epsilon, X_0) \supset C_{\bar{v}_1}(\alpha, X_1)$ and hence $C_{\bar{u}_0}(\alpha, X_0) \supset C_{\bar{v}_1}(\alpha, X_1)$.

We have to prove that $C_\eta(\alpha, s) \supset C_\eta(\alpha, s + t)$ whenever $t \geq 2d_\gamma$ and $|\theta| < \epsilon$.

Let $X_1, X_2, \dots, X_k$ be the images in $\mathbb{H}^3$ of the bending points of η_τ strictly between the points $\eta(s)$ and $\eta(s+t)$ and let $X_0 = \eta_\tau(s), X_{k+1} = \eta_\tau(s+t)$. Since $t \geq 2d_\gamma$, if $k < 2$, the result follows from (i), noting that in the proof of (i) it is not important that X_1 be a bending point but only that $d(X_0, X_1) \geq d_\gamma$. Suppose therefore that $k \geq 2$. For $i = 0, \dots, k$ let σ_i be the segment of η_τ between X_i and X_{i+1} and for $i = 1, \dots, k$ let $\bar{u}_i = \bar{v}(\sigma_{i-1}, X_i), \bar{v}_i = \bar{v}(\sigma_i, X_i)$.

From (i) and (ii) above we see that

$$C_{\bar{v}_{i-1}}(\alpha, X_{i-1}) \supset C_{\bar{v}_i}(\alpha, X_i), i = 2, \dots k$$

and that

$$C_{\bar{v}_0}(\alpha, \eta_\tau(s)) \supset C_{\bar{v}_2}(\alpha, X_2), \quad C_{\bar{v}_k}(\alpha, X_k) \supset C_{\bar{v}_{k+1}}(\alpha, \eta_\tau(s+t)),$$

where $\bar{v}_0$ and $\bar{v}_{k+1}$ are tangent vectors along σ_0 and σ_k at X_0, X_{k+1} respectively.

In addition, if X_0 is a bending point then $C_{\bar{u}_0}(\alpha, \eta_\tau(s)) \supset C_{\bar{v}_1}(\alpha, X_1)$ and if X_{k+1} is a bending point then $C_{\bar{v}_{k-1}}(\alpha, X_{k-1}) \supset C_{\bar{v}_{k+1}}(\alpha, X_1)$, where u_0 and v_{k+1} are tangent vectors along the segments of η_τ at which meet σ_0 or σ_k at X_0, X_{k+1} respectively. Thus in all cases it follows that $C_\eta(\alpha, s) \supset C_\eta(\alpha, s+t)$.

Since $\sigma_0, \sigma_1 \subset C_\eta(\alpha, s)$, and $\sigma_i \subset C_{\bar{v}_i}(\alpha, X_i), i = 1, \dots, k$, it also follows that $\eta_\tau([s, s+t]) \subset C_\eta(\alpha, s)$. Since $t > 0$ is arbitrary, this shows that $\eta_\tau([s, \infty)) \subset C_\eta(\alpha, s)$. $\square$

6.2. Proof of theorem 6.1. As a consequence of the cone lemma we have the first part of theorem 6.1.

PROPOSITION 6.5. *Let $G_0 \in \mathcal{F}$ and let γ be a simple closed geodesic on $\mathbb{D}/G_0$. Let $\tau = i\theta, \theta \in \mathbb{R}$ and let $\phi_\tau : \mathbb{D} \to \mathbb{D}_\tau \subset \mathbb{H}^3$ be the pleated surface defined by the rational quakebend $Q_\tau(\gamma)$. Suppose $0 < \alpha < \frac{\pi}{2}$. Let $\epsilon = \epsilon(\alpha, \gamma)$ be as in the cone lemma, and suppose in addition that $\epsilon < \pi - 2\alpha$. Then if $|\theta| < \epsilon$, the developed surface $\mathbb{D}_\tau$ is embedded in $\mathbb{B}^3$, and ϕ_τ extends continuously to a map $\partial\mathbb{D} \to \partial\mathbb{B}^3 \simeq \hat{\mathbb{C}}$.*

PROOF. First we prove $\mathbb{D}_\tau$ is embedded. Clearly, it suffices to see that the developed image η_τ of any geodesic $\eta \subset \mathbb{D}$ is embedded. The condition $\epsilon < \pi - 2\alpha$ ensures that the cones $C_{\bar{v}}(\alpha, X)$ based at any point X on η_τ in (any of) the forward and backward tangent directions $\bar{v}$ to η at X are disjoint and separate the forward and backward trajectories from X. Thus η_τ can never intersect itself at X.

Next we show that the image of any geodesic in $\mathbb{D}$ has a definite endpoint on $\partial\mathbb{B}^3$. Let $\eta : [0, \infty) \to \mathbb{D}$ be a geodesic with initial point $Z \in \mathbb{D}$ and endpoint $X \in \partial\mathbb{D}$. Set $\eta_\tau(0) = Z$. Choose an increasing sequence $s_0, s_1, \dots \in [0, \infty)$ such that $\eta(s_n)$ is not a bending point of η for any n, and such that $s_n - s_{n-1} > 2d_\gamma, n = 1, 2, \dots$. If the image $\eta_\tau([0, \infty))$ were contained in a compact neighborhood of $\eta_\tau(Z) \in \mathbb{B}^3$, then we could find a subsequence s_{n_r} such that the vertices and the directions of the central axes of the cones $C_\eta(\alpha, s_{n_r})$ accumulated at a point whose distance from $\eta_\tau(Z) \in \mathbb{B}^3$ is finite. By the cone lemma however, these cones are nested and their vertices are at least d_γ apart so that they cannot accumulate inside $\mathbb{B}^3$. It follows that η_τ leaves any compact set in $\mathbb{B}^3$. The accumulation set of η_τ is therefore contained in the intersections of the shadows of $C_\eta(\alpha, s_n)$. Let B_r be the sphere of Euclidean radius r centered at O. The Euclidean diameter of the shadow of a cone $C_{\bar{v}}(\alpha, Y)$ with vertex Y on B_r and $\bar{v}$ pointing out from B_r is $O(1-r)$. We conclude that η_τ has a unique limit point on $\partial\mathbb{B}^3$ which we denote by $\phi_{\tau,Z}(X)$.

We claim that $\phi_{\tau,Z}(X)$ is independent of the initial point Z. Suppose η_1, η_2 are geodesics in $\mathbb{D}$ with different initial points Z_1, Z_2 and the same endpoint $X \in \partial\mathbb{D}$. Since η_1, η_2 are asymptotic in $\mathbb{D}$, and since ϕ_τ is a local isometry off the bending points, if K is any compact set in $\mathbb{B}^3$ we can find points $Y_1 \in \eta_1, Y_2 \in \eta_2$ outside K such that $|Y_1 - Y_2|$ is arbitrarily small and such that the vertices and the directions of the axes of the cones $C_{\eta_1}(\alpha, Y_1), C_{\eta_2}(\alpha, Y_2)$ are arbitrarily close. Thus their shadows are both arbitrarily close and arbitrarily small and $\phi_{\tau,Z_1}(X) = \phi_{\tau,Z_2}(X)$. We write $\phi_\tau(X)$ in place of $\phi_{\tau,Z}(X)$.

It is now easy to see that the extension of ϕ_τ to $\partial\mathbb{D}$ is injective. Given any two points $X, Y \in \partial\mathbb{D}$, we define $\phi_\tau(X), \phi_\tau(Y)$ as $\phi_{\tau,Z}(X), \phi_{\tau,Z}(Y)$ where Z is any point on the geodesic $\eta : (-\infty, \infty) \to \mathbb{D}$ with endpoints $X, Y \in \partial\mathbb{D}$. As in the first paragraph of this proof, since $\epsilon < \pi - 2\alpha$, the forward and backward cones at any point of η_τ separate the forward and backward trajectories so the limit points $\phi_\tau(X), \phi_\tau(Y)$ are distinct.

We now show that the extension is also continuous. Let $X \in \partial\mathbb{D}$ and let $X_n \in \mathbb{D} \cup \partial\mathbb{D}, n \geq 1$ be a sequence converging to X in the Euclidean metric on $\mathbb{D} \cup \partial\mathbb{D}$. Let $\eta : [0, \infty) \to \mathbb{D}$ be the geodesic from some point $Z \in \mathbb{D}$ to X. We may suppose that Z is a non-bending point and that ϕ_τ is a local isometry at Z. We may also suppose that Z is close to X in the Euclidean metric and that $\bar{v}$, the unit tangent to the image of η at $\phi_\tau(Z)$, points out of the ball of radius $\rho_{\mathbb{H}^3}(O, \phi_\tau(Z))$ centered at O.

For $n \geq 0$ let η_{X_n} be the geodesic from Z to X_n. Let $\psi_n = \angle XZX_n, n \geq 1$. Since ϕ_τ is a local isometry at Z, the unit tangent $\bar{v}_n$ to $\phi_\tau(\eta_{X_n})$ at $\phi_\tau(Z)$ makes an angle ψ_n with $\bar{v}$. By the cone lemma, $\phi_\tau(\eta_{X_n})$ is contained in the cone $C_n = C_{\bar{v}_n}(\alpha, \phi_\tau(Z))$ and by definition, $\phi_\tau(X_n)$ is contained in the shadow of C_n. Clearly $\psi_n \to 0$ as $n \to \infty$. Hence the cones C_n converge to the cone $C = C_v(\alpha, \phi_\tau(Z))$ whose shadow contains $\phi_\tau(X)$. By choosing Z sufficiently close to X we can ensure that the diameter of the shadow of C is arbitrarily small, giving the continuity. $\square$

Now we are able to complete the proof of theorem 6.1.

THEOREM 6.6. *Let $G_0 \in \mathcal{F}$ and let γ be a simple closed geodesic on $\mathbb{D}/G_0$. Let $\tau \in \mathbb{C}$ and let G_τ be the deformed group defined by the rational quakebend $Q_\tau(\gamma)$. Then there exists $\epsilon > 0$, depending only on the hyperbolic length of γ, such that if $|\tau| < \epsilon$, the induced map $G_0 \to G_\tau$ is a type preserving isomorphism and G_τ is quasifuchsian.*

REMARK 6.7. *In this theorem, we have dropped the hypothesis that τ be purely imaginary, therefore Q_τ is no longer necessarily a pure bend.*

PROOF. Set $\tau = t + i\theta$ and let $V = V(\gamma) \in G_0$ be an element whose axis projects to γ.

Suppose first Q_τ is not a pure bend. Earthquake the base group G_0 by a distance t along γ; that is, move from the group G_0 with Fenchel-Nielsen coordinates (λ_V, τ_V) to the group G_t with coordinates $(\lambda_V, \tau_V + t)$. Then G_t is still in $\mathcal{F}$ and $Q_{t+i\theta}(G_0) = Q_{i\theta}(G_t)$. Thus by taking our starting point as G_t and replacing $t + i\theta$ by $i\theta$, we may assume without loss of generality that $\tau = i\theta$ and that Q_τ is a pure bend. We call the new starting group G_0 and as always take $\phi_\tau(O) = O$.

We claim the homomorphism $G_0 \to G_\tau$ is injective and type preserving. By the group invariance, fixed points of $g \in G_0$ are mapped to the fixed points of g_τ. Since $g \neq id$ has no fixed point inside $\mathbb{D}$ neither does g_τ. Any parabolic $g \in G_0$ is conjugate to the commutator of the generators and thus by construction, g_τ is also parabolic. By the injectivity of ϕ_τ the images of hyperbolic elements have distinct fixed points and hence are loxodromic.

Finally we show G_τ quasifuchsian. By proposition 6.5, ϕ_τ extends to $\partial\mathbb{D}$ injectively and continuously so that $\phi_\tau(\partial\mathbb{D})$ is a Jordan curve J. Hence $J = \Lambda(G_\tau)$ and $\Omega(G_\tau)$ contains two simply connected components, $\Omega^\pm_\tau$. By[**17**], p. 217, we only have to show that $\Omega^\pm_\tau$ are G_τ-invariant. If they were not, G_τ and thus G_0 would have to contain an element of order 2 which is not the case. □

REMARK 6.8. *It would have been sufficient to prove only that ϕ_τ extends injectively to the fixed point set of G_0 and to note that the generators of G_τ depend holomorphically on τ; the extension would thus define a holomorphic motion on the fixed points and by the λ-lemma it would follow that ϕ_τ is continuous and G_τ is quasifuchsian. Our proof however, is more elementary.*

The following corollary will be useful later.

COROLLARY 6.9. *Let $G_0 \in \mathcal{F}$ and let γ be a simple closed geodesic on $\mathbb{D}/G_0$. Let $\tau \in \mathbb{C}$ and let $\mathbb{D}_\tau$ be the developed surface associated to the quakebend $Q_\tau(\gamma)$. If τ, ϵ satisfy the hypotheses of proposition 6.5, then $\overline{\mathbb{D}}_\tau$ separates $\mathbb{B}^3 \cup \hat{\mathbb{C}}$ into two pieces that meet $\hat{\mathbb{C}}$ in $\Omega^\pm_\tau$ and the extension of ϕ to $\partial\mathbb{D}$ is a homeomorphism $\partial\mathbb{D} \to \Lambda_\tau$.*

PROOF. We identify $\mathbb{D}$ with the equatorial plane in $\mathbb{B}^3$ as in section 5.3 so that $\partial\mathbb{D}$ is identified with the equator E. Denote by f_τ the quasiconformal homeomorphism of $\hat{\mathbb{C}}$ that conjugates G_0 to G_τ. By the group invariance, the extension of ϕ_τ to $\partial\mathbb{D}$ and the restriction of f_τ to E agree on the fixed points of elements of G_0 and hence on E. Since D_τ is embedded, the developing map $\phi_\tau : \mathbb{D} \to \mathbb{D}_\tau$ is a homeomorphism. As a quasiconformal homeomorphism of $\hat{\mathbb{C}}$, f_τ induces a homeomorphism from, say, the northern hemisphere, to Ω^+_τ. Therefore we may glue ϕ_τ and f_τ together along E to get a map from $\partial\mathbb{B}^3 \simeq S^2$ into $\mathbb{B}^3$ separating it into two pieces. □

7. The main theorem: Rational case

In this section we prove our main theorem, theorem 3.1, in the case when the bending laminations μ, ν are both rational.

7.1. Forcing the pleating on one side. We begin by showing that the results of the section 6 control the pleating locus on one side of $\partial\mathcal{C}$.

LEMMA 7.1. *Let $G_0 \in \mathcal{F}$ and let γ be a simple closed geodesic on $\mathbb{D}/G_0$. Let $\tau \in \mathbb{C}$ and let $\mathbb{D}_\tau$ be the developed surface associated to the quakebend $Q_\tau(\gamma)$. Suppose that τ, ϵ satisfy the hypotheses of proposition 6.5. Then one of the two half spaces cut out by $\mathbb{D}_\tau$ in $\mathbb{B}^3 \cup \hat{\mathbb{C}}$, is convex.*

PROOF. Embedding $\mathbb{D}$ in $\mathbb{H}^3$ as the equatorial plane E, we may fix an orientation on $\mathbb{D}$ that we call "up". From the construction of a rational quakebend, all the bending angles are in the same direction relative to this orientation, which we shall take to be "down". (If $\Im\tau > 0$, this agrees with the conventions described in section 5.3.) The developed surface $\mathbb{D}_\tau$ is the image of E under the developing map ϕ_τ, and therefore inherits an orientation that we also describe by the words "up" and "down". With this convention, let H be the halfspace, cut out by $\mathbb{D}_\tau$ in $\mathbb{B}^3$, and lying "below" $\mathbb{D}_\tau$. We claim that H is convex.

Let $x, y \in H$ and let β be the $\mathbb{H}^3$ geodesic joining them. Suppose that β is not completely contained inside H, and let Π be a plane containing β. Then $\beta \cup (\partial H \cap \Pi)$ is a geodesic polygon Γ with n sides, for some $n > 1$. The bending angles on the bending lines of $\mathbb{D}_\tau$ are all measured in the same direction and by our choice of convention, the interior angles in Γ are all greater than π. Since the sum of the angles in a hyperbolic n-gon is less than $\pi(n-2)$, this is impossible. This completes the proof. □

PROPOSITION 7.2. *Let $G_0 \in \mathcal{F}$ and let γ be a simple closed geodesic on $\mathbb{D}/G_0$. Let $\tau \in \mathbb{C}$ and let $\mathbb{D}_\tau$ be the developed surface associated to the quakebend $Q_\tau(\gamma)$. Suppose that τ, ϵ satisfy the hypotheses of proposition 6.5. Then $\mathbb{D}_\tau$ is one of the components of $\partial\mathcal{C}(G_\tau)$.*

PROOF. Let $\bar{H}$ be the closure in $\mathbb{B}^3 \cup \hat{\mathbb{C}}$ of the convex half space H cut out by $\mathbb{D}_\tau$ as in lemma 7.1. By corollary 6.9, $\partial\mathbb{D}_\tau = \Lambda(G_\tau)$. Thus $\Lambda(G_\tau)$, and hence the convex hull $\mathcal{C}(G_\tau)$, are contained in $\bar{H}$.

The bending lines of $\mathbb{D}_\tau$ are the lifts to $\mathbb{H}^3$ of the bending lamination γ. The endpoints of these axes all lie in $\Lambda(G_\tau)$. Clearly, $\mathbb{D}_\tau$ is contained in the convex hull of its bending lines, hence $\mathbb{D}_\tau \subset \mathcal{C}(G_\tau)$. Since by construction, $\mathbb{D}_\tau$ is the boundary of H in $\mathbb{H}^3$, it follows that $\mathbb{D}_\tau \subset \partial\mathcal{C}(G_\tau)$. Because $\mathbb{D}_\tau$ is a pleated surface it is clearly both open and closed in $\partial\mathcal{C}(G_\tau)$. Since $\mathbb{D}_\tau$ is connected, it must be one of the components of $\partial\mathcal{C}(G_\tau)$. □

COROLLARY 7.3. *Let γ be a simple closed curve on $\mathcal{T}_1$. Then there exists $G \in \mathcal{Q}F$ such that $[pl(G)^+] = \gamma$.*

PROOF. This is immediate from the proposition above. □

7.2. Forcing the pleating on both sides. We now turn to constructing a quasifuchsian group in which we simultaneously prescribe both pl^+ and pl^-. We begin with a simple but crucial observation proved in [**11**], lemma 4.6.

LEMMA 7.4. *Suppose that β is a simple closed curve on $\mathcal{T}_1$ and suppose that $\beta \subset pl^{\pm}(G)$ for some $G \in \mathcal{Q}F$. Let $M \in PSL(2,\mathbb{C})$ be an element corresponding to β in G. Then M is purely hyperbolic, that is,* $\operatorname{Tr} M \in \mathbb{R}$ *and* $|\operatorname{Tr} M| > 2$.

REMARK 7.5. *In general, the condition that* $\operatorname{Tr} M$ *be purely hyperbolic is not sufficient for β to be in the pleating locus of G. However, as we shall see below, in the special case of punctured tori with small bending angles, it is. The following proposition depends crucially on the fact that we are working with a punctured torus and hence that on each component of $\partial\mathcal{C}$, the bending angles along all lines in the pleating locus are equal.*

PROPOSITION 7.6. *Let $G_0 \in \mathcal{F}$, let γ, γ' be distinct simple closed geodesics on $\mathbb{D}/G_0$ and let $\tau \in \mathbb{C}$. Let $G = G_\tau$ be the deformed group associated to the quakebend $Q_\tau(\gamma)$. Then there exists $\eta > 0$ such that if $|\Im\tau| < \eta$, and if the element V'_τ corresponding to γ' in G_τ is purely hyperbolic then $([pl^+(G_\tau)], [pl^-(G_\tau)]) = (\gamma, \gamma')$.*

PROOF. Let V be the element corresponding to γ in G_0, and let $W \in G_0$ be a complementary generator. As above, the group $G_\tau = G_\tau(V, W)$ and the developed surface D_τ are constructed using the Fenchel-Nielsen construction relative to the generators V, W.

Let V' be the element corresponding to γ' in G_0, and let $W' \in G_0$ be a complementary generator. Let W'_τ be the element corresponding to W' in G_τ. Let $(\lambda_{V'}, \tau_{V'})$ be complex Fenchel-Nielsen coordinates relative to (V', W') in the construction of section 4.3. Let us consider the construction carefully for the group G_τ. By theorem 6.6 we may assume that τ is small enough so that $G_\tau \in \mathcal{QF}$ and hence that V'_τ is loxodromic. By assumption $\operatorname{Tr} V'_\tau \in \mathbb{R}$; hence V'_τ must be hyperbolic and $\lambda_{V'} > 0$. It follows that $\operatorname{Tr} {W'}_\tau^{-1} V'_\tau W'_\tau \in \mathbb{R}$; moreover since (V', W') is a generating pair, $V'_\tau {W'}_\tau^{-1} {V'}_\tau^{-1} W'_\tau$ is parabolic and also has real trace. Thus the axes of V'_τ and ${W'}_\tau^{-1} V'_\tau W'_\tau$ are coplanar and the skew polygon P in the construction of section 4.2, is planar.

As explained at the end of section 5.3, because P is planar, the quakebend $Q_{\tau'}(\gamma')$ along the curve γ' is well defined and determines a developed surface $\mathbb{D}_{\tau'}$, where the parameter $\tau' = \tau_{V'}$ is the complex Fenchel-Nielsen twist parameter relative to the generator V' for the group G_τ (relative to the initial group G_0). Note that groups $G_\tau(V, W)$ and $G_{\tau'}(V', W')$ are the same group described in different ways. We may therefore apply all the above results to $D_{\tau'}$, provided we know that $|\Im\tau'|$ is sufficiently small.

Recall from section 4.3, that τ'_V is a holomorphic function of (λ_V, τ_V). As we vary G_τ by a quakebend on γ, λ_V is fixed. Therefore τ'_V is a holomorphic function of the quakebend parameter $\tau = \tau_V$ alone. Moreover, in Fuchsian space, $\Im\tau = \Im\tau' = 0$. Therefore, when $|\Im\tau_V|$ is small, so is $|\Im\tau'_V|$. By proposition 6.5 we may choose $|\Im\tau|$ sufficiently small so that the developed surface $D_{\tau'}$ is embedded in $\mathbb{H}^3$ and hence, by proposition 7.2, is one of the components of $\partial\mathcal{C}(G_\tau)$. Since we also know from proposition 7.2 that for small $|\Im\tau|$, D_τ is also a component, and since the bending lines γ, γ' are distinct so that $D_\tau \neq D_{\tau'}$, the result follows. □

To complete the proof of theorem 3.1 we have only to show that, given distinct simple geodesics γ, γ' on $\mathcal{T}_1$, it is possible to perform a (small) quakebend along γ in such a way that the trace of the element corresponding to γ' remains real. This is what we do next.

7.3. Forcing real trace. Let γ, γ' be distinct simple geodesics on $\mathcal{T}_1$. We need to study the behavior of the trace of the group element corresponding to γ' as we quakebend along γ.

We first consider earthquakes where the parameter τ is real. If we start from group $G_0 \in \mathcal{F}$, and apply an earthquake $Q_t, t \in \mathbb{R}$, then the deformed group G_t is also in $\mathcal{F}$. We recall the following key result proved in [**14**]:

THEOREM 7.7 (Kerckhoff). *Let $G_0 \in \mathcal{F}$ and let γ be a simple closed geodesic on $\mathbb{D}/G_0$. Let $Q_t, t \in \mathbb{R}$, be an earthquake along γ. Let β be any closed geodesic transverse to γ, and let β_t be the geodesic corresponding to β in the deformed group G_t. Then the hyperbolic length $l_\beta(t)$ of β_t is a strictly convex function of t, with a unique minimum $t_\beta \in \mathbb{R}$.*

We also note that under the same hypotheses it is a result of Wolpert [**24**], (see also [**7**]), that the second derivative $\frac{d^2 l_\beta(t)}{dt^2}$ does not vanish at t_β.

In consequence of these results, extending the real parameter t to the complex parameter τ, we get:

Corollary 7.8. *Let $G_0 \in \mathcal{F}$ and let γ be a simple closed geodesic on $\mathbb{D}/G_0$. Let $Q_\tau(\gamma)$ denote the quakebend by $\tau = t + i\theta \in \mathbb{C}$ along γ. Let γ' be any simple closed geodesic distinct from γ, and let $V' \in G_0$ be an element corresponding to γ'. Then there is a neighborhood $\mathcal{U}$ of $\{\tau = t + 0i, t \in \mathbb{R}\}$ and a path $\xi : (-\epsilon, \epsilon) \mapsto \mathbb{C}$ such that*

- $\xi(0) = t_{\gamma'}$; $\quad \Im\xi(s) \neq 0$ *for* $s \neq 0$ *and*
- $\operatorname{Tr} V'(\xi(s)) \in \mathbb{R}$, $|\operatorname{Tr} V'(\xi(s))| > 2$; $\quad \operatorname{Tr} V'(\tau) \notin \mathbb{R}$ *for* $\tau \in \mathcal{U} - Im(\xi)$, $\Im\tau \neq 0$.

Proof. Note that from the formula $\operatorname{Tr} V' = 2\cosh\lambda_{V'}$; it follows that the conditions $\lambda_{V'} > 0$ and $\operatorname{Tr} V' \in \mathbb{R}$, $|\operatorname{Tr} V'| > 2$ are equivalent. For a group $G_t \in \mathcal{F}$, the length function $l_{\gamma'}(t)$ of Kerckhoff's theorem is nothing other than twice the the multiplier $\lambda_{V'}(\tau)$ restricted to $\mathbb{R}$, with $t = \Re\tau$. By theorem 4.3, $\lambda_{V'}(\tau)$ is a holomorphic function of τ; in particular the Kerckhoff minimum point $t_{\gamma'}$ is the unique critical point of $\lambda_{V'}(\tau)$ in $\mathbb{R}$.

Expanding $\lambda_{V'}$ at a point $a \in \mathbb{R}$ we get

$$\lambda_{V'}(\tau) = \lambda_{V'}(a) + \frac{d\lambda_{V'}}{d\tau}\Big|_a(\tau - a) + \text{ higher order terms.}$$

If a is not the critical point of $\lambda_{V'}$, then the only tangent direction to the real locus $\{\tau \in \mathbb{C} : \lambda_{V'}(\tau) \in \mathbb{R}\}$ is along the real axis. At the critical point $t_{\gamma'}$, $\frac{d\lambda_{V'}}{d\tau} = 0$ and by Wolpert's result, $\frac{d^2\lambda_{V'}}{d\tau^2}|_a > 0$. Thus the real locus $\lambda_{V'} \in \mathbb{R}$ has two branches that meet orthogonally at $t_{\gamma'}$. The path ξ of the statement is the non-real branch. $\square$

Remark 7.9. *The fact that ξ meets $\mathbb{R}$ orthogonally means that, infinitesimally, the quakebend that keeps $\lambda_{V'}$ real is a pure bend; also, since $\frac{d^2\lambda_{V'}}{d\tau^2}|_{t_{\gamma'}}$ is positive it follows that, in accordance with intuition, $\lambda_{V'}$ decreases in a neighborhood of $t_{\gamma'}$.*

Our main result for rational laminations, theorem 3.2, now follows immediately from proposition 7.6 and corollary 7.8.

8. The general case

In this section we complete the proof of theorem 3.1 by showing how to extend our results to the case of an arbitrary pair of distinct measured geodesic laminations $\mu, \nu \in PML(\mathcal{T}_1)$. The key point is to prove a suitable extension of the cone lemma; the theorem then follows much as before with the additional ingredient of a result about the continuity of the pleating locus in $\mathcal{QF}$ proved in [**13**]. We begin by collecting some facts we need about general quakebends.

8.1. The quakebend cocycle for arbitrary laminations. One cannot immediately extend the definition of complex Fenchel-Nielsen coordinates to coordinates relative to non-rational measured laminations. One can however extend the construction of a rational quakebend, with its associated developed surface, to that of a quakebend along an arbitrary measured lamination μ with starting point an arbitrary Fuchsian group G_0. This was done by Thurston, see [**7**]. Here we quickly review the results we need.

Let $G_0 \in \mathcal{F}$ and let ω be a *complex valued* measured geodesic lamination on $\mathbb{D}/G_0$: this means that the values taken by the transverse measure ω may be complex. Chapter 3 of [**7**] contains a detailed discussion of how to define the *quakebend cocycle* B_ω and the corresponding pleated surface $\phi_\omega : \mathbb{D} \to D_\omega \subset \mathbb{H}^3$

that determine the quakebend Q_ω along ω. It follows immediately from section 3.7 of [**7**] that Q_ω determines a group $G_\omega \subset PSL(2,\mathbb{C})$ such that ϕ_ω intertwines the actions of G_0 on the hyperbolic disc $\mathbb{D}$ and G_ω in $\mathbb{H}^3$.

For our purposes, we shall use this construction applied to complex measured laminations of the form $\omega = \tau\mu$, where μ is a measured lamination on $\mathcal{T}_1$ in the usual sense and $\tau \in \mathbb{C}$. This means that the measure of a segment κ transverse to $|\mu|$ is $\tau\mu(\kappa)$. One obtains a group $G_{\tau\mu} \in PSL(2,\mathbb{C})$ and a map $\phi_{\tau\mu} : \mathbb{D} \to \mathbb{D}_{\mu\tau}$ that for rational μ gives just the group G_τ and map ϕ_τ defined in section 5.3. By Lemma 3.8.1 of [**7**], the family of quakebends $Q_\tau = Q_{\tau\mu}$ induced by the quakebend cocycle $B_{\tau\mu}$ depends holomorphically on τ.

It will be sufficient to restrict our discussion to the case of *pure bends*, where $\tau = i\theta$, $\theta \in \mathbb{R}$. For simplicity of notation, in this section and the next, we shall consider the unit quakebend $Q_{i\mu}$ in which we take $\tau = i$. We shall only introduce the parameter θ in the proof of the generalized cone lemma in section 8.3.

The *quakebend cocyle* $B_{i\mu}(x,y)$ is a cocycle $B_{i\mu} : \mathbb{D}\times\mathbb{D} \to PSL(2,\mathbb{C})$, with $B_{i\mu}(x,y)B_{i\mu}(y,z) = B_{i\mu}(x,z), x,y,z \in \mathbb{D}$ and $B_{i\mu}(x,x) = \mathrm{id}$, $x \in \mathbb{D}$. The developing map $\phi_{i\mu} : \mathbb{D} \to \mathbb{H}^3$ is defined by $\phi_{i\mu}(O) = O$, $\phi_{i\mu}(x) = B_{i\mu}(O,x)(x)$ where $\mathbb{D}$ is embedded in $\mathbb{H}^3$ as the equatorial plane and O is a chosen base point as in section 5.3.

The cocyle $B_{i\mu}(x,y)$ is defined as a limit of products of rotations about axes that approximate the leaves of the bending lamination $|\mu|$. The following approximation, proved in Lemma 3.4.4 of [**7**], is the first step in proving this limit is well defined.

Let l be an oriented geodesic in $\mathbb{H}^3$. Let $E(l,\psi)$ denote the rotation by ψ about l, where ψ is measured in the plane orthogonal to l and facing along l in the direction of orientation of l. Following [**7**], we use the operator norm on $SL(2,\mathbb{C})$ acting by linear maps on $\mathbb{C}^2$: namely, if $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{C})$ then $\|M\| = \max\{|a|+|b|,|c|+|d|\}$.

Lemma 8.1. *Let K be a compact subset of $\mathbb{H}^2$ and let $M > 0$. Then there exists a constant c_0 with the following property. Let κ be any oriented geodesic segment in K and let $\mathcal{L}$ be any set of lines that intersect κ, all oriented to the left when moving along κ. Let $l_1,\dots,l_k$ be any collection of lines in $\mathcal{L}$, enumerated in the order in which they meet κ; also let $l \in \mathcal{L}$. Let $\zeta_1,\dots\zeta_k \in [0,2\pi)$ be such that $\sum_{i=1}^{k}\zeta_i = \zeta$. Then*

$$\|E(l_1,\zeta_1)\dots E(l_k,\zeta_k) - E(l,\zeta)\| \le c_0 l(\kappa)|\zeta|.$$

The next step is to get uniform control of our approximations.

Lemma 8.2. *Let $G_0 \in \mathcal{F}$ and let μ be a measured geodesic lamination on $\mathbb{D}/G_0$. There exist constants $d_\mu > 0$, $K_\mu > 0$ such that if κ is any geodesic segment on $\mathbb{D}/G_0$ of length less than d_μ, then $\mu(\kappa) < K_\mu$.*

Proof. Let $|\mu|$ denote the geodesic lamination that as a point set underlies the measured lamination μ. As is well known, (see for example [**21**]) none of the leaves of $|\mu|$ end at the cusp of $\mathbb{D}/G_0$ and in fact there is a horoball neighborhood $\mathcal{U}$ of the cusp such that $|\mu|\cap\mathcal{U} = \emptyset$. Thus we can find $d = d_\mu > 0$ such that for any $x \in |\mu|$, the hyperbolic ball $B_{d_\mu}(x)$ of radius d_μ and center x is embedded in $\mathbb{D}/G_0$. Now cover $\mathbb{D}/G_0 - \mathcal{U}$ by a finite number of such balls $B_i, i = 1,\dots,N$ and let κ be a geodesic segment on $\mathbb{D}/G_0$ of length less than d_μ. If $|\mu| \cap \kappa = \emptyset$ we have nothing to prove, so pick $x \in |\mu| \cap \kappa$. Then $\kappa \subset B_{d_\mu}(x)$ and hence is embedded. Thus we can partition κ into at most N segments, each of which is contained in one of the balls B_i.

We claim that there is an upper bound, denoted $\mu(B_i)$, to the μ measure of a geodesic arc κ intersecting B_i. This follows since we can clearly find a finite number of transverse arcs such that their union intersects all leaves of $|\mu| \cap \kappa$, (see the discussion in the proof of theorem 4.5 in [**13**]). Let $M = \max\{\mu(B_i) : 1 \leq i \leq N\}$. Setting $K_\mu = MN$, the result follows. □

Now let $x, y \in \mathbb{D}$ and let κ be the geodesic segment joining x to y. The cocycle $B_{i\mu}(x,y)$ is defined as the limit of products $E(l_1,\zeta_1)\dots E(l_k,\zeta_k)$ where the lines l_i are chosen as in lemma 8.1. The maximum distance between the l_i along κ tends to zero, and $\sum_{i=1}^{k} \zeta_i = \mu(\kappa)$. Convergence of the products follows from lemmas 8.1 and 8.2.

The following lemma follows directly from the lemmas above and gives the key approximation we shall need in the next section.

Corollary 8.3. *Let $G_0 \in \mathcal{F}$ and let μ be a measured geodesic lamination on $\mathbb{D}/G_0$. There exists a constant c_1, depending on μ and G_0, with the following property. Let κ be a geodesic segment in $\mathbb{D}$ with endpoints x, y and length less than $2d_\mu$, transverse to the lift of $|\mu|$ to $\mathbb{D}$. Let $q \in \kappa \cap |\mu|$ and let l be the leaf of μ through q. Then $\|B_{i\mu}(x,y) - E(l,\mu(\kappa))\| \leq c_1 l(\kappa)$.*

Remark 8.4. *For later reference, we note that it is clear that the constants d_μ, K_μ, c_1 of the above lemma and corollary can be chosen uniformly for the family of groups G_t obtained by applying the earthquakes $Q_{t\mu}$ to the initial group G_0 for suitable small $t \in \mathbb{R}$.*

8.2. Approximating developed geodesics. Let $\eta : [0,\infty) \to \mathbb{D}$ be an oriented geodesic. Recall that the image of a geodesic in $\mathbb{D}$ under the developing map associated to a rational quakebend is made up of geodesic segments joined at points that we called its *bending points*. As above, denote the image of η under the quakebend map by $\eta_{i\mu} = \phi_{i\mu}(\eta)$. A *bending point* of η with respect to the irrational lamination μ is a point in $\eta \cap |\mu|$. Clearly a segment of η containing no bending points maps to a geodesic segment in $\mathbb{H}^3$. We call such segments the *flat pieces* of η. If $\eta(s)$ lies in the interior of a flat piece κ of η then we can define a tangent vector $\dot\eta_{i\mu}(s)$ as the unit speed tangent to the segment $\phi_{i\mu}(\eta)$ at s in the obvious way.

In general, if μ is irrational, a segment of η contains an infinite number of bending points. Since the measure μ is non-atomic, the bending angle at each point of $|\mu|$ is zero. Lemma 8.6 below enables us to define a tangent vector at each point of $\eta_{i\mu}$.

It will be convenient to introduce a metric D_T on the tangent space $\mathrm{T}\mathbb{H}^3$ to $\mathbb{H}^3$. Let $(P_i, \bar v_i) \in \mathrm{T}\mathbb{H}^3$ where $\bar v_i$ is a tangent vector at $P_i \in \mathbb{H}^3$. We define

$$D_T((P_1,\bar v_1),(P_2,\bar v_2)) = \rho_{\mathbb{H}^3}(P_1,P_2) + \|T_{P_2}(\bar v_1) - \bar v_2\|$$

where T_{P_2} is the parallel transport along the geodesic joining P_1 to P_2 and $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^3$.

We begin with an easy modification of Lemma 3.3.2 of [**7**].

LEMMA 8.5. *Given a compact subset* $K \subset \mathbb{H}^3$*, and a compact subset* $L \subset PSL(2,\mathbb{C})$*, there exists* $c_2 > 0$ *such that if* $P \in K$*,* $M, M' \in L$*, and* $\bar{v}$ *is a unit tangent vector at* x*, then*

$$D_T((M(P), M(\bar{v})), ((M'(P), M'(\bar{v}))) < c_2 \|M - M'\|.$$

PROOF. Essentially the same proof as in [**7**] applies: namely write $M^{-1}M'$ as a product of a rotation about P followed by a translation and note that parallel transport commutes with hyperbolic isometries; we have, for some c_2:

$$\rho_{\mathbb{H}^2}(M(P), M'(P)) + \|T_{P'}(M(\bar{v})) - M'(\bar{v})\| < c_2 \|M - M'\|.$$

$\square$

For simplicity of notation, when we use the metric D_T on the tangent space to $\mathbb{H}^3$ in what follows, we only write in the vectors and leave the base points understood.

LEMMA 8.6. *Let* μ *be an irrational lamination and let* $\eta : [0, \infty) \to \mathbb{D}$ *be an oriented geodesic. Let* $\eta(s), \eta(s')$ *lie in the interior of flat pieces of* η *and let* $\dot{\eta}_{i\mu}(s), \dot{\eta}_{i\mu}(s')$ *be the tangents to* $\eta_{i\mu}$ *at the points* $\phi_{i\mu}(\eta(s)), \phi_{i\mu}(\eta(s'))$*. Then* $D_T(\dot{\eta}_{i\mu}(s), \dot{\eta}_{i\mu}(s')) \to 0$ *as* $|s - s'| \to 0$*.*

PROOF. Let $I, I' \subset [0, \infty)$ be open intervals containing s, s' respectively and such that $\eta(I), \eta(I')$ are contained in flat pieces of η. Let $\bar{v}, \bar{v}'$ be unit tangent vectors to η at $\eta(s), \eta(s')$ respectively. Then by definition, $\dot{\eta}_{i\mu}(s) = B_{i\mu}(O, \eta(s))(\bar{v})$ and $\dot{\eta}_{i\mu}(s') = B_{i\mu}(O, \eta(s'))(\bar{v}')$. We need to show that $D_T(\dot{\eta}_{i\mu}(s), \dot{\eta}_{i\mu}(s')) \to 0$ as $|s - s'| \to 0$.

Using the cocycle relation we have

$$\dot{\eta}_{i\mu}(s') = B_{i\mu}(O, \eta(s)) B_{i\mu}(\eta(s), \eta(s'))(\bar{v}').$$

Hence it is sufficient to show that $D_T(\bar{v}, B_{i\mu}(\eta(s), \eta(s'))(\bar{v}')) \to 0$ as $|s - s'| \to 0$. Since clearly $D_T(\bar{v}, \bar{v}') \to 0$ as $|s - s'| \to 0$, it is enough, by lemma 8.5, to show that $\|B_{i\mu}(\eta(s), \eta(s'))\| \to \text{id}$.

Now use lemma 8.3 to find a rotation $E = E(l, \mu(\eta([s, s'])))$ about a line l that intersects η on the interval $[s, s']$ and such that $\|B_{i\mu}(s, s') - E\| \leq c_1 |s - s'|$. By lemma 8.5, $D_T(B_{i\mu}(s, s')(\bar{v}')), E(\bar{v}')) < c_1 c_2 |s - s'|$.

As is well known ([**20**]), and see the proof of lemma 3.4, measured laminations on $\mathcal{T}_1$ are maximal. Since by assumption the lamination $|\mu|$ is not rational, so that the transverse measure μ is non-atomic, as s' converges to s, we have that $\mu(\eta([s, s'])) \to 0$. It follows that $E(l, \mu(\eta([s, s'])))$ and hence $B_{i\mu}(\eta(s), \eta(s')) \to$ id. $\square$

8.3. The cone lemma. Lemma 8.6 shows that we may extend the definition of the tangent vectors to the developing image of a geodesic $\eta \subset \mathbb{D}$ from the flat pieces to any point of $\eta_{i\mu}$. Denote by $\dot{\eta}_{i\mu}(s)$, the unit tangent vector to $\eta_{i\mu}$ at the point $\eta_{i\mu}(s)$.

Now assume $\tau = i\theta$, $\theta \in \mathbb{R}$ and perform the quakebend $Q_{i\theta\mu}$. Recall from section 6.1 that the forward cone $C_{\bar{v}}(\alpha, P)$ is the solid cone of angle α with vertex at P whose central axis is the $\mathbb{H}^3$ geodesic in the direction of $\bar{v}$. We write $C_{\eta,\theta}(\alpha, s)$ for the cone $C_{\dot{\eta}_{i\theta\mu}(s)}(\alpha, \eta_{i\theta\mu}(s))$.

We now generalize the cone lemma to the case of a quakebend about an arbitrary lamination.

LEMMA 8.7 (Cone lemma; general case). *Let $G_0 \in \mathcal{F}$ and let $\tau = i\theta, \theta \in \mathbb{R}$. Let μ be a measured geodesic lamination on $\mathbb{D}/G_0$ and let $\alpha \in (0, \frac{\pi}{2})$. Then there exists $\epsilon > 0$, depending on μ and α, such that if $Q_{i\theta\mu}$ is the quakebend along μ, and if $\eta : [0,\infty) \to \mathbb{D}$ is an oriented geodesic transverse to the lift of $|\mu|$ to $\mathbb{D}$, then for all $|\theta| < \epsilon$,*

$$C_{\eta,\theta}(\alpha, s) \supset C_{\eta,\theta}(\alpha, s+t)$$

whenever $t \geq d_\mu$, where d_μ is the constant of lemma 8.2, and

$$\eta_{i\theta\mu}([s,\infty)) \subset C_{\eta,\theta}(\alpha, s), \ \ \forall s \geq 0.$$

PROOF. Let $\eta : [0,\infty) \to \mathbb{D}$ be a geodesic with image $\eta_{i\mu}$ in $\mathbb{D}_{i\mu}$. We state two claims and show they imply the lemma. Then we prove the claims. The claims are:

(1) $C_{\eta,\theta}(\alpha, s) \supset C_{\eta,\theta}(\alpha, t)$ whenever $s + d_\mu \leq t \leq s + 2d_\mu$ and

(2) $\eta_{i\theta\mu}([s, s+d_\mu]) \subset C_{\eta,\theta}(\alpha, s)$ whenever $s \geq 0$.

To deduce the lemma from these two claims we first show that $C_{\eta,\theta}(\alpha, s) \supset C_{\eta,\theta}(\alpha, t)$ whenever $t \geq s + d_\mu$. Let $t \in [s + d_\mu, \infty)$ and find $k \in \mathbb{N}, k \geq 2$ such that $s + (k-1)d_\mu \leq t < s + kd_\mu$. Then by (1),

$$C_{\eta,\theta}(\alpha, s) \supset C_{\eta,\theta}(\alpha, s+d_\mu) \supset \ldots \supset C_{\eta,\theta}(\alpha, s+(k-2)d_\mu) \supset C_{\eta,\theta}(\alpha, s+t).$$

This proves the first statement of the lemma.

Now we prove $\eta_{i\mu\theta}([s,\infty)) \subset C_{\eta,\theta}(\alpha, s)$. From (2),

$$\eta_{i\theta\mu}([s, s+d_\mu]) \subset C_{\eta,\theta}(\alpha, s).$$

Combining this with the chain of inclusions of cones above proves that

$$\eta_{i\theta\mu}([s,\infty)) \subset C_{\eta,\theta}(\alpha, s), \ \ \forall s > 0.$$

This is the second statement of the lemma.

It remains to prove the claims (1) and (2).

First we prove (1). Let $\zeta = \theta\mu([s, s+2d_\mu])$. Choose a line $l \subset \mathbb{D}$ through the point $\eta(s + d_\mu)$ which is either disjoint from all the leaves of $|\mu|$, or coincident with a leaf of $|\mu|$. Orient l consistently with the leaves of $|\mu|$ as in corollary 8.3 and let $E = E(l, \zeta)$ be the rotation by ζ about l consistent with this orientation. Set $B = B_{i\theta\mu}(O, \eta(s))$. Observe that the cone $C = C_{\eta,\theta}(\alpha, s)$ is just the image under B of the cone $C_{\dot\eta(s)}(\alpha, \eta(s))$, the cone of angle α with vertex $\eta(s)$ and central axis coinciding with η. (Here we are as usual assuming that $\mathbb{D}$ is embedded as the equatorial plane in $\mathbb{H}^3$, so that $C_{\dot\eta(s)}(\alpha, \eta(s))$ is a cone in $\mathbb{H}^3$.)

We make an approximation $\beta : [s, s+2d_\mu] \to \mathbb{H}^3$ to the segment $\eta_{i\theta\mu}([s, s+2d_\mu])$ consisting of the two geodesic segments $\sigma_0 = B(\eta([s, s+d_\mu]))$ and $\sigma_1 = BE(\eta([s+d_\mu, s+2d_\mu]))$. (This means that we approximate $\eta_{i\theta\mu}([s, s+2d_\mu])$ by the image under the bending cocycle B of the segment $\eta([s, s+d_\mu])$ of η and the rotation $E(\eta([s+d_\mu, s+2d_\mu]))$ of the segment $\eta([s+d_\mu, s+2d_\mu])$.)

Let X_0, X_1 be the initial points of σ_0, σ_1 respectively and let $\bar{v}_1$ be the unit tangent vector to σ_0 at X_1. Notice that $X_0 = B(\eta(s)) = \eta_{i\theta\mu}(s)$ and that X_0 is the vertex of the cone C, while its central axis $B(\eta)$ is coincident with the segment σ_0.

Let $\bar{v}_2$ be the unit tangent vector to σ_1 at its endpoint X_2 and let $\bar{v}_3$ be the unit tangent vector to $\eta_{i\theta\mu}$ at $X_3 = \eta_{i\theta\mu}(s+2d_\mu)$. Our aim is to show that $(X_2, \bar{v}_2)$

and $(X_3, \bar{v}_3)$ are close in the metric D_T. This will enable us to control the position of the cone $C_{\bar{v}_3}(\alpha, X_3)$.

Let X_4 be a point on the axis of the cone C at distance $2d_\mu$ from X_0, and let $\bar{v}_4$ be the forward pointing unit tangent vector at X_4. We have $X_0 = B(\eta(s)), X_1 = B(\eta(s+d_\mu)), X_4 = B(\eta(s+2d_\mu))$ and $X_2 = BE(\eta(s+2d_\mu))$. Thus $BEB^{-1}(X_4) = X_2$ and $BEB^{-1}(\bar{v}_4) = \bar{v}_2$.

From the cocycle identity we have

$$B_{i\theta\mu}(O, \eta(s+2d_\mu)) = B_{i\theta\mu}(O, \eta(s))B_{i\theta\mu}(\eta(s), \eta(s+2d_\mu)).$$

Writing $H = B_{i\theta\mu}(\eta(s), \eta(s+2d_\mu))$ we find

$$X_3 = B_{i\theta\mu}(O, \eta(s+2d_\mu))(\eta(s+2d_\mu)) = BH(\eta(s+2d_\mu)) = BHB^{-1}X_4.$$

Applying the same reasoning to the tangent vector at X_3 we get also

$$\bar{v_3} = BHB^{-1}\bar{v}_4.$$

By corollary 8.3 we have

$$\|B_{i\theta\mu}(\eta(s), \eta(s+2d_\mu)) - E(l, \theta\mu([s, s+2d_\mu]))\| \leq 2c_1|\theta|d_\mu$$

so that

$$\|BHB^{-1} - BEB^{-1}\| \leq 2c_2|\theta|d_\mu$$

for a suitable constant c_2.

We now want to apply lemma 8.5 and need to check that suitable compactness conditions hold. First note that

$$\begin{aligned} &D_T((X_2, \bar{v}_2), (X_3, \bar{v}_3)) \\ &\qquad = D_T((BEB^{-1}X_4, BEB^{-1}\bar{v}_4), (BHB^{-1}X_4, BHB^{-1}\bar{v}_4)) \\ &\qquad\qquad = D_T((EB^{-1}X_4, EB^{-1}\bar{v}_4), (HB^{-1}X_4, HB^{-1}\bar{v}_4)). \end{aligned}$$

Let $J \in PSL(2, \mathbb{C})$ be an isometry which carries $\eta(s)$ to $O \in \mathbb{H}^3$. Clearly, $B^{-1}X_4$ lies in a compact neighborhood of $B^{-1}X_0 = \eta(s)$, and hence $JB^{-1}X_4$ lies in a compact neighborhood of $O \in \mathbb{H}^3$, whose diameter is bounded by a constant depending only on d_μ. Also JEJ^{-1} is a rotation about the line $J^{-1}(l)$ which lies near O, hence JEJ^{-1} and JHJ^{-1} lie in a compact neighborhood of the identity in $PSL(2, \mathbb{C})$. We can therefore apply lemma 8.5, where in the notation of the statement of the lemma, $JB^{-1}X_4$ is x and JEJ^{-1}, JHJ^{-1} are M, M', to obtain

$$D_T((JX_2, J\bar{v}_2), (JX_3, J\bar{v}_3)) < c_3|\theta|$$

for a constant c_3 depending only on d_μ.

Now, as in the proof of the cone lemma in the rational case, we may choose $\epsilon > 0$ so that $C_{\bar{v}_2}(\alpha, X_2) \subset C_{\bar{v}_0}(\alpha, X_0)$ whenever $|\theta| < \epsilon$. We want to adjust this choice to guarantee that, in addition,

$$C_{\bar{v}_3}(\alpha, X_3) \subset C_{\bar{v}_0}(\alpha, X_0).$$

Clearly this is equivalent to guaranteeing that

$$JC_{\bar{v}_3}(\alpha, X_3) \subset JC_{\bar{v}_0}(\alpha, X_0).$$

Recall that in the proof of the cone lemma we used the function $f(\psi, b)$, giving the exterior angle at C of a hyperbolic triangle ABC with one vertex B at ∞, side AC of hyperbolic length b and angle $\angle BAC = \psi$.

Since $f(\alpha, d_\mu) > \alpha$, the shadow of the cone $JC_{\bar{v}_2}(\alpha, X_2)$ on $\partial\mathbb{B}^3$ lies strictly inside the shadow of $JC_{\bar{v}_0}(\alpha, X_0)$, with bounds depending only on α and d_μ. (This is the point at which it is important that we have translated the cones to lie near O.) For fixed α, the center and radius of the shadow of a cone $JC_{\bar{v}}(\alpha, P)$ on $\partial\mathbb{B}^3$ are continuous functions of $(JP, J\bar{v})$ with respect to the metric D_T on $\mathrm{T}\mathbb{H}^3$. Combining this with the estimate $D_T((X_3, \bar{v}_3), (X_2, \bar{v}_2)) < c_3|\theta|$ above, we may adjust our choice of ϵ to find $\epsilon' > 0$, depending only on α and μ, such that $JC_{\bar{v}_3}(\alpha, X_3) \subset JC_{\bar{v}_0}(\alpha, X_0)$ whenever $|\theta| < \epsilon'$. This proves (1).

Finally we must prove (2). For $t \in [s, s + d_\mu]$, we want to show that $\eta_{i\theta\mu}(t) \in C_{\eta,\theta}(\alpha, s)$. We make an approximation to $\eta_{i\theta\mu}([s,t])$ by joining two geodesic segments as above, replacing $[s, s+2d_\mu]$ by $[s,t]$ so that the two approximating segments are joined at the image of the midpoint of $[s,t]$. Subject to this modification, we label points X_i and vectors $\bar{v}_i$ as above. In particular, $X_3 = \eta_{i\theta\mu}(t)$.

With this notation, we have to prove that X_3 lies inside the cone $C = C_{\bar{v}_0}(\alpha, X_0)$. Let X_5 be the foot of the perpendicular from X_3 to the axis of $C_{\bar{v}_0}(\alpha, X_0)$, let $l = \rho_{\mathbb{H}^3}(X_0, X_5)$ and let $f = \rho_{\mathbb{H}^3}(X_3, X_5)$. The hyperbolic distance h from X_5 to the boundary of C, is given by $\sinh h = \sin\alpha \sinh l$. We shall show that for sufficiently small θ, $\rho_{\mathbb{H}^3}(X_5, X_3) < h$, which will give the result.

Let $\psi = \angle X_3 X_0 X_5$. Then $\tanh f = \tan\psi \sinh l$ and as above, $\rho_{\mathbb{H}^3}(X_2, X_3) < c_3|\theta|$. The rotation angle $\angle X_2 X_1 X_5$, by construction equal to $|\theta|\mu([s,t])$, is less than $c_4|\theta|$ for some constant c_4. Now f is bounded above by $d_\mu + c_3|\theta|$ so that $\sinh f = \tanh f \cosh f \le \cosh(2d_\mu) \tanh f$ for small enough $|\theta|$. Thus since $|\psi| < \angle X_2 X_1 X_5 < c_4|\theta|$, we have $\sinh f \le c_6 \theta \sinh l$. Choosing $|\theta|$ sufficiently small in relation to the constants c_i and α gives the result. $\square$

8.4. General pleating loci on one side. We are now ready to discuss the extension of our results about the existence of quasifuchsian groups with a given pleating locus on one side of $\partial\mathcal{C}$ to the case of an arbitrary measured lamination $\mu \in PML$. The first step is to prove the generalization of theorem 6.1. We have:

THEOREM 8.8. *Let $G_0 \in \mathcal{F}$ and let $\mu \in PML$ be a projective measured lamination on $\mathbb{D}/G_0$. Let $Q_{\tau\mu}$ be the associated quakebend by τ along μ. Then there exists $\epsilon > 0$, depending on μ, such that if $|\tau| < \epsilon$ then $\mathbb{D}_{\tau\mu}$ is embedded and G_τ is quasifuchsian.*

PROOF. The only place in section 6 where we made use of the fact that μ is rational was in the proof of the cone lemma 6.3. Replacing this by the generalized cone lemma above proves the result. Note that we are using the uniformity of the constants d_μ, K_μ, c_1 for groups $G_{t\mu}$ with small t as in remark 8.4. $\square$

We now turn to the results of section 7.1. We first need a generalization of lemma 7.1.

LEMMA 8.9. *Let $G_0 \in \mathcal{F}$ and let $\mu \in PML$. Let $\tau \in \mathbb{C}$ and let $\mathbb{D}_{\tau\mu}$ be the developed surface associated to the quakebend $Q_{\tau\mu}$. Suppose that τ, ϵ satisfy the hypotheses of theorem 8.8. Then one of the two half spaces cut out by $\mathbb{D}_{\tau\mu}$ in $\mathbb{B}^3 \cup \hat{\mathbb{C}}$, is convex.*

PROOF. We modify the proof of lemma 7.1 using a suitable finite approximation to the quakebend $Q_{\tau\mu}$.

By exactly the same argument as before, the developed surface $\mathbb{D}_\tau$ inherits an orientation from $\mathbb{D}$ which we describe by the words "up" and "down'. We shall approximate the bending cocycles $B(\xi, \zeta), \xi, \zeta \in \mathbb{D}$ by finite products of rotations as described as described in detail in [**7**] chapter 3, see also section 8.1 above. The bending angles in all the finite approximations are in the same direction, which we shall again take to be "down". As before, let H be the halfspace, cut out by $\mathbb{D}_\tau$ in $\mathbb{B}^3$, and lying "below" $\mathbb{D}_\tau$. We claim that H is convex.

Following the proof of lemma 7.1, it will be sufficient to show that if $x, y \in \mathbb{D}_\tau$ then the $\mathbb{H}^3$ geodesic β joining x, y is contained in H.

Let ϕ_τ be the developing map associated to the quakebend Q_τ. Let $x', y' \in \mathbb{D}$ be the preimages under ϕ_τ of x, y, and let $\sigma \subset \mathbb{D}$ be the geodesic segment joining x' to y'. Then $\sigma_\tau = \phi_\tau(\sigma)$ is a path on $\mathbb{D}_\tau$ joining x to y.

We want to approximate σ_τ by a piecewise geodesic path $\tilde{\sigma}$ in $\mathbb{H}^3$, sufficiently closely that we can apply the same argument as in lemma 7.1 to show that if $\beta \not\subset H$, there is an impossible polygon formed by $\tilde{\sigma}$ and β.

To do this we approximate the bending cocycle $B(x', y')$, as in [**7**] 3.5.3, by a finite product of rotations $E_1 \ldots E_r$ about r lines $l_1, \ldots l_r \subset \mathbb{D}$ whose intersections $P_1, \ldots P_r$ with σ meet σ in order going from x' to y' and which form a partition $\mathcal{P}$ of σ into intervals of small length.

The product $E_1 \ldots E_r$ induces a finite quakebend in a neighborhood of the line segment σ, carrying it into a broken geodesic segment $\tilde{\sigma} \subset \mathbb{H}^3$. We denote the associated developing map by ψ.

Let $B = B(O, x')$ and for $z \in \sigma_\tau$ write $z' = \phi_\tau^{-1}(z) \in \sigma$. Note that $z = BB(x', z')(z')$. Let $P_1, \ldots, P_k$ be the points of the partition $\mathcal{P}$ meeting σ between x' and z'. Then $\psi(z') = BE_1 \ldots E_k(z')$ approximates $z = \phi_\tau(z') = BB(x', z')(z')$. More precisely, if $\bar{v}$ is the unit tangent vector to σ at z', and if $\psi\bar{v}, \phi_\tau\bar{v}$ are its images uder the induced map into $\mathrm{T}\mathbb{H}^3$, then by lemma 8.5,

$$
\begin{aligned}
&D_T((\psi(z'), \psi\bar{v}), (\phi_\tau(z'), \phi_\tau\bar{v})) \\
&\qquad\qquad < c\|E_1 \ldots E_k - B(x', z')\| \leq c'\|E_1 \ldots E_r - B(x', y')\|
\end{aligned}
$$

for suitable constants c, c', where for the last inequality we use the method of lemma 3.4.2 of [**7**] .

By choosing the approximating finite quakebend sufficiently close to Q_τ, which we may do, as in [**7**] 3.5.3, by making the size of the partition $\mathcal{P}$ sufficiently fine, we make $\|E_1 \ldots E_r - B(x', y')\|$ arbitrarily small; and hence complete the proof as described above. ☐

We now have the following direct generalization of proposition 7.2.

PROPOSITION 8.10. *Let $G_0 \in \mathcal{F}$ and let $\mu \in PML$. Let $\tau \in \mathbb{C}$ and let $\mathbb{D}_{\tau\mu}$ be the developed surface associated to the quakebend $Q_{\tau\mu}$. Suppose that τ, ϵ satisfy the hypotheses of theorem 8.8. Then $\mathbb{D}_{\tau\mu}$ is one of the components of $\partial\mathcal{C}(G_\tau)$.*

PROOF. Besides the above generalization of lemma 7.1, the only place in the proof of proposition 7.2 where we use the rationality of μ is to conclude that the endpoints of the bending lines of D_τ lie in Λ_τ. In the general case this follows from corollary 6.9. The result follows. ☐

8.5. Proof of theorem 3.1. We are finally able to complete the proof of our main theorem 3.1.

PROOF. We first consider the case in which the lamination μ is arbitrary but ν is rational, represented by a simple closed curve γ' on $\mathcal{T}_1$. We assume $\mu \neq \nu$ and show that there is a group $G \in \mathcal{QF}$ such that $[\mathrm{pl}^+(G)] = \mu, [\mathrm{pl}^-(G)] = \nu$.

We first note that Kerckhoff's theorem, theorem 7.7, holds for earthquakes along arbitrary laminations; thus we may replace the simple closed curve γ in corollary 7.8 by the measured lamination μ and conclude that the length $l_{\gamma'}$ of the geodesic corresponding to the lamination ν is a convex function of the earthquake parameter along μ. This function still has a unique minimum; further by results in [**7**] lemma 3.8.1 it extends to a complex analytic function of the earthquake parameter τ. Thus if W is any element in G_0 corresponding to γ', we argue as before that we can quakebend on μ along a path through the minimum point $t_{\gamma'}$ of $l_{\gamma'}$ in such a way that the resulting group G_τ is quasifuchsian, $[pl^+(G)] = \mu$ and $\mathrm{Tr}\, W$ remains real. Then again by lemma 7.6, $[pl^-(G)] = \nu$.

Finally, to finish the proof of theorem 3.1 in the general case, we have only to show that we may replace the rational lamination ν by an arbitrary lamination $\nu \in PML$, $\mu \neq \nu$. To do this we appeal to the continuity of the pleating laminations proved in [**13**].

It is a well known fact (see e.g.[**4**]) that there is a natural identification $j : PML(\mathcal{T}_1) \to \mathbb{R} \cup \{\infty\}$ in which $PML_0(\mathcal{T}_1)$ is identified with $\mathbb{Q} \cup \{\infty\}$. Using this identification, we can find rational laminations ν_1, ν_2, corresponding to elements $W_1, W_2 \in G_0$, such that $j(\nu_1) \leq j(\nu) \leq j(\nu_2)$, and since $\mu \neq \nu$ we may further suppose that $\mu \notin [j(\nu_1), j(\nu_2)]$. Let τ be the quakebend parameter on μ measured from some base group $G_0 \in \mathcal{F}$ and let t_1, t_2 be the minimum points for the length functions l_{W_1}, l_{W_2} respectively. Let $\xi_i : [0, \epsilon) \to \mathbb{C}$ be paths as in corollary 7.8, with $\xi_i(0) = t_i$, $\Im \xi_i(s) > 0$ for $s > 0$ and such that $\mathrm{Tr}\, W_i(\xi_i(s)) \in \mathbb{R}$, $|\,\mathrm{Tr}\, W_i(\xi_i(s))| > 2$, for $s \in [0, \epsilon)$.

Using theorem 8.8, it follows by compactness that there exists a neighborhood $\mathcal{W}$ of $[t_1, t_2]$ in $\mathbb{C}$ containing the paths $\xi_i([0, \epsilon))$, and such that such that $G_\tau \in \mathcal{QF}$ and $[pl^+(G_\tau)] = \mu$ for $\tau \in \mathcal{W}$. Let $\beta : [0, 1] \to \mathcal{W} - \mathbb{R}$ be a path with initial point in ξ_1 and endpoint in ξ_2. The map $\sigma : [0, 1] \to PML(\mathcal{T}_1)$; $s \mapsto [pl^-(G_{\beta(s)})]$, defines a path in PML from ν_1 to ν_2. Since $[pl^+(G_\tau)] = \mu$ for $\tau \in \mathcal{W}$, the image of σ is contained in $PML - \{\mu\}$. By Theorem 4.6 of [**13**], the map $\mathcal{QF} \to PML \times PML$, defined by $G \mapsto [(pl^+(G)], [pl^-(G)])$ is continuous. Using the identification of PML with $\mathbb{R} \cup \{\infty\}$, we see that $PML - \{\mu\}$ is an interval and it follows that there exists $t_0 \in [t_1, t_2]$ such that $[pl^-(G_{t_0 + i\delta})] = \nu$. This completes the proof of theorem 3.1 in all cases. $\square$

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