Contemporary Mathematics

Parabolic Perturbation of the family $\lambda \tan z$

Linda Keen and Shenglan Yuan

ABSTRACT. In this paper we study parabolic bifurcation in holomorphic dynamics and apply our result to the family $\lambda \tan z$ to complete the proof that there are infinite cascades of non-standard period doubling bifurcations along the imaginary axis.

1. Introduction

In conformal dynamics one is interested in describing the equivalence classes of topologically conjugate functions in natural one parameter families such as $\{P_\lambda = z^2 + \lambda, \lambda \in \mathbb{C}\}$. In this family, examples of such classes are the components of the interior and exterior of the Mandelbrot set where the dynamics are characterized by the combinatorics of the super attracting periodic cycle of the function at the center of the component. These components, are called the hyperbolic components. These components come in pairs: the pair shares a common boundary point at which the orbits of the asymptotic values land on a pole. The boundary of a hyperbolic component also contains many other points that are shared with another component. At such a boundary point there are one or two parabolic periodic cycles; in one of the components at this point, called the root, the parabolic cycle becomes attracting and in the other, called the bud, it becomes repelling and another attracting cycle of a higher period develops. There are other special boundary points called cusps. At these there are parabolic cycles, but they are not boundary points of any other component.

Another natural one parameter family of transcendental meromorphic functions is the family $\{T_\lambda = \lambda \tan z, \lambda \in \mathbb{C} \setminus \{0\}\}$. The dynamical properties of this family were first studied by Devaney and Keen in [**3, 4**] and pursued by Keen and Kotus in [**7, 8**] whose work was motivated by the computer pictures of the parameter plane created by W. Jiang in his unpublished thesis [**6**]. These pictures indicate that pairs of hyperbolic components share a common boundary point with a parabolic cycle. In [**6**], bud components were observed at points of rational argument on the unit circle.

On the imaginary axis a new period doubling phenomenon called non-standard bifurcation was observed: instead of a single parabolic cycle bifurcating into an

2000 *Mathematics Subject Classification.* 30D05, 58F08.

Key words and phrases. Iteration, meromorphic functions, hyperbolic components, parabolic bifurcation, tangent family.

Paritally supported by a PSC-CUNY Grant.

attracting cycle of double the period, one saw a parabolic cycle bifurcating into two distinct attracting cycles of the same period. The question of whether a parabolic cycle that is not at a cusp always has either a standard or non-standard bifurcation was proved only under the assumption that the multiplier of the cycle is a monotone function of the parameter as it passes through the parabolic point.

In this paper we prove this assumption is unnecessary; we give an independent proof of the existence of the bifurcation. As a corollary, we deduce that for λ on the imaginary axis there are infinite cascades of period doubling analogous to those for quadratic polynomials on the real axis. This also completes the discussion in [**8**] of a Sharkovskii-like ordering for the hyperbolic components intersecting the imaginary axis. In particular, the inequalities in [**8**] on the number of components of given period become equalities and we obtain a complete description of the deployment of the hyperbolic components on the imaginary axis.

The main tool we use is the local normal form for a function in the neighborhood of a parabolic fixed point of $(\lambda \tan z)^{\circ p}$ (see [**1, 5, 9**]). This gives us bifurcations at rational points on the unit circle, theorem 1. The arguments for the imaginary axis are more delicate. To prove the main result, theorem 4, we first prove a local theorem, theorem 3 to obtain the properties of a perturbation of the form $(1+\delta)\lambda \tan z$. Then we apply this theorem to the two types of root components, those with two attracting cycles of period $2n$ and those with a single attracting cycle of period $4n$, $n > 1$, to obtain standard and non-standard bifurcations respectively.

The paper is organized as follows. In section 2, we give the basic results about perturbation near a parabolic fixed point z_0. In section 3 we discuss the basic facts we need about the dynamics of the tangent family. In section 4, we study the bifurcations on the unit circle. In section 5, we study the bifurcation along the imaginary axis and prove theorems 3 and 4. In section 6, we apply our results: we prove that there exists an infinite cascade of non-standard period doubling bifurcations along the imaginary axis; we also obtain the full description of the ordering of components on the imaginary axis.

2. Parabolic Points

In holomorphic dynamics, a cycle $\langle z\rangle = \{z, f(z), f^2(z), \ldots, f^{n-1}(z), f^n(z) = z\}$, of period n, of a holomorphic or meromorphic function f has a multiplier $\rho = (f^n)'(z)$. This multiplier is independent of the point of the cycle. The cycle is called *attracting* if $|\rho| < 1$; it is *indifferent* if $|\rho| = 1$, and it is *repelling* if $|\rho| > 1$. An indifferent cycle is *rationally indifferent* if ρ is a root of unity, otherwise it is *irrationally indifferent.* A rationally indifferent cycle is called *parabolic* if f^i is not the identity map for any i.

We recall some standard facts about parabolic periodic points. Suppose that in some neighborhood of the parabolic fixed point z_0, which we assume to be the origin, f is given by a power series of the form

$$f(z) = z - z^{m+1} + \mathrm{O}(\mathrm{z}^{\mathrm{m}+2})$$

with $m \geq 2$. Then, [**1, 2, 9**], f is locally holomorphically conjugate to a map of the form

$$\tilde{f} = z - z^{m+1} + bz^{2m+1} + \mathrm{O}(\mathrm{z}^{2\mathrm{m}+2})$$

If z_0 is a periodic parabolic fixed point with period p and multiplier ρ, then in a neighborhood of z_0, f can be expressed as

$$f(z) = \rho(z - z^{m+1} + z^{2m+1}) + O(z^{2m+2})$$

The dynamics in a neighborhood of z_0 are described in the Leau Flower theorem, (see for example, [**2, 9**]) which says that there are m disjoint simply connected domains, called *attracting petals*, with common boundary point z_0, that fall into $k = \frac{m}{p}$ distinct forward invariant cycles under f.

The following lemmas give the basic local picture of perturbations of parabolic points.

LEMMA 1. *Assume that α is a real number and that $f(z)$ has normal form $f(z) = z - z^{m+1} + \alpha z^{2m+1} + \mathrm{O}(z^{2m+2})$ in a neighborhood of $z_0 = 0$. For any ϵ such that $\frac{1}{4|\alpha|} > \epsilon > 0$, the perturbation of $f(z)$ defined by $f_{1+\epsilon}(z) = (1+\epsilon)\cdot f(z)$ has a repelling fixed point at z_0 and m attracting fixed points near z_0.*

PROOF. Clearly $f(z)$ has a fixed point at z_0 with multiplicity $m+1$ and simple fixed points close to the m^{th} roots of $\frac{1}{\alpha}$. It is also obvious that $f_{1+\epsilon}(z)$ has a simple fixed point at z_0. Let x_i denote the m^{th} roots of $\frac{\epsilon}{1+\epsilon}$ where $i = 1, 2, \ldots, m$. Set

$$F(z) = \epsilon - (1+\epsilon)z^m + (1+\epsilon)\alpha z^{2m}$$

Then when $\alpha > 0$, we have $F(x_i) > 0$ and when $\alpha < 0$ we have $F(x_i) < 0$. Let y_i denote the m^{th} roots of $\frac{2\epsilon}{1+\epsilon}$ where $i = 1, 2, \ldots, m$ so that $F(y_i) = -\epsilon + \alpha\frac{4\epsilon^2}{(1+\epsilon)}$. Either $\alpha > 0$ or $\alpha < 0$ and, by hypothesis $\epsilon < \frac{1}{4\alpha}$; in either case we see that $F(y_i) < 0$. Since $F(0) > 0$, we deduce that $F(z)$ has m simple zeros inside the disk with center z_0 and radius $\sqrt[m]{\frac{2\epsilon}{1+\epsilon}}$. Let z_i denote the m zeros of $F(z)$ inside the disk which are the simple fixed points of $f_{1+\epsilon}(z)$. Then,

$$f'_{1+\epsilon}(z_i) = (1+\epsilon)(1-(m+1)z_i^m + (2m+1)\alpha z_i^{2m}) = (1+\epsilon)(1 - \frac{(m+1)\epsilon}{1+\epsilon} + m\alpha z_i^{2m})$$

and $z_i^{2m} < (\frac{2\epsilon}{1+\epsilon})^2$; therefore $f'_{1+\epsilon}(z_i) < 1$. □

LEMMA 2. *Assume again that for some real α, $f(z)$ has the normal form $f(z) = z - z^{m+1} + \alpha z^{2m+1} + \mathrm{O}(z^{2m+2})$ in a neighborhood of $z_0 = 0$. If $\alpha > 0$, choose any ϵ such that $1 > \epsilon > 0$, whereas if $\alpha < 0$ choose any ϵ such that $0 < \epsilon < \frac{1}{1-4\alpha}$. Then the perturbation of $f(z)$ given by $f_{1-\epsilon}(z) = (1-\epsilon)\cdot f(z)$ has an attracting fixed point at z_0 and m repelling fixed points near z_0.*

PROOF. Clearly $f(z)$ has a fixed point at z_0 with multiplicity $m+1$ and a simple fixed point close to each of the m^{th} roots of $\frac{1}{\alpha}$. Also, $f_{1-\epsilon}(z)$ has a simple fixed point at z_0. Let x_i denote the m^{th} roots of

$$\frac{1 - \epsilon - \sqrt{(1-\epsilon)^2 + 4\epsilon(1-\epsilon)\alpha}}{2(1-\epsilon)\alpha}$$

where $i = 1, 2, \ldots, m$. Then

$$f'_{1-\epsilon}(x_i) = 1 + m\epsilon + (1-\epsilon)m\alpha x_i^{2m}$$

Now when $\alpha > 0$, $1 - \epsilon > 0$ so that x_i^m is real and $(1-\epsilon)m\alpha x_i^{2m} > 0$. Therefore $f'_{1-\epsilon}(x_i) > 1$. On the other hand, when $\alpha < 0$, $\epsilon < \frac{1}{1-4\alpha}$, so that

$\sqrt{1-\epsilon}+\sqrt{1-\epsilon+4\epsilon\alpha}>\sqrt{1-\epsilon}$. Computing the multiplier at x_i we have

$$f'_{1-\epsilon}(x_i)=1+m\epsilon+(1-\epsilon)m\alpha\frac{2(1-\epsilon)^2+4\epsilon(1-\epsilon)\alpha-2(1-\epsilon)\sqrt{(1-\epsilon)^2+4\epsilon(1-\epsilon)\alpha}}{4(1-\epsilon)^2\alpha^2}$$

$$=1+2m\epsilon+m\frac{-2\epsilon\sqrt{1-\epsilon}}{(\sqrt{1-\epsilon}+\sqrt{1-\epsilon+4\epsilon\alpha})}$$

so that again $f'_{1-\epsilon}(x_i)>1$. □

Lemma 3. *Assume that $f(z)$is given in normal form near a periodic parabolic point $z_0=0$ with period p and multiplier ρ, a primitive $p-th$ root of unity so that for some real α,*

$$f(z)=\rho(z-z^{m+1}+\alpha z^{2m+1})$$

For any sufficiently small $\epsilon>0$ satisfying $|1-m\epsilon|<1$ and $\epsilon<\frac{1}{4}\tilde{\tilde{\alpha}}$, where $\tilde{\tilde{\alpha}}$ depends on m,p and α, there exists a perturbation $f_{1+\epsilon}(z)=(1+\epsilon)\cdot f(z)$ of $f(z)$, such that $f_{1+\epsilon}(z)$ has a repelling fixed point at z_0 and $k=\frac{m}{p}$ attracting periodic points of period p near z_0.

Proof. There exists a holomorphic conjugation h such that $g=h^{-1}fh=z-z^{m+1}+\alpha z^{2m+1}+\mathrm{O}(\mathrm{z}^{2m+2})$ and $g^p=h^{-1}f^p(z)h=z-pz^{m+1}+\tilde{\alpha}z^{2m+1}+\mathrm{O}(\mathrm{z}^{2m+2})$. We see easily that $z_0=0$ is a parabolic fixed point of $f^p(z)$ with multiplicity $m+1$ and there are simple fixed points close to the m^{th} roots of $\frac{\tilde{\alpha}}{p}$.

Let us look at the perturbation $\tilde{g}=(1+\epsilon)^{\frac{1}{p}}\cdot g$ where we take the positive real p^{th} root. Set $\tilde{f}=h\tilde{g}h^{-1}$ We compute that:

$$\tilde{g}^p=(1+\epsilon)z-((1+\epsilon)+(1+\epsilon)^{\frac{m}{p}+1}+\ldots+(1+\epsilon)^{m-\frac{m}{p}+1})z^{m+1}+\tilde{\tilde{\alpha}}z^{2m+1}+\mathrm{O}(\mathrm{z}^{2m+2})$$

where $\tilde{\tilde{\alpha}}$ is real. Since $(\tilde{g}^p)'(0)=(h^{-1}\tilde{f}^ph)'(0)=1+\epsilon$, $z_0=0$ is a simple repelling fixed point of $\tilde{f}^p$. Arguing as we did in lemma 1, we set

$$G(z)=\epsilon-((1+\epsilon)+(1+\epsilon)^{\frac{m}{p}+1}+\ldots+(1+\epsilon)^{m-\frac{m}{p}+1})z^m+\tilde{\tilde{\alpha}}z^{2m}+\mathrm{O}(\mathrm{z}^{2m+1})$$

We let x_i, $i=1,\ldots,m$ be the m^{th} roots of

$$\frac{\epsilon}{(1+\epsilon)+(1+\epsilon)^{k+1}+\ldots+(1+\epsilon)^{(k-1)p+1}}$$

and y_i, $i=1,\ldots m$ be the m^{th} roots of

$$\frac{2\epsilon}{(1+\epsilon)+(1+\epsilon)^{k+1}+\ldots+(1+\epsilon)^{(k-1)p+1}}$$

where as above, $k=\frac{m}{p}$. We compute that for ϵ satisfying $\epsilon<\frac{1}{4}\tilde{\tilde{\alpha}}$, $G(x_i)$ and $G(y_i)$ have opposite signs and we conclude that there are m simple roots of $G(z)$ in a disk centered at z_0 with radius depending on ϵ.

These roots closely approximate the roots z_i, $i=1,\ldots,m$ of $\tilde{f}^p$ that lie in a perhaps slightly bigger disk. We compute $(\tilde{f}^p)'(z_i)=1-m\epsilon+\mathrm{O}(\epsilon^2)$. If $|1-m\epsilon|<1$, the $z_i, i=1,\ldots,m$ are attracting periodic points. Thus f bifurcates into a function with one simple repelling fixed point at z_0 and m attracting periodic points that split into k cycles of period p. □

3. Basic facts about the Tangent Family

Let us recall some definitions and basic facts about the dynamics of meromorphic functions in general and the tangent family in particular.

For a complex analytic function f, the Julia set carries the interesting dynamical information and is defined by any of the following equivalent conditions:

1. The Julia set $J(f)$ is the complement of the set on which the iterates of f form a normal family in the sense of Montel; its complement is called the Fatou set.

2. $J(f)$ is the closure of the set of repelling periodic points of f.

3. In addition, if f is meromorphic, there are points whose forward orbit lands on a pole and are thus preimages of ∞. Such points are called prepoles and $J(f)$ is the closure of the set of prepoles of f.

It is well known that the parameter plane of the quadratic family $P_c(z) = z^2 + c$ is decomposed into the Mandelbrot set $M = \{c \in \mathbb{C} | J(P_c) \text{ is connected}\}$ and its complement $\mathbb{C} \setminus M$ where $J(P_c)$ is a Cantor set. Let

$$M' = \{c \in C | P_c \text{ has a finite attracting periodic cycle}\}$$

The *hyperbolic set* is the union of M' and $\mathbb{C} \setminus M$.

The orbits of critical points and asymptotic values play an important role in the study of Julia sets. The quadratic polynomials P_c have two critical points: one at infinity, which is fixed, and one at the origin. For $P_c \in \mathbb{C} \setminus M$, the critical point at zero is attracted to infinity and the Julia set is a Cantor set. Thus, the Mandelbrot set is the same as the set of c for which the orbit of the critical point zero remains bounded. In the connected components of M' the dynamics are determined by the combinatorics of the orbit of the critical point at zero.

For the tangent family $T_\lambda(z) = \lambda \tan z$, there are no critical points; there are two asymptotic values, $\pm\lambda i$ and the dynamics are controlled by the orbits of these values. Recall that these values are omitted but that

$$\lim_{y \to \pm\infty} T_\lambda(x + iy) = \pm\lambda i$$

Since any attracting or parabolic cycle must attract the orbit of an asymptotic value, [**4**], there are either two distinct cycles, each attracting one asymptotic value, or a single cycle attracting both. Note that the symmetries in this family

$$T_\lambda(-z) = -T_\lambda(z) \text{ and } T_{-\lambda}(z) = -T_\lambda(z)$$

$$T'_\lambda(-z) = T'_\lambda(z) \text{ but } T_{-\lambda}(z) = -T'_\lambda(z)$$

imply that if there are two cycles they are symmetric and both are attracting or parabolic with the same multiplier, if there is just one attracting or parabolic cycle, the points of the cycle are symmetric in pairs.

The *hyperbolic set* for this family is defined as

$$\mathcal{H} = \{\lambda \in \mathbf{C} | T_\lambda \text{ has a finite attracting periodic cycle}\}$$

In analogy with quadratic polynomials, the connected components of $\mathcal{H}$ can be characterized by the combinatorics of the orbits of the asymptotic values (see [**7**]).

The unit disk minus the origin is the only hyperbolic component of $\mathcal{H}$ for which $\lambda \tan z$ has only one attracting fixed point at 0. The multiplier at 0 is $T'_\lambda(0) = \lambda \sec^2(0) = \lambda$. Both asymptotic values are attracted to 0 and the connected component of the attracting basin of 0 is the full Fatou set; the Julia set is a

Cantor set, [**3**]. The set $\{0 < |\lambda| < 1\}$ is thus analogous to the complement of the Mandelbrot set.

In [**6, 7, 8**] the deployment of the remaining components of $\mathcal{H}$ are studied. We summarize the results we need from those papers here. The results in [**7, 8**] confirm the computer pictures of W. Jiang in [**6**].

Because there are no critical points, there are no superattractive cycles; thus the components have no centers. Instead, they occur naturally in pairs (Ω_p, Ω_p') where in Ω_p there are two distinct attractive cycles of period p and in Ω_p' there is one attractive cycle of period $2p$. By symmetry, the two cycles in Ω_p have the same multiplier. These paired components share a boundary point called *the virtual center*. At a virtual center, the asymptotic values are prepoles; that is $T_\lambda^p(\pm\lambda i) = \infty$. The asymptotic values may be thought of as images of infinity (along paths whose imaginary parts tend to infinity). In this sense, the asymptotic values belong to a *virtual cycle*. The multiplier of this cycle is the limit of the multipliers of the attractive cycles in the hyperbolic component as the parameter approaches the virtual center, and it is zero — hence the name virtual center.

In Ω_p the attracting periodic points of the cycle are holomorphic functions of λ. As λ passes from Ω_p through the center into the paired component Ω_p', in each cycle, one of the periodic points passes through a pole and its image changes sign. The two cycles thus intertwine and become a single cycle of double the period.

In [**8**] it is proved that for each hyperbolic component Ω there is a holomorphic conjugation $\phi : \Omega \to H$ to the upper half plane H such that if $M(\lambda)$ is the multiplier of the attractive cycle(s) then $M = \exp(i\phi)$. It follows that along $\phi^{-1}(n\pi + iy)$ the multipliers are real and at the boundary points $\lim_{y\to 0} \phi^{-1}(n\pi + iy)$ there are parabolic cycles.

4. Bifurcations on the Unit Circle

The computer pictures of the λ-plane by W. Jiang show the component pairs bud off the unit circle along the endpoints of rational internal rays where the argument of λ is a root of unity. We prove here that the bifurcations at these endpoints actually occur as shown in the pictures.

If λ is real, it is easy to see, [**3**], that for $\lambda > 1$, $\lambda \tan z$ has a repelling fixed point at zero and two attracting fixed points $\pm ti$, t real, while if $\lambda < -1$, zero is a repelling fixed point and the points $\pm ti$ form an attracting cycle of period two.

First let us state a lemma that we need.

LEMMA 4. *Let $T_\lambda(z) = \lambda \tan z$, where λ is a primitive n-th root of unity, $n > 2$. Then in a neighborhood of zero we can write*

$$T_\lambda^n = z + a_n z^{2n+1} + \mathrm{O}(\mathrm{z}^{2n+2})$$

when n is an odd number and

$$T_\lambda^n = z + a_n z^{n+1} + \mathrm{O}(\mathrm{z}^{n+2})$$

when n is an even number.

PROOF. We know that, since the tangent is an odd function, in a neighborhood of zero,

$$T_\lambda(z) = \lambda(z + a_1 z^3 + a_2 z^5 + a_3 z^7 + \ldots + a_n z^{2n+1} + \ldots)$$

When n is even,

$$T_\lambda^n(z) = \lambda^n z + (\lambda^2 + \lambda^4 + \ldots + \lambda^{2n}) a_1 z^3 + \mathrm{O}(\mathrm{z}^4).$$

The multiplier at the fixed point 0 is $T'_\lambda(0) = \lambda \sec^2(0) = \lambda$. Set $n = 2k$ We have the identity

$$(\lambda^2)^k - 1 = (\lambda^2 - 1)((\lambda^2)^{k-1} + (\lambda^2)^{k-2} + \ldots + 1) = 0,$$

and since $\lambda \neq \pm 1$

$$(\lambda^2)^{k-1} + \ldots + 1 = 0$$

Thus $\lambda^2 + \lambda^4 + \ldots + \lambda^{2n} = 0$.

By a similar calculation we can show that, when $i < n/2$, a_i has a factor of $1 + \lambda^2 + \lambda^4 + \ldots + \lambda^{n-2}$; we conclude $a_i = 0$ for all $i < \frac{n}{2}$.

When n is odd, we again have a similar calculation, and we can show that a_i has a factor of $1 + \lambda + \lambda^2 + \ldots + \lambda^{n-1}$ and this must also be 0. Thus $a_i = 0$ when $i < n$. $\square$

PROPOSITION 1. *Let $T_\lambda = \lambda \tan z$ and suppose $\lambda = i$. If $\epsilon < 0$, the perturbation of T_λ given by $T_{\lambda(1+\epsilon)^{\frac{1}{4}}}$ has an attracting fixed point at 0 and repelling periodic points of period 4. If $\epsilon > 0$, the perturbation $T_{\lambda(1+\epsilon)}$ has a single repelling fixed point at 0 and an attracting periodic cycle of period 4.*

PROOF. By lemma 4, we have $T_i(z) = z + a_5 z^5 + \mathrm{O}(z^6)$; that is, $T_i^4(z)$ has a parabolic fixed point at zero with multiplicity 5.

For $\epsilon < 0$, applying lemmas 2 and 3, we deduce that $T^4_{\lambda(1+\epsilon)^{\frac{1}{4}}}$ has a single attractive fixed point at 0 and a repelling point of period 4 near 0.

If $\epsilon > 0$ we apply lemmas 1 and 3 to deduce that $T_i(z)$ bifurcates into a repelling fixed point at zero and an attracting periodic cycle with period 4. $\square$

In general, we have the following:

THEOREM 1. *Let $T_\lambda = \lambda \tan z$ and $\lambda = e^{2\pi i \frac{p}{q}}$, where $(p, q) = 1$. If $\epsilon < 0$ the perturbation $T_{\lambda(1+\epsilon)^{\frac{1}{q}}}$ has an attracting fixed point at 0 and a repelling periodic point of period q and if $\epsilon > 0$, $T_{\lambda(1+\epsilon)^{\frac{1}{q}}}$ has a repelling fixed point at 0 and an attracting periodic point of period q. That is, the unit disk is a root component with bud components at all the points on the circle with rational argument.*

PROOF. First apply lemma 4 to conclude that $T_\lambda^q(z) = z + a_0 z^{2q+1} + \mathrm{O}(z^{2q+2})$ and hence that T_λ^q has a parabolic fixed point at zero with multiplicity $2q + 1$. Now applying lemmas 1 and 3, we deduce that for $\epsilon > 0$, the perturbed function $(1 + \epsilon)T_\lambda$ has a repelling fixed point at zero and an attracting periodic cycle near zero with period q. $\square$

5. Bifurcations along the imaginary axis

For the family T_λ the imaginary axis $\Im$ in the parameter plane plays an important role which is analogous to the role the real axis plays for the quadratic family. Here we summarize the results we need from [KK2].

PROPOSITION 2. *If $\lambda \in \Im$ and $z \in \mathbf{R}$, then for every $n \in \mathbf{N}$, we either have $T_\lambda^{2n}(z) \in \mathbf{R}$, and $T_\lambda^{2n+1} \in \Im$, or $T_\lambda^{2n+1}(z) = \infty$.*

PROPOSITION 3. *If $\lambda \in \Im$ belongs to a hyperbolic component, then the period of the attracting cycle (or cycles) is even.*

PROPOSITION 4. *If $\lambda \in \Im$, then the multiplier $M(\lambda)$ of any parabolic or attracting periodic cycle of period $2n$ is always real and*

$$sgn(M(\lambda)) = sgn(\lambda^{2n}) = (-1)^n \tag{1}$$

A direct corollary of equation (1) is

PROPOSITION 5. *Let $\lambda \in \Im$ belong to a hyperbolic component. If there is a single cycle of period $4n$, then $0 < M(\lambda) < 1$, while if there are two cycles of period $2n$, then either n is odd and $-1 < M(\lambda) < 0$, or n is even and $0 < M(\lambda) < 1$.*

In the components Ω'_{2n} with a single cycle of period $4n$, note that we can choose a particular square root of the multiplier as follows. We set

$$m(\lambda) = \frac{df_\lambda^{2n}}{dz}(z_i(\lambda)).$$

Another immediate corollary of equation (1) is

COROLLARY 2. *If $\lambda \in \Omega'_{2n} \cap \Im$ then* $\operatorname{sgn} m(\lambda) = (-1)^n$.

We call $m(\lambda)$ the *half multiplier*; note that it is well defined independent of i.

PROPOSITION 6. *Suppose a hyperbolic component Ω intersects the imaginary axis $\Im$. Then it has a unique virtual center on the imaginary axis, and $\Omega \cap \Im$ is an interval whose endpoints are the virtual center λ_0 and a parameter λ_1 such that*

$$\lim_{\lambda \to \lambda_1} M(\lambda) = \pm 1,$$

where the limit is taken inside Ω.

We call λ_1 the *real parabolic boundary point*

We have the following proposition (corollary 8.5 of [**8**]) that characterizes attracting petals at the real parabolic boundary points.

PROPOSITION 7.

i) *Suppose λ_1 is the real parabolic boundary point of a component Ω_{2n} intersecting the imaginary axis. If $M(\lambda_1) = 1$, then there is exactly one attracting petal at each periodic point in the parabolic cycle, whereas if $M(\lambda_1) = -1$, then there are two attracting petals at each periodic point in the parabolic cycle.*
ii) *Suppose λ_1 is the real parabolic boundary point of a component Ω'_{2n} intersecting the imaginary axis. If $m(\lambda_1) = 1$, then there are two attracting petals at each periodic point in the parabolic cycle, whereas if $m(\lambda_1) = -1$, then there is only one attracting petal at each periodic point in the parabolic cycle.*

We now turn our attention to the local behavior in the neighborhood of the real parabolic parameter λ_1 on the imaginary axis when there are two attracting petals. In theorem 8.8 of [**8**], it is proved that standard and non-standard bifurcations occur at such points under the assumption that the multiplier is monotonic there. The following theorem shows that such an assumption is unnecessary.

THEOREM 3. *Suppose λ_1 is a real parabolic boundary point of a hyperbolic component intersecting the imaginary axis and suppose further that*

there are two invariant attracting petals at each point of the single parabolic cycle of period $2n$. Then for small $\epsilon > 0$, the perturbed function $((1+\epsilon)^{\frac{1}{2n}}\lambda_1 \tan z)^{2n}$

has a repelling fixed point and two attracting fixed points, close to, and on opposite sides of it. In addition, for $\epsilon < 0$, the perturbed function $((1+\epsilon)^{\frac{1}{2n}}\lambda_1 \tan z)^{2n}$ has a single attracting cycle of period $2n$.

Remark 1. *We have made the assumption that there is a single periodic cycle to avoid dealing with two cases in the proof. If there are two periodic cycles of period n and two attracting petals, the n^{th} iterate also fixes the periodic point, but interchanges the attracting petals. The $2n^{th}$ iterate, however, maps each attracting petal into itself as in the other case. The argument below will apply to this doubled iterate. When $\epsilon > 0$ it will show that there are two new attracting fixed points of the $2n^{th}$ iterate. Since in this case the multiplier of the parabolic cycle is negative, the multiplier of the attracting cycle(s) is negative. It follows that both new points belong to the same attracting cycle; this is a standard bifurcation.*

Proof. We need only consider local behavior in the neighborhood of a periodic point. The parameter λ_1 is fixed in the discussion so we simplify our notation and set $T(z) = \lambda_1 \tan z$. Let $(z_0, z_1, \ldots z_{2n-1})$ be the periodic cycle. Because λ_1 is on the imaginary axis, the points in the cycle alternate between the real and imaginary axes; that is, if we assume z_0 is real, then for k even, $z_k \in \Re$ and $z_{k+1} \in \Im$.

Now let $h_k(z) = z - z_k$ and note that $h_{2n} = h_0$. Define

$$S_k(\zeta) = h_{k+1} \circ T \circ h_k^{-1} = T(\zeta + z_k) - z_{k+1}$$

Each S_k has 0 as a fixed point.

Set

$$\mathbb{S} = S_{2n-1} \circ S_{2n-2} \circ \ldots S_0 = h_0 T^{2n} h_0^{-1}$$

Because there are two attracting petals at each fixed point of T^{2n} the same is true for $\mathbb{S}$. Introducing another conjugation if necessary, we can expand $\mathbb{S}$ about 0 as

$$\mathbb{S}(\zeta) = \zeta - \zeta^3 + \alpha\zeta^5 + \mathrm{O}(\zeta^6) \tag{2}$$

Consider the derivatives, $S_k'(0) = T'(z_k) = \mu_k$. We know that

$$\Pi_0^{2n-1}\mu_k = 1 \tag{3}$$

but each $\mu_k \neq 1$. In fact, since the multiplier is independent of the point in the cycle,

$$\Pi_j^{2n-1+j \bmod 2n}\mu_k = (-1)^{2n} = 1$$

Also although $\mathbb{S}''(0) = 0$, the second derivatives $S_k''(0)$ are not 0. We have the local expansion

$$S_k(\zeta) = \mu_k\zeta + \upsilon_k\zeta^2 + \rho_k\zeta^3 + \mathrm{O}(\zeta^4)$$

Note that if k is even and $\zeta \in \Re$ then $S_k(\zeta) \in \Im$ whereas if k is odd and $\zeta \in \Im$ then $S_k(\zeta) \in \Re$. It follows that the derivatives μ_k and υ_k also alternate between the real and imaginary axes.

Let a denote $(1+\epsilon)^{\frac{1}{2n}}$. Set

$$S_{a_k} = aS_k = a(\mu_k\zeta + \upsilon_k\zeta^2 + \rho_k\zeta^3 + \mathrm{O}(\zeta^4)$$

and write

$$\mathbb{S}_a = S_{a_{2n-1}} \circ S_{a_{2n-2}} \circ \ldots S_{a_0}$$

To get a local expansion for $\mathbb{S}_a(\zeta)$ set

$$S_{a_0} = A_0\zeta + B_0\zeta^2 + C_0\zeta^3 + \mathrm{O}(\zeta^4)$$

where
$$A_0 = a\mu_0, \quad B_0 = a\upsilon_0, \quad C_0 = a\rho_0$$
Recursively set $a_k = a\mu_k$, $b_k = a\upsilon_k$, $c_k = a\rho_k$ and
$$S_{a_k} \circ \dots S_{a_0} = A_k\zeta + B_k\zeta^2 + C_k\zeta^3 + \mathrm{O}(\zeta^4)$$
and find
$$A_k = a_k A_{k-1}, \quad B_k = b_k A_{k-1}^2 + a_k B_{n-1}, \quad C_k = c_k A_{k-1}^3 + 2A_{k-1}B_{k-1} + a_k C_{k-1}$$
Using equation 3 we find
$$A_{2n-1} = (1+\epsilon) \tag{4}$$
Expanding, we have
$$B_{2n-1} = a^{4n-1}(\mu_0 \dots \mu_{2n-2})^2 \upsilon_{2n-1} + a^{4n-2}(\mu_0 \dots \mu_{2n-2})^2 \upsilon_{2n-1}\mu_{2n} + \dots$$
$$+ a^{2n}\upsilon_0\mu_1 \dots \mu_{2n-1}$$
By the symmetry of the cycle, for $k = 0, \dots n-1$, we have $\mu_k = \mu_{k+n}$ and $\upsilon_k = -\upsilon_{k+n}$. Using this observation we obtain
$$B_{2n-1} = (1+\epsilon)((1+\epsilon)^{\frac{1}{2}} - 1)\sum_{k=0}^{n-1}(1+\epsilon)^{\frac{k-1}{2n}} \frac{\upsilon_{k+n}}{\mu_{k+n}} \frac{1}{\mu_{k+n} \dots \mu_{2n-1}} \tag{5}$$
It follows that $B_{2n-1} = \mathrm{O}(\epsilon)$. Applying equations (4) and (5) we deduce that $C_{2n-1} = \mathrm{O}(|1+\epsilon|)$.

Now comparing
$$\mathbb{S}_a(\zeta) = (1+\epsilon)\zeta + B_{2n-1}\zeta^2 + C_{2n-1}\zeta^3 + \mathrm{O}(\zeta^4) \tag{6}$$
with equation (2), and using equation (5), we see that as $\epsilon \to 0$, $B_{2n-1} = \mathrm{O}(\epsilon) \to 0$ and $C_{2n-1} \to -1$. Moreover, since $\mathbb{S}_a(\zeta)$ is real for ζ real, we deduce that B_{2n-1} and C_{2n-1} must be real.

Note that for $\epsilon < 0$, $\zeta = 0$ is an attracting fixed point and for $\epsilon > 0$ it is repelling.

Now consider the function
$$\frac{\mathbb{S}_a - \zeta}{\zeta} = \epsilon + B_{2n-1}\zeta + C_{2n-1}\zeta^2 + \mathrm{O}(\zeta^3)$$
which has two roots $u_{1,2}$ near zero. These are approximate roots of $\epsilon + B_{2n-1}\zeta + C_{2n-1}\zeta^2 = 0$ and, since for ϵ suitably small we have $C_{2n-1} < 0$, the roots are real and close to 0.

Computing the multipliers at u_i, $i = 1, 2$, we have
$$M(u_i) = 1 + \epsilon - 2\epsilon + C_{2n-1}(u_i)^2 = 1 - \epsilon + C_{2n-1}(u_i)^2 < 1$$
Thus, for $\epsilon > 0$, these real roots are both attracting fixed points for $\mathbb{S}_a$ and so the repelling fixed point at 0 must be between them. Therefore $\mathbb{S}_a$ is a perturbation of $\mathbb{S}$ in which the parabolic fixed point at 0 has become repelling and two new attracting fixed points u_i have appeared.

We now need to compare $\mathbb{S}_a$ with $(aT)^{2n}$. Set $\delta = a - 1 \approx \epsilon/2n$ and denote
$$R_k(\zeta) = aT(z_k + \zeta) - z_{k+1} = h_{k+1}(aT)h_k^{-1} = Sa_k(\zeta) + \delta z_{k+1}$$
We can write
$$R_k(\zeta) = \delta z_{k+1} + \mu_k\zeta + \upsilon_k\zeta^2 + \mathrm{O}(\zeta^3)$$
Now set
$$\mathbb{R}_a(\zeta) = R_{2n-1} \circ R_{2n-2} \circ \dots R_0(\zeta) = h_0(aT)^{2n}h_0^{-1}$$

Expanding and collecting terms we can compute that

$$\mathbb{R}_a(\zeta) = \mathbb{S}_a(\zeta) + \mathrm{O}(\delta)$$

Moreover, since $\mathbb{R}_a(\zeta)$ is real for ζ real, the graph of $\mathbb{R}_a$ restricted to $\mathbb{R}^2$ is a small translation of the graph of $\mathbb{S}_a$ and both functions have the same dynamics. That is, if $\epsilon < 0$, $\mathbb{R}_a$ has an attracting periodic point of period $2n$ whereas if $\epsilon > 0$ we conclude that $\mathbb{R}_a$ has a real repelling fixed point, w_0, close to 0 and two real attracting fixed points, u_0 and v_0, on either side of it. Using the conjugacy h_0 we find that $z_0 + w_0$ is a repelling fixed point of $(aT)^{2n}$ and $z_0 + u_0$, $z_0 + v_0$ are attracting fixed points. □

Now we can prove our main result

Theorem 4. *Suppose n is even and $\lambda_1 = q'_{2n}i$ is the real parabolic boundary point of a hyperbolic component Ω'_{2n}. Then there is a non-standard bifurcation at $q'_{2n}i$; that is, as $|q'_{2n}|$ increases, the parabolic cycle of $q'_{2n}i \tan z$ of period $2n$ bifurcates into one repelling cycle of period $2n$ and two attracting cycles of period $2n$ so that there is a bud component Ω_{4n} off the root component Ω'_{2n} at λ_1.*
Next suppose n is odd and $\lambda_1 = q_{2n}i$ is the boundary point of a hyperbolic component Ω_{2n}. Then there is a standard bifurcation at $q_{2n}i$; that is, as $|q_{2n}|$ increases, each parabolic cycle of period $2n$ bifurcates into one repelling cycle of period $2n$ and one attracting cycle of period $4n$ so that there is a bud component Ω_{4n} off the root component Ω_{2n} at λ_1.

Proof. We first need to show that in either of the cases there are two attracting petals at each point of the parabolic cycle.

In the first case, if n is even and the component is Ω'_{2n}, then $M(q'_{2n}i) = 1$. By proposition 7, if z_0 is a parabolic periodic point of $q'_{2n}i \tan z$ there must be exactly two attracting petals at z_0. This is proved by noting that each attracting petal must contain the orbit of at least one asymptotic value and these orbits lie either on the real or imaginary axes. Moreover, since the multiplier is positive, the orbits of the two asymptotic values lie on opposite sides of the periodic point and the attracting petals at the periodic points fall into two forward invariant cycles.

In the second case, if n is odd and the component is Ω_{2n}, the multiplier for each cycle satisfies $M(q'_{2n}i) = -1$. Again, applying proposition 7, if z_0 is a parabolic periodic point of $q_{2n}i \tan z$ there must be exactly two attracting petals at z_0. This time, the argument proceeds by observing that since the multiplier is negative, the orbit of exactly one of the asymptotic values, say q_{2n}, lies on both sides of the periodic point z_0 (and the orbit of the other value $-q_{2n}$ lies on both sides of $-z_0$).

We now apply theorem 3 in each of these cases:

In the first case z_0 is a is a parabolic fixed point of $(q'_{2n}i \tan z)^{4n}$. It is part of a single parabolic cycle. In the perturbation, by theorem 3, (where we replace n by $2n$), if $\epsilon < 0$, the perturbed function belongs to Ω'_{2n}. For $\epsilon > 0$, however, z_0 has become a repelling fixed point w_0 and two attracting fixed points u_0 and v_0 of $((1+\epsilon)^{\frac{1}{4n}}(q'_{2n}i \tan z))^{4n}$ have been created. We may assume these points are all real and $u_0 < w_0 < v_0$.

Suppose, for the sake of argument that the orbit of the asymptotic value q'_{2n} accumulated to z_0 in the attracting petal on the right and the orbit of $-q'_{2n}$ accumulated to z_0 in the attracting petal on the the left. By continuity we deduce that after the perturbation, the orbit of $q'_{2n}(1+\epsilon)^{\frac{1}{4n}}$ accumulates to v_0 and the orbit of

$-q'_{2n}(1+\epsilon)^{\frac{1}{2n}}$ accumulates to u_0. This implies that v_0 and u_0 belong to different cycles, each of period $4n$. Hence the bifurcation is non-standard and the perturbed function belongs to a component Ω_{4n}. (See also the proof of theorem 8.8 of [**8**].)

In the second case $(q_{2n}i\tan z)^{2n}$ has two distinct parabolic cycles of order $2n$. Let z_0 be a parabolic fixed point on the real axis of $(q_{2n}i\tan z)^{2n}$ with multiplier -1. We can apply theorem 3 with $\epsilon < 0$ to obtain a perturbed function inside Ω_{2n}.

For $\epsilon > 0$, we can again apply theorem 3, (again using $2n$ instead of n), to the doubled iterate $(q_{2n}i\tan z)^{4n}$. The only differences in this case are first, that $\upsilon_k = \upsilon_{k+n}$ and second, that $\Pi_j^{2n-1+j \bmod 4n}\mu_k = -1$. These two changes in sign cancel each other out and the formula for B_{4n-1} remains the same.

As indicated in the remark above, z_0 has become a repelling fixed point w_0 and two attracting fixed points u_0 and v_0 of $((q_{4n}(1+\epsilon)^{\frac{1}{2n}}i)\tan z)^{4n}$ have been created. Again we may assume these points are all real and $u_0 < w_0 < v_0$. In this case only one of the asymptotic values, say q_{2n}, accumulates to z_0; that is, both the right and left attracting petals of z_0 contain points of the orbit of q_{2n} and neither contains points of the orbit of $-q_{2n}$. The orbit of $-q_{2n}$ accumulates to $-z_0$. Again, by continuity we deduce that after the perturbation, the orbit of the new asymptotic value $q_{2n}(1+\epsilon)^{\frac{1}{4n}}$ accumulates to both the left side of u_0 and the right side of v_0 so that both the new points belong to the same cycle of period $4n$. The point $-w_0$ is also a repelling periodic point and the points $-u_0, -v_0$ belong to another newly created attracting cycle. Thus the bifurcation in this case is standard and the perturbed function belongs to a bud component Ω_{4n} □

6. Applications

In order to apply our result to obtain the existence of an infinite cascade of period doubling, and a Sharkovskii type ordering of the components we need some notation. Although discussion here is for parameters on the upper imaginary axis, $iq, q > 0$, by the symmetry of the parameter plane, we obtain results for the full axis by reflection.

In the discussion below of the ordering of component pairs we will also care about the order of the components in the pair. We will write (Ω_p, Ω'_p) if

$$\sup_{\lambda \in \Omega_p \cap \Im} |\lambda| \leq \inf_{\lambda \in \Omega'_p \cap \Im} |\lambda|$$

and (Ω'_p, Ω_p) if

$$\sup_{\lambda \in \Omega'_p \cap \Im} |\lambda| \leq \inf_{\lambda \in \Omega_p \cap \Im} |\lambda|.$$

We order the component pairs intersecting the imaginary axis in terms of their distance from the origin. We say $(\Omega_p, \Omega'_p) \lhd (\Omega_q, \Omega'_q)$

- if both pairs intersect $\Im^+$ and

$$\sup_{\lambda \in (\Omega_p \cup \Omega'_p) \cap \Im^+} |\lambda| \leq \inf_{\lambda \in (\Omega_q \cup \Omega'_q) \cap \Im^+} |\lambda|$$

- or both pairs intersect $\Im^-$ and

$$\sup_{\lambda \in (\Omega_p \cup \Omega'_p) \cap \Im^-} |\lambda| \leq \inf_{\lambda \in (\Omega_q \cup \Omega'_q) \cap \Im^-} |\lambda|$$

As our first application we obtain the existence of a period doubling cascade. More precisely, if we denote the interval $[i(\frac{\pi}{2} + k\pi), i(\frac{\pi}{2} + (k+1)\pi)]$ by I_k, we have

THEOREM 5. *If* $\lambda_k = i(\frac{\pi}{2} + k\pi)$ *then* λ_k *is the virtual center of a component pair* (Ω_2', Ω_2) *and there exists a sequence of component pairs such that*

$$(\Omega_2', \Omega_2) \lhd (\Omega_4, \Omega_4') \lhd (\Omega_8, \Omega_8') \lhd \ldots \lhd (\Omega_{2^n}, \Omega_{2^n}') \ldots$$

such that at each symbol $\lhd$*, the left component is a root and the right component a bud. Moreover, these pairs accumulate to a point* $q^* i$ *with* $q^* \leq (k+1)\pi$*.*

PROOF. The existence of the pairs (Ω_2', Ω_2) was proved in [**8**]. The standard bifurcation at the real parabolic boundary point of Ω_2 and the non-standard bifurcations at the real parabolic boundary points of the Ω_{2^n}', $n \geq 2$ exist by theorem 4. The intersection of the closures of these components with $\Im$ must be a closed interval J. We know, however, that for $(k+1)\pi i \tan z$, the asymptotic value lands on a repelling fixed point, the origin. Thus the Julia set is the whole Riemann sphere and this function cannot be in the interior or a boundary point of a hyperbolic component intersecting the imaginary axis. Therefore $J \subset [\lambda_k, (k+1)\pi i]$ and the component pairs in the cascade accumulate to some point $q^* i$ such that $\lambda_k < q^* \leq (k+1)\pi i$. □

Theorem 7.2 of [**8**] says

THEOREM 6. *There exist component pairs of every even period in every* I_k*. Moreover, in each* I_k *there are at least* $2k(2k-1)^{n-2}$ *pairs* $(\Omega_{2n}, \Omega_{2n}')$ *(or* $(\Omega_{2n}', \Omega_{2n})$*). In fact, these are ordered inductively: to each of the perhaps more than* $2k-1$ *pairs of component pairs* $(\Omega_{2n}, \Omega_{2n}')$ *(or* $(\Omega_{2n}', \Omega_{2n})$*), there are at least* $2k$ *component pairs* $(\Omega_{2n+2}', \Omega_{2n+2})$ *(or* $(\Omega_{2n+2}, \Omega_{2n+2}')$*).*

We can now strengthen this theorem as follows

THEOREM 7 (Sharkovskii ordering). *There exist component pairs of every even period in every* I_k*. Moreover, in each* I_k *there are exactly* $2k(2k-1)^{n-2}$ *pairs* $(\Omega_{2n}, \Omega_{2n}')$ *(or* $(\Omega_{2n}', \Omega_{2n})$*). In fact, these are ordered inductively: to each of the* $2k-1$ *pairs of component pairs* $(\Omega_{2n}, \Omega_{2n}')$ *(or* $(\Omega_{2n}', \Omega_{2n})$*), there are exactly* $2k$ *component pairs* $(\Omega_{2n+2}', \Omega_{2n+2})$ *(or* $(\Omega_{2n+2}, \Omega_{2n+2}')$*).*

PROOF. The proof of theorem 6, is based on the proof of the existence of standard and non-standard bifurcations at real parabolic points. This existence was proved only under the assumption that the multiplier at the real parabolic point was a monotonic function of the parameter. If this assumption were false, extra components would exist. Using theorem 4 in the proof for the existence of the bifurcations, we conclude these extra components do not exist. Hence the inequalities become equalities. □

References

[1] A. Beardon, *Iteration of Rational Mappings*, Springer-Verlay, N.Y.

[2] L. Carleson and T. W. Gamelin, *Complex Dynamics*, Springer-Verlag, N.Y., 1993. MR94h:30033

[3] R. L. Devaney and L. Keen, *Dynamics of tangent*, Proc. Maryland Special Year in Dynamics, vol. 1342, Springer, 1987. MR 90e:58093

[4] R. L. Devaney and L. Keen, *Dynamics of meromorphic maps: maps with polynomial Schwarzian derivative*, Ann. Sci. Ecole Norm. Sup., 22 4^e série (1989), 55-79. MR 90e:58071

[5] A. L. Epstein, *Infinitiesimal Thurston Rigidity and the Fatou-Shishikura Inequality*, Preprint.

[6] W.H.Jiang, *The Parameter Space of* $\lambda tan(z)$, CUNY Ph. D. Thesis (1991), Unpublished.

[7] L. Keen and J. Kotus, *Dynamics of the family* $\lambda tan(z)$, Conformal Geometry and Dynamics An Electronic Journal of the American Mathematical Society, Vol 1, Pages 28–57,(August 13,1997)

[8] L. Keen and J. Kotus, *On period doubling phenomena and Sharkovskii type ordering for the family* $\lambda tan(z)$, Contemporary Mathematics, Volume 303, 2002, Page 51-78.

[9] J.Milnor, *Dynamics in One Complex Variable* Introductory Lectures, Vieweg, 2^{nd} Edition 2000.

Department of Mathematics, Lehman College and Graduate Center, CUNY, Bronx NY 10468

E-mail address: linda.keen@lehman.cuny.edu

Department of Mathematics, CUNY Graduate Center,, New York, NY 10016

E-mail address: syuan@gc.cuny.edu