A FRAMEWORK TOWARDS UNDERSTANDING THE CHARACTERIZATION OF HOLOMORPHIC MAPS

YUNPING JIANG

DEDICATED TO PROFESSOR JOHN MILNOR ON HIS 80^{TH} BIRTHDAY

ABSTRACT. This paper gives a review of our work on the characterization of geometrically finite rational maps and then outlines a framework for characterizing holomorphic maps. Whereas Thurston's methods are based on estimates of hyperbolic distortion in hyperbolic geometry, the framework suggested here is based on controlling conformal distortion in spherical geometry. The new framework enables one to relax two of Thurston's assumptions, first, that the iterated map has finite degree and, second, that its post-critical set is finite. Thus, it makes possible to characterize certain rational maps for which the post-critical set is not finite as well as certain classes of entire and meromorphic coverings for which the iterated map has infinite degree.

1. CHARACTERIZATION

Suppose $f : S^2 \to S^2$ is an orientation-preserving branched covering of the two-sphere S^2. We just call it a branched covering. We use $f^n = \underbrace{f \circ \ldots \circ f}_{n}$ to denote the n^{th}-composition. Let $d = \deg f > 1$ be the degree of f, let

$$\Omega_f = \{c \mid \deg_c f \geq 2\}$$

be the set of all branched points, where $\deg_c f$ means the local degree of f at c. Let

$$P_f = \overline{\cup_{n\geq 1} f^n(\Omega_f)}$$

be the post-critical set. A rational map which is a quotient of two polynomials, is a holomorphic branched covering.

2000 *Mathematics Subject Classification.* 58F23, 30D05.

Definition 1.1 (Combinatorial Equivalence)**.** *Two branched coverings f and g are said to be combinatorially equivalent if there are two homeomorphisms $\phi, \psi : S^2 \to S^2$ such that:*

$$\begin{array}{ccc} (S^2, P_f) & \xrightarrow{\phi} & (S^2, P_g) \\ \downarrow f & & \downarrow g \\ (S^2, P_f) & \xrightarrow{\psi} & (S^2, P_g) \end{array}$$

commutes and $\phi \stackrel{homotopy}{\sim} \psi$ rel P_f. (The term "rel P_f" means that $\psi|P_f = \phi|P_f$ and the homotopy preserves this equation, refer to Definition 4.1.)

A basic problem in complex dynamics is that

Problem 1. *Can a branched covering f be combinatorially equivalent to a rational map?*

Consider the complex plane $\mathbb{C}$ and consider the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ which is the two-sphere equipped with the standard conformal structure. Let

$$M(\mathbb{C}) = \{\mu \mid \mu \text{ is a measurable function on } \mathbb{C} \text{ such that } \|\mu\|_\infty < 1\}$$

be the open unit ball of the complex Banach space $\mathcal{L}^\infty(\mathbb{C})$. Each μ in $M(\mathbb{C})$ is called a Beltrami coefficient and can be viewed as a conformal structure on $\widehat{\mathbb{C}}$. The Beltrami equation

$$w_{\overline{z}} = \mu w_z \tag{1}$$

has a unique quasiconformal homeomorphism solution fixing $0, 1, \infty$ (refer to [Ahlfors-Bers]). We always use w^μ to denote this solution.

Assume f is locally quasiconformal except for Ω_f. In other words, we assume that f is quasiregular, that is, $f = R \circ g$, where g is a quasiconformal homeomorphism and R is a rational map. Given a Beltrami coefficient μ (viewed as a conformal structure on $\widehat{\mathbb{C}}$), the pull-back $f^*\mu$ of μ by f can be calculated as

$$f^*\mu(z) = \frac{\mu_f(z) + \mu(f(z))\theta(z)}{1 + \overline{\mu_f(z)}\mu(f(z))\theta(z)}$$

where $\mu_f(z) = f_{\overline{z}}(z)/f_z(z)$ is a Beltrami coefficient and where $\theta(z) = \overline{f_z(z)}/f_z(z)$.

Let $E \supseteq P_f$ be a closed subset of $\widehat{\mathbb{C}}$ such that $f(E) \subseteq E$. Suppose $0, 1, \infty \in E$. Then $f^*\mu \sim_E \mu$ means that $(w^{f^*\mu})^{-1} \circ w^{\mu} \stackrel{homotopy}{\sim} id$ rel E

(the term "rel E" means that $(w^{f^*\mu})^{-1} \circ w^{\mu}|E = id$ and the homotopy preserves this equation, refer to Definition 4.1).

Problem 2. *Can we view $f : (\widehat{\mathbb{C}}, E) \to (\widehat{\mathbb{C}}, E)$ as a holomorphic map? More precisely, can we find a Beltrami coefficient μ such that $f^*\mu \sim_E \mu$ rel E?*

A special case is when $E = \widehat{\mathbb{C}}$. The answer to Problem 2 in this special case is the uniform quasiconformality, that is,

Theorem 1.1 ([Sullivan;2]). *There is a Beltrami coefficient μ (in other words, a conformal structure on $\widehat{\mathbb{C}}$) such that $f^*\mu = \mu$ if and only if $\sup_{n \geq 0} K(f^n) \leq K_0 < \infty$, where $K(f^n)$ means the maximal quasiconformal dilatation of f^n.*

Remark 1.1. *Sullivan proved the above theorem for two-dimensional quasiconformal semi-groups acting on the Riemann sphere. Hinkkanen* [Hinkkanen] *proved that a similar method can also be applied to prove the above theorem for any quasi-entire function, $f = e \circ g$, and any quasi-meromorphic function, $f = m \circ g$, where g is a quasiconformal homeomorphism and e and m are an entire function and a meromorphic function on the complex plane, as a two-dimensional semi-group acting on the complex plane. Sullivan* [Sullivan;1] *and Tukia* [Tukia] *also proved the above theorem for two-dimensional quasiconformal groups acting on the Riemann sphere.*

2. Obstruction

In general, answers to Problems 1 and 2 are not easy. There is a topological obstruction which we call a Thurston obstruction in this study.

Suppose $E \supseteq P_f$ is a closed subset of the Riemann sphere (it is not necessary to assume that $0, 1, \infty \in E$). A simple closed curve γ in $S^2 \setminus E$ is called non-peripheral if every component of $S^2 \setminus \gamma$ contains at least two points from E. A *multi-curve* $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$ is a set of finitely many pairwise disjoint, non-homotopic, and non-peripheral curves in $S^2 \setminus E$.

Since $E \supseteq P_f$, $\Omega_f \subseteq f^{-1}(E)$ and $f : S^2 \setminus f^{-1}(E) \to S^2 \setminus E$ is a covering map of finite degree and every component of $f^{-1}(\gamma)$ is a simple closed curve. A multi-curve Γ is said to be *f-stable* if for any $\gamma \in \Gamma$, every non-peripheral component of $f^{-1}(\gamma)$ is homotopic to an element of Γ rel E.

For a stable multi-curve Γ, define f_Γ as follows: For each $\gamma_j \in \Gamma$, let $\gamma_{i,j,\alpha}$ denote the components of $f^{-1}(\gamma_j)$ homotopic to γ_i in $S^2 \setminus E$ and $d_{i,j,\alpha}$ be the degree of $f|_{\gamma_{i,j,\alpha}}: \gamma_{i,j,\alpha} \to \gamma_j$. Define

$$f_\Gamma(\gamma_j) = \sum_i \Big(\sum_\alpha \frac{1}{d_{i,j,\alpha}} \Big) \gamma_i.$$

Then $f_\Gamma : \mathbb{R}^\Gamma \to \mathbb{R}^\Gamma$ is a linear transformation with a non-negative matrix $A = (f_\Gamma)$. There exists a largest non-negative eigenvalue $\lambda(A)$ of A with a non-negative vector.

Definition 2.1 (Thurston obstruction)**.** *A stable multi-curve Γ is called a* Thurston obstruction *if the largest eigenvalue $\lambda(A)$ is greater than or equal to* 1*.*

Definition 2.2 (Levy Cycle)**.** *If there is a finite set of non-peripheral curves $\Gamma = \{\gamma_0, \gamma_1, \ldots, \gamma_n\}$ which is f-invariant in the sense that there is a component γ_i' of $f^{-1}(\gamma_{i+1})$ such that $\gamma_i' \sim \gamma_i$ rel P_f and $f : \gamma_i' \to \gamma_{i+1}$, with i taken $\mod n$, is a homeomorphism, then Γ is called a* Levy cycle*.*

Remark 2.1. *Usually a Levy cycle is only a multi-curve but it can be extended to a stable multi-curve. A stable Levy cycle is a special kind of Thurston obstruction. On the other hand, for a topological polynomial (not necessarily post-critically finite) or a post-critically finite branched covering of degree two, a Thurston obstruction always contains a Levy cycle (see* [Levy, Rees] *and also refer to* [Tan, Godillon]*). Among all Thurston obstructions (if they exist), the canonical obstruction is the most interesting one (see Definition 4.4).*

3. Review

A branched covering f is called post-critically finite if P_f is a finite subset. In this case, an answer to Problems 1 and 2 is Thurston's Theorem as follows.

Proposition 1. *Any post-critically finite branched covering f is combinatorially equivalent to a quasi-regular post-critically finite branched covering $g = R \circ h$ where R is a rational map and h is a quasiconformal homeomorphism, in other words, g is locally quasiconformal except for Ω_g.*

Proof. Recall that Ω_f is the set of branched points of f in $\widehat{\mathbb{C}}$. Consider the topological surface $S = \widehat{\mathbb{C}} \setminus \Omega_f$. For every $p \in S$, let U_p be a small neighborhood about p such that $\phi_p = f|_U : U \to f(U) \subset \widehat{\mathbb{C}}$ is injective. Then $\alpha = \{(U_p, \phi_p)\}_{p\in S}$ defines an atlas on S with charts (U_p, ϕ_p). If $U_p \cap U_q \neq \emptyset$, then $\phi_p \circ \phi_q^{-1}(z) = z : \phi_q(U_p \cap U_q) \to \phi_p(U_p \cap U_q)$. Thus all transition maps are conformal ($1-1$ and analytic) and the atlas α defines a Riemann surface structure on S which we again denote by α. Denote this Riemann surface by (S, α). From the uniformization theorem, (S, α) is conformally equivalent to the Riemann surface $\widehat{\mathbb{C}} \setminus A$ with the standard complex structure induced by $\widehat{\mathbb{C}}$, where A consists of $\#(\Omega_f)$ points. Thus we have a homeomorphism $\widehat{h} : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that $\widehat{h} : (S, \alpha) \to \widehat{\mathbb{C}} \setminus A$ is conformal. Thus $R = f \circ \widehat{h}^{-1} : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is holomorphic with critical points at $\widehat{h}(\Omega_f)$. Since the set P_f is finite, following the standard procedure in quasiconformal mapping theory, there is a K-quasiconfornal homeomorphism $h : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that h is isotopic to $\widehat{h}$ rel P_f. The map $g = R \circ h$ is a quasi-regular map and combinatorially equivalent to f. This completes the proof of the proposition. $\square$

Associated to the finite set $E = P_f$, one can introduce an orbifold structure as follows:

Define the signature $\nu_f : S^2 \to \mathbb{N} \cup \{\infty\}$ as

$$\nu_f(x) = \begin{cases} 1, & x \notin \cup_{n\geq 1} f^{\circ n}(\Omega_f); \\ lcm\{\deg_y f^n \mid \ n > 0, f^n(y) = x\}, & otherwise, \end{cases}$$

where lcm means the least common multiple and $\deg_y f^n$ means the local degree of f^n at y. The orbifold associated to f is $\mathcal{O}_f = (S^2, \nu_f)$ and the Euler characteristic of $\mathcal{O}_f$ is defined as

$$\chi(\mathcal{O}_f) = 2 - \sum_{x\in S^2} \left(1 - \frac{1}{\nu_f(x)}\right).$$

It is known that $\chi(\mathcal{O}_f) \leq 0$ for any post-critically finite branched covering f. The orbifold $\mathcal{O}_f$ is called hyperbolic if $\chi(\mathcal{O}_f) < 0$ and parabolic if $\chi(\mathcal{O}_f) = 0$.

Theorem 3.1 ([Thurston], in [Douady-Hubbard]). *A post-critically finite branched covering f with hyperbolic orbifold is combinatorially equivalent to a rational map if and only if it has no Thurston obstruction. Moreover, the rational map is unique up to conjugation by an automorphism of the Riemann sphere.*

Remark 3.1. *The paper* [Douady-Hubbard] *is a good source to learn Thurston's theorem, where a complete and detailed and comprehensive proof is presented. Another source to learn this theorem is video tapes of Sullivan's lectures given at the CUNY Graduate Center in 1981-1986* [Sullivan;3].

Furthermore,

Theorem 3.2 ([Pilgrim])**.** *If a post-critically finite branched covering f is not combinatorially equivalent to a rational map* (*i.e., a negative answer to Problem 1*)*, then it must have the canonical obstruction* (*see Definition 4.4*)*. The canonical obstruction is a Thurston obstruction.*

Things become very different when one turns to geometrically finite branched coverings. We started this study in [Cui-Jiang-Sullivan;M] and then summarized in [Cui-Jiang-Sullivan]. The latter puts our study into two parts, the first part concentrates on a local combinatorial theory and the second part focus on a global combinatorial theory. A branched covering f is called geometrically finite if $\#(P_f) = \infty$ but the accumulation set P_f' is finite. In this case, $P_f' = \{a_1, \ldots, a_k\}$ consists of finitely many periodic orbits. In [Cui-Jiang], it is proved that this definition of a geometrically finite branched covering is equivalent to the traditional definition of a geometrically finite rational map when the branched covering is holomorphic. Recall that the traditional definition of a geometrically finite rational map is that a rational map is geometrically finite if the intersection of the post-critical set and the Julia set is a finite set. There was little progress towards the understanding of the characterization of geometrically finite rational maps (for example, [Brown]) until our work below. Note that for a general rational map R, we have the following.

Theorem 3.3 ([McMullen])**.** *Suppose R is a rational map. Take $E = P_R$. Let Γ be a f-stable multi-curve on $\widehat{\mathbb{C}} \setminus E$. Then $\lambda(\Gamma) \leq 1$. Only in the following two cases, $\lambda(\Gamma)$ may be 1:*

1) *R is post-critically finite with $\#(E) = 4$ and the orbifold $\mathcal{O}_f$ is $(2, 2, 2, 2)$. Moreover, R is a double covered by an integral torus endomorphism.*
2) *E is an infinite set and Γ includes the essential curves in a finite system of annuli permuted by R. These annuli lie in Siegel disks or Herman rings (refer to* [Milnor]*) for R and each annulus is a connected component of $\widehat{\mathbb{C}} \setminus E$.*

A geometrically finite rational map has no Thurston obstruction (refer to the proof of [Douady-Hubbard, Theorem 4.1] or refer to Theorem 3.3 since it has no Siegel disks and Herman rings). Furthermore, we found a counterexample in the geometrically finite case which prevents a similar theorem to Thurston's theorem under the combinatorial equivalence.

Theorem 3.4 (Counterexample [Cui-Jiang])**.** *There exists a geometrically finite branched covering f having no Thurston obstruction and giving a negative answer to Problem 1* (*i.e., it is not combinatorially equivalent to a rational map*)*.*

The idea to construct this counterexample is to start with $q(z) = \lambda z + z^2$, $0 < |\lambda| < 1$. Then 0 is its attractive fixed point such that the critical point $c = -\lambda/2$ is in the immediate basin of 0. The post-critical orbit $\{q(c), \ldots, q^n(c), \ldots\}$ has a limiting point 0. Since $q(z)$ is a geometrically finite rational map, it has no Thurston obstruction. Apply Dehn twists along this post-critical orbit such that the topological structure near this limiting point has infinite complexity and such that there is no new Thurston obstruction being introduced. Remember that according to Kœnig's Theorem (see [Milnor]), the topological structure of a holomorphic map near an attractive fixed point must be very simple. Thus the newly constructed geometrically finite branched covering can not be combinatorially equivalent to a rational map. Pictures in [Cui-Jiang, Figures 1-3] provide the further idea about this construction.

Based on this counterexample, we define a semi-rational branched covering and a sub-hyperbolic semi-rational branched covering.

Definition 3.1 (Semi-rational and sub-hyperbolic [Cui-Jiang])**.** *Suppose $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a geometrically finite branched covering of degree $d \geq 2$. We say f is* semi-rational *if*

1) *f is holomorphic in a neighborhood of P'_f;*
2) *each cycle $< p_0, \ldots, p_{k-1} >$ of period $k \geq 1$ in P'_f is either attracting, that is, $0 < |(f^k)'(p_0)| < 1$, or super-attracting, that is, $(f^k)'(p_0) = 0$ or parabolic, that is, $[(f^k)'(p_0)]^q = 1$ for some integer $1 \leq q < \infty$.*
3) *For each parabolic cycle in P'_f, every attracting petal associated to this cycle contains at least one point in a critical orbit.*

Furthermore, if in addition P'_f contains only attracting and super-attracting periodic cycle, then we call f sub-hyperbolic semi-rational.

Furthermore, we proved that

Theorem 3.5 ([Cui-Jiang])**.** *Any semi-rational branched covering is always combinatorially equivalent to a sub-hyperbolic semi-rational branched covering.*

Therefore, we concentrated our study on the sub-hyperbolic semi-rational case. In the same paper [Cui-Jiang], we gave the following definition of the *CLH-equivalence* (combinatorial and locally holomorphic equivalence) among all sub-hyperbolic semi-rational branched coverings.

Definition 3.2 (The CLH-equivalence [Cui-Jiang])**.** *Suppose f and g are two sub-hyperbolic semi-rational branched coverings. We say that they are CLH-equivalent if there are two homeomorphisms $\phi,\ \varphi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that*

1) *ϕ is isotopic to φ rel P_f,*
2) *$\phi \circ f = g \circ \varphi$, and*
3) *$\phi|U_f = \varphi|U_f$ is holomorphic on some open set $U_f \supset P'_f$.*

Remark 3.2. *In the study of the family $x :\mapsto x + a + \frac{b}{2\pi}\sin(2\pi x)$, a similar concept called the strong combinatorial equivalence is defined in* [Epstein-Keen-Tresser] *but the meaning is a little different from the above definition.*

Similar to Proposition 1, we have

Proposition 2. *Any sub-hyperbolic semi-rational branched covering f is CLH-equivalent to a quasi-regular sub-hyperbolic semi-rational branched covering $g = R\circ h$ where R is a rational map and h is a quasiconformal homeomorphism, in other words, g is locally quasiconformal except for Ω_g.*

Proof. Suppose $P'_f = \{a_1, \ldots, a_n\}$. Let D_i be a round disk of radius $r_i > 0$ centered at a_i such that there is no point P_f on ∂D_i and such that $f|D_i$ is holomorphic. Let $U = \cup_{i=1}^{n} D_i$. Let $P = \widehat{\mathbb{C}} \setminus U$. Then $\#(P) < \infty$.

Recall that Ω_f is the set of branched points of f in $\widehat{\mathbb{C}}$. Consider the topological surface $S = \widehat{\mathbb{C}} \setminus \Omega_f$. For every $p \in S$, let U_p be a small neighborhood about p such that $\phi_p = f|_U : U \to f(U) \subset \widehat{\mathbb{C}}$ is injective. When $p \in D_i$, we pick $U_p \subset D_i$. Then $\alpha = \{(U_p, \phi_p)\}_{p\in S}$

defines an atlas on S with charts (U_p, ϕ_p). If $U_p \cap U_q \neq \emptyset$, then $\phi_p \circ \phi_q^{-1}(z) = z : \phi_q(U_p \cap U_q) \to \phi_p(U_p \cap U_q)$. Thus all transition maps are conformal $(1-1$ and analytic) and the atlas α defines a Riemann surface structure on S which we again denote by α. Denote this Riemann surface by (S, α). From the uniformization theorem, (S, α) is conformally equivalent to the Riemann surface $\widehat{\mathbb{C}} \setminus A$ with the standard complex structure induced by $\widehat{\mathbb{C}}$, where A consists of $\#(\Omega_f)$ points. Thus we have a homeomorphism $\widehat{h} : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that $\widehat{h} : (S, \alpha) \to \widehat{\mathbb{C}} \setminus A$ is conformal. Moreover, $\widehat{h}|U : U \to h(U)$ is also conformal. Thus $R = f \circ \widehat{h}^{-1} : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is holomorphic with critical points at $\widehat{h}(\Omega_f)$. Since the set P is finite, following the standard procedure in quasiconformal mapping theory, there is a K-quasiconfonal homeomorphism $h : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that h is isotopic to $\widehat{h}$ rel $P \cup U$. The map $g = R \circ h$ is a quasi-regular map and CLH-equivalent to f. This completes the proof of the proposition. $\square$

Furthermore, we proved the following theorem.

Theorem 3.6 ([Cui-Jiang-Sullivan] ([Cui-Tan]), [Zhang-Jiang])**.** *Suppose f is a sub-hyperbolic semi-rational branched covering. Then it is CLH-equivalent to a rational map if and only if f has no Thurston obstruction. Moreover, the rational map is unique up to the conjugation by an automorphism of the Riemann sphere.*

Remark 3.3. *The idea in the proof given in* [Cui-Jiang-Sullivan] *and the idea in the proof given in* [Zhang-Jiang] *are very different. The former is to decompose a sub-hyperbolic semi-rational branched covering along a stable multi-curve into finitely many post-critically finite branched coverings and to check they have no Thurston obstruction if the original map does not have (Thurston obstruction) and then to apply Thurston's theorem. A good source to learn the former proof is* [Cui-Tan] *where the proof was rewritten with more detailed and comprehensive explanations. The latter is first to develop Thurston's idea working on an induced map from the associated Teichmüller space into itself. In this case, we have an infinite dimensional Teichmüller space. Similar to Thurston's idea, we study an iterated sequence by the induced map. Each element in this iterated sequence represents a punctured Riemann sphere with infinitely many punctures. We then compare the geometry of this punctured Riemann sphere with the geometry of another Riemann sphere with finitely many punctures (see* [Zhang-Jiang, Lemma 5.6]*). Here the shielding ring lemma (Lemma 1) guarantees* *this comparison. Note that the punctured Riemann sphere with finitely many punctures, which is used in the comparison, has no relationship with the punctured Riemann sphere with infinitely many punctures dynamically. This comparison enables us eventually to prove that the induced map strongly contracts the Teichmüller distance along this iterated sequence.*

Both methods in [Cui-Jiang-Sullivan] ([Cui-Tan]) *and in* [Zhang-Jiang] *can be further developed. The former method of "decomposition along a stable multi-curve" can be used to study the renormalization of rational maps as well as their combinatorics (for examples, trees, wandering continuum) and branched coverings with "Siegel disks" and "Herman rings", for examples, the recent work in* [Zhang;1, Zhang;2, Wang] *for the characterization of rational maps with Siegel disks and Herman rings. The latter method of "iterations in the Teichmüller space" can be used to characterize several kinds of holomorphic maps beyond post-critical finite rational maps, including hyperbolic maps, exponential maps and some other meromorphic maps, and maps with "Siegel disks" and "Herman rings" (see §4).*

Furthermore, we proved in a later paper that

Theorem 3.7 ([Chen-Jiang])**.** *If a sub-hyperbolic semi-rational branched covering f is not CLH-equivalent to a rational map* (*i.e., a negative answer to Problem 1 under the CLH-equivalence*)*, then it must have the canonical Thurston obstruction* (*see Definition 4.4*)*. The canonical obstruction is a Thurston obstruction.*

4. Geometry

In this section, a framework is outlined for the further study of Problems 1 and 2 (under either the combinatorial equivalence or the CLH-equivalence). We divide this study into three steps. The first step is to introduce the important concept, *bounded geometry* and to study Problems 1 and 2 under this condition. The second step is to prove the equivalence between the bounded geometry condition and Thurston's topological condition. The third step is to prove that the equivalence between the bounded geometry condition and the canonical topological condition.

The bounded geometry condition is useful in the study of this direction and should be fully developed. It is an analytic condition but can be very well connected with Thurston's topological condition and the canonical topological condition in the critically finite case and in the sub-hyperbolic semi-rational case as we will show. With the bounded geometry condition we will be able to work in this direction based on calculations in *spherical geometry*. This enables us to work on Teichmüller spaces $T(E)$ of closed subsets E on the Riemann sphere directly. This is different from Thurston's original work based on calculations in hyperbolic geometry (refer to [Douady-Hubbard]), which first works on moduli spaces and then lifts results obtained on moduli spaces to Teichmüller spaces (this process needs assumptions that finite degree and finiteness on the post-critical set (refer to [Douady-Hubbard, Lemma 5.2])). The bounded geometry condition can be also defined for quasi-entire functions or quasi-meromorphic functions and used in the study of Problems 1 and 2 (under either the combinatorial equivalence or the CLH-equivalence) for these maps without involving moduli spaces and Lemma 5.2 in [Douady-Hubbard] directly.

The idea of bounded geometry is very important in complex dynamics. Besides its application to single dynamics, it could be applied to the parameter space. For examples, infinitely renormalizable quadratic polynomials and hyperbolic rational maps with Sierpinsky curve Julia sets.

Since our definition of bounded geometry relates to Teichmüller spaces $T(E)$ of closed subsets E on the Riemann sphere, let us briefly mention the definition of $T(E)$ and some related properties.

Suppose E is a closed subset of $\widehat{\mathbb{C}}$. Suppose $0, 1, \infty \in E$ (which allows us to consider it is as a subset in spherical geometry rather than consider it as a subset in hyperbolic geometry which treats two subsets are the same if one is the image of another one under a Möbius transformation). Recall that we use w^μ to denote the unique quasiconformal homeomorphism with $0, 1, \infty$ fixed and with Beltrami coefficient μ in $M(\mathbb{C})$ (refer to the Beltrami equation (1)).

Definition 4.1 (Teichmüller space of a closed subset)**.** *Two elements $\mu, \nu \in M(\mathbb{C})$ are said to be E-equivalent, denoted as $\mu \sim_E \nu$, if*

$$(w^\mu)^{-1} \circ w^\nu \overset{homotopy}{\sim} id \quad rel \quad E.$$

(The term "rel E" means that we have a continuous map $H(t, z) : [0, 1] \times \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that (a) $H(0, z) = z$ and $H(1, z) = (w^\mu)^{-1} \circ w^\nu(z)$ for all $z \in \widehat{\mathbb{C}}$ and (b) $H(t, z) = z$ for all $0 \le t \le 1$ and all $z \in E$.) The

Teichmüller space of E is defined as

$$T(E) = \{[\mu]_E\}$$

the space of all E-equivalence classes.

The space $T(E)$ is very useful for people working on complex dynamics and the study of $T(E)$ is closely related to the study of holomorphic motions of E. From [Earle-McMullen], [Lieb], [Epstein], [Earle-Mitra], [Mitra], [Gardiner-Lakic], and [Jiang-Mitra], [Beck-Jiang-Mitra-Shiga], [Gardiner-Jiang-Wang], [Jiang-Mitra-Shiga], [Jiang-Mitra-Wang], we know a lot about $T(E)$ now. The reader may find many similarities and differences between $T(E)$ and classical Teichmüller spaces of Riemann surfaces. An excellent book on the classical Teichmüller theory which is comprehensive for people working on complex dynamics is [Hubbard]. We now give a very brief summary of some basic properties of $T(E)$.

i) $T(E)$ has a complex manifold structure such that the projection

$$P_E(\mu) = [\mu]_E : M(\mathbb{C}) \to T(E)$$

is holomorphic.

ii) The projection P_E is a holomorphic split submersion, that is, for any $\tau \in T(E)$, $\exists U$ (neighborhood) about τ and a holomorphic section $s_\tau : U \to M(\mathbb{C})$ such that $P_E \circ s_\tau = id$.

iii) $\exists$ a continuous global section $S : T(E) \to M(\mathbb{C})$ such that $P_E \circ S = id$. Therefore, $T(E)$ is contractible.

iv) If $\dim T(E) \geq 2$, there is no global holomorphic section.

v) $T(E)$ is biholomorphically equivalent to $\prod_i T(S_i, \partial S_i) \times M(E)$, where S_i are components of $\widehat{\mathbb{C}} \setminus E$ and $T(S_i, \partial S_i)$ is the Teichmüller space of the hyperbolic Riemann surface S_i with the ideal boundary ∂S_i.

vi) We have the lifting property for $T(E)$: any holomorphic map $f : \Delta = \{z \mid |z| < 1\} \to T(E)$ can be lift to a holomorphic map $\tilde{f} : \Delta \to M(\mathbb{C})$ such that $P_E \circ \tilde{f} = f$, that is, the following diagram

$$\begin{array}{ccc} & & \mathcal{M}(\mathbb{C}) \\ & \tilde{f} \nearrow & \downarrow P_E \\ \Delta & \xrightarrow{f} & T(E) \end{array}$$

commutes.

vii) Teichmüller's metric d_T on $T(E)$ coincides with Kobayashi's pseudo-metric d_K on $T(E)$.

Proposition 3. *Suppose $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is quasiregular (or a quasi-entire or a quasi-meromorphic). Suppose E is a closed subset of $\widehat{\mathbb{C}}$ such that $E \supseteq V_f = f(\Omega_f)$ and $E \subseteq f^{-1}(E)$ (in the quasi-entire and quasi-meromorphic case, V_f must contains all asymptotic values and essential singularities). If $\mu \sim_E \nu$, then $f^*\mu \sim_E f^*\nu$.*

Proof. From the assumption, $\mu \sim_E \nu$, we can find a continuous map $H(t,z) : [0,1] \times \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that

1) $H(0,z) = z$ for all $z \in \widehat{\mathbb{C}}$;
2) $H(1,z) = (w^\mu)^{-1} \circ w^\nu(z)$ for all $z \in \widehat{\mathbb{C}}$; and
3) $H(t,z) = z$ for all $z \in E$ and all $0 \le t \le 1$.

Since $V_f \subseteq E$, we have that $\Omega_f \subseteq f^{-1}(E)$. This implies that $f : \widehat{\mathbb{C}} \setminus f^{-1}(E) \to \widehat{\mathbb{C}} \setminus E$ is a covering map. The homotopy $H(t,z) : [0,1] \times (\widehat{\mathbb{C}} \setminus E) \to \widehat{\mathbb{C}} \setminus E$ rel ∂E can be lift to a homotopy $\widetilde{H}(t,z) : [0,1] \times (\widehat{\mathbb{C}} \setminus f^{-1}(E)) \to \widehat{\mathbb{C}} \setminus f^{-1}(E)$ rel $\partial f^{-1}(E)$ such that

$$H(t, f(z)) = f(\widetilde{H}(t,z)), \quad \forall z \in \widehat{\mathbb{C}} \setminus f^{-1}(E), \ \ 0 \le t \le 1$$

and

$$\widetilde{H}(t,z) = z, \quad \forall z \in \partial f^{-1}(E), \ \ 0 \le t \le 1.$$

Define $\widetilde{H}(t,z) = z$ for all $z \in f^{-1}(E)$ and $0 \le t \le 1$. Then the new defined map, which we still denote as $\widetilde{H}$, is a continuous map $\widetilde{H}(t,z) : [0,1] \times \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that

a) $\widetilde{H}(z,0) = z$ for all $z \in \widehat{\mathbb{C}}$ and
b) $\widetilde{H}(z,t) = z$ for all $z \in E$ and all $0 \le t \le 1$.

Therefore, it is a homotopy from the identity to $\widetilde{H}_1(z) = \widetilde{H}(1,z)$.

Let $H_1(z) = H(1,z)$. Since $H_1 \circ f = f \circ \widetilde{H}_1$ and $H_1 = (w^\mu)^{-1} \circ w^\nu$ is quasiconformal, $\widetilde{H}_1$ is quasiconformal.

Now by using two commuting equations,

$$(w^\mu)^{-1} \circ w^\nu(z) \circ f = f \circ \widetilde{H}_1 \quad \text{and} \quad g \circ w^{f^*\mu} = w^\mu \circ f,$$

where g is holomorphic, we have that

$$g \circ w^{f^*\mu} \circ \widetilde{H}_1 = w^\mu \circ f \circ \widetilde{H}_1 = w^\mu \circ (w^\mu)^{-1} \circ w^\nu(z) \circ f = w^\nu(z) \circ f.$$

Since g is holomorphic,

$$\mu_{w^{f^*\mu} \circ \widetilde{H}_1} = \mu_{g \circ w^{f^*\mu} \circ \widetilde{H}_1} = \mu_{w^\nu(z) \circ f} = f^*\nu.$$

Since both $w^{f^*\mu} \circ \widetilde{H}_1$ and $w^{f^*\nu}$ fix $0, 1, \infty$, we get

$$w^{f^*\mu} \circ \widetilde{H}_1 = w^{f^*\nu}.$$

In other words,
$$\widetilde{H}_1 = (w^{f^*\mu})^{-1} \circ w^{f^*\nu}.$$
Thus $\widetilde{H}(t,z) : [0,1] \times \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a homotopy from the identity to $(w^{f^*\mu})^{-1} \circ w^{f^*\nu}$ rel $f^{-1}(E)$. But $E \subseteq f^{-1}(E)$, so the last statement implies that $f^*\mu \sim_E f^*\nu$. This completes the proof. $\square$

From the above proposition, we can induce a map
$$\sigma_f(\tau) = [f^*\mu]_E : T(E) \to T(E), \quad \mu \in \tau.$$
Since
$$\sigma_f(\tau) = P_E \circ f^* \circ s_\tau,$$
σ_f is holomorphic. Thus it is weakly contracting due to the property that Kobayashi's metric and the Teichmüller metric are the same. That is,
$$d_T(\sigma_f(\tau), \sigma_f(\tau')) \leq d_T(\tau, \tau'), \quad \forall \tau, \tau' \in T(E).$$
Suppose f is holomorphic on E and $\{0, 1, \infty\} \subset E \subseteq f^{-1}(E)$. For any μ with $\mu|E = 0$, we have that $f^*\mu|E = 0$.

Problem 3. *Can we find a (unique) fixed point $\tau = [\mu]_E$ with $\mu|E = 0$ of σ_f?*

A positive answer to Problem 2 (under either combinatorially equivalence or CLH-equivalence) is equivalent to a positive answer to Problem 3 (by recalling Theorem 1.1).

For any $\tau_0 = [\mu_0]_E \in T(E)$ with $\mu_0|E = 0$, let $\tau_n = \sigma_f(\tau_{n-1}) = [\mu_n]_E \in T(E)$ where $\mu_n = f^*\mu_{n-1}$ with $\mu_n|E = 0$ for all $n \geq 1$. Let w^{μ_n} be the corresponding quasiconformal homeomorphism solution of the Beltrami equation (1) with $0, 1, \infty$ fixed. Then we can define a sequence of subsets $E_n = w^{\mu_n}(E)$ such that $0, 1, \infty \in E_n$. Roughly speaking, f has bounded geometry if we have a subsequence E_{n_i} "converging" to E_∞ as i goes to ∞ such that E_∞ is homeomorphic to E. We will give a more precise definition in the critically finite case and in the sub-hyperbolic semi-rational case. We use d_{SP} to mean the spherical distance on $\widehat{\mathbb{C}}$.

Definition 4.2 (Bounded geometry in the critically finite case)**.** *Suppose f is a critically finite branched covering. Suppose $E = P_f = \{0, 1, \infty, p_1, \ldots, p_{k-3}\}$. We say f has bounded geometry if there are a constant $C > 0$ and a point $\tau_0 \in T(E)$ such that*
$$d_{SP}(a, b) \geq C, \quad \forall a, b \in E_n = w^{\mu_n}(E), \quad n \geq 0.$$

Lemma 1 (Shielding ring lemma [Zhang-Jiang])**.** *Suppose f is a sub-hyperbolic semi-rational branched covering. Let $P'_f = \{a_i\}$ be the set of accumulation points of P_f. There exists a collection of finite number of open disks $\{D_i\}$ centered at a_i and a collection of finite number of annuli $\{A_i\}$ (we call them the* shielding rings*) such that*

i) $\overline{A_i} \cap P_f = \emptyset$;
ii) $A_i \cap D_i = \emptyset$, *but one components of ∂A_i is the boundary of D_i;*
iii) $(\overline{D_i \cup A_i}) \cap (\overline{D_j \cup A_j}) = \emptyset$ *for $i \neq j$;*
iv) *f is holomorphic on $\overline{D_i} \cup A_i$; and*
v) *every $f(\overline{D_i} \cup A_i)$ is contained in D_{i+1} for $1 \leq i \leq k-1$ and $f(\overline{D_k} \cup A_k)$ is contained in D_1 where k is the period of a_i*

Suppose $\{(D_i, A_i)\}$ are domains and annuli constructed in the shielding ring lemma. Take

$$\mathcal{D}_0 = \cup_{i=1}^k \overline{D_i} \quad \text{and} \quad P_0 = P_f \setminus \mathcal{D}_0.$$

Then $\#(P_0) < \infty$. Take

$$E = P_0 \cup \mathcal{D}_0. \tag{2}$$

For any $\tau_0 = [\mu_0]_E$ such that $\mu_0 | \mathcal{D}_0 = 0$, we have that $\mu_n | \mathcal{D}_0 = 0$ for all $n \geq 1$. Then $E_n = w^{\mu_n}(E)$ for $n = 0, 1, \ldots$. Note that

$$E_n = P_n \cup \mathcal{D}_n$$

where

$$P_n = w^{\mu_n}(P_0) \quad \text{and} \quad \mathcal{D}_n = w^{\mu_n}(\mathcal{D}_0).$$

Every component D in $\mathcal{D}_0$ is a round disk with center a. The image $D_n = w^{\mu_n}(D)$ is a topological disk in $\mathcal{D}_n$. We call $a_n = w^{\mu_n}(a)$ the center of D_n. Then $\mathcal{D}_n$ consists of finitely many disjoint topological disks. Suppose $0, 1, \infty \in E$. Then $0, 1, \infty \in E_n$ for all $n > 0$.

Definition 4.3 (Bounded geometry in the sub-hyperbolic semi-rational case)**.** *We say f has bounded geometry if there are a constant $C > 0$ and a point $\tau_0 = [\mu_0]_E \in T(E)$ with $\mu_0 | \mathcal{D}_0 = 0$ such that*

i) *$d_{SP}(a, b) \geq C$, for any two points $a, b \in P_n = w^{\mu_n}(P_0)$ and for all $n \geq 0$.*
ii) *$d_{SP}(A, B) \geq C$, for any two different components A and B of $\mathcal{D}_n = w^{\mu_n}(\mathcal{D}_0)$ and for all $n \geq 0$.*
iii) *$d_{SP}(a, A) \geq C$, for any point $a \in P_n$ and any component A of $\mathcal{D}_n$ and for all $n \geq 0$.*
iv) *Each component D_n of $\mathcal{D}_n$ contains a round disk of radius C centered at the center point a_n of D_n.*

Remark 4.1. *Both Definition 4.2 and Definition 4.3 are not preserved under actions of Möbius transformations.*

Consider $\widehat{\mathbb{C}} \setminus E$ with the standard conformal structure $\tau_0 = [0]_E$ as the hyperbolic Riemann surface R_0 and consider $\widehat{\mathbb{C}} \setminus E$ with the conformal structure $\tau_n = [\mu_n]_E$ as the hyperbolic Riemann surface R_n. Then R_n is Teichmüller equivalent to $\widehat{\mathbb{C}} \setminus E_n$.

For any non-peripheral simple closed curve γ in $\widehat{\mathbb{C}} \setminus E$, $w^{\mu_n}(\gamma)$ is a non-peripheral simple closed curve in $\widehat{\mathbb{C}} \setminus E_n$. We use $l_n(\gamma)$ to denote the hyperbolic length of a unique geodesic which is homotopic to $w^{\mu_n}(\gamma)$ in $\widehat{\mathbb{C}} \setminus E_n$. Under the condition of bounded geometry, we can find a positive constant $b > 0$ such that

$$l_n(\gamma) \geq b, \quad \forall n \geq 0, \ \forall \gamma \text{ (non-peripheral simple closed curve)}. \tag{3}$$

However, to get from (3) to bounded geometry, we need the assumption that $0, 1, \infty \in E$ (consequently, $0, 1, \infty \in E_n$).

In both the critically finite and the sub-hyperbolic semi-rational cases, *non bounded geometry* is equivalent to saying that there is a sequence of non-peripheral simple closed curves γ_{n_i} in $\widehat{\mathbb{C}} \setminus E$ such that

$$l_{n_i}(\gamma_{n_i}) \to 0, \quad \text{as} \quad i \to \infty.$$

In particular, we have the following definition. (Refer to [Pilgrim] for the post-critical finite case and to [Chen-Jiang] for the sub-hyperbolic semi-rational case.)

Definition 4.4 (Canonical obstruction)**.** *Let*

$$\Gamma_c = \{\gamma \mid l_n(\gamma) \to 0, \quad n \to \infty\}$$

where γ is a non-peripheral simple closed curve in $\widehat{\mathbb{C}} \setminus E$. If $\Gamma_c \neq \emptyset$, then it is called the canonical obstruction for f.

Clearly, if $\Gamma_c \neq \emptyset$, then f has no bounded geometry. Moreover, we have that

Theorem 4.1 (Equivalent statements for the critically finite case)**.** *Suppose f is a critically finite branched covering with $\#(P_f) \geq 4$ and has a hyperbolic orbifold. Then the following are equivalent:*

1. *f can be viewed as a holomorphic map (i.e, it is combinatorially equivalent to a unique rational map).*
2. *f has bounded geometry.*
3. *$\Gamma_c = \emptyset$.*

(4) *f has no Thurston obstruction.*

Similarly, we have that

Theorem 4.2 (Equivalent statements for the sub-hyperbolic semi-rational case)**.** *Suppose f is a sub-hyperbolic semi-rational branched covering. Then the following are equivalent:*

(1) *f can be viewed as a holomorphic map (i.e, it is CLH-equivalent to a unique rational map).*
(2) *f has bounded geometry.*
(3) *$\Gamma_c = \emptyset$.*
(4) *f has no Thurston obstruction.*

Proofs of these two theorems can now be put into one framework through bounded geometry. Let $E = P_f$ in the post-critically finite case and let $E = P_0 \cup \mathcal{D}_0$ as defined in Equation (2) in the sub-hyperbolic semi-rational case.

First prove that $\sigma_f^k : T(E) \to T(E)$ is contracting for some $k \geq 1$, that is, $d_T(\sigma_f^k(\tau), \sigma_f^k(\tau')) < d_T(\tau, \tau')$ for any $\tau, \tau' \in T(E)$. In the post-critically finite case, $k = 2$, and in the sub-hyperbolic semi-rational case, $k = 1$. Secondly, prove the family of rational maps $\{g_n = w^{\mu_n} \circ f \circ (w^{\mu_{n+1}})^{-1}$ forms a compact subset in the space of all rational maps of degree $d = \deg(f)$ with the uniform convergence topology under the assumption of bounded geometry. Thirdly, consider the operator $d\sigma_f^k$ from the cotangent space $T^*_{\tau_{n+k}} T(E)$ to the cotangent space $T^*_{\tau_n} T(E)$ and show that it is strictly contracting by using bounded geometry and the formula for $d\sigma_f : T^*_{\sigma_f(\tau)} T(E) \to T^*_\tau T(E)$ defined as a transfer operator

$$\mathcal{L}q(z) = \sum_{g_n(w)=z} \frac{q(w)}{(g_n'(w))^2}$$

where $q(w)$ is the coefficient of an integrable holomorphic quadratic differential on the hyperbolic Riemann surface $\widehat{\mathbb{C}} \setminus E_{n+1}$. All the above steps imply that σ_f^k is strongly contracting on the sequence $\{\tau_n\}_{n=0}^{\infty}$ in $T(E)$, that is, there is a constant $0 < c < 1$ such that

$$d_T(\sigma_f^k(\tau_{n+1}), \sigma_f^k(\tau_n)) \leq c d_T(\tau_{n+1}, \tau_n).$$

Then prove that the bounded geometry is equivalent to the no Thurston obstruction condition. And finally, by using some properties of Thurston obstructions (if they exist), prove that Γ_c is not empty.

In this framework, we first prove the equivalence (1) $\Leftrightarrow$ (2) and then prove (2) $\Leftrightarrow$ (3) and (2) $\Leftrightarrow$ (4). A detailed explanation will be given in a later expository paper.

The existing proof of (3) $\Rightarrow$ (2) (refer to [Pilgrim, Chen-Jiang]) uses a Thurston obstruction as a bridge to argue that if f has no bounded geometry, then f has a Thurston obstruction, furthermore, by using this Thurston obstruction, prove that $\Gamma_c \neq \emptyset$. However, we are more interested in a proof without using those properties of Thurston obstructions, that is,

Problem 4. *Find a direct proof of* (3) $\Rightarrow$ (2) *in both Theorem 4.1 and Theorem 4.2. In other words, without using a Thurston obstruction as a bridge show that if f has no bounded geometry, then $\Gamma_c \neq \emptyset$.*

This is an interesting problem because for a quasi-entire or a quasi-meromorphic function, it is not clear to us that the equivalence between the bounded geometry condition and the no Thurston obstruction condition due to infinite degree and the existence of asymptotic values and essential singularities (refer to sectiom 6).

5. Geometrization

Given the canonical obstruction $\Gamma_c = \{\gamma_1, \ldots, \gamma_n\} \neq \emptyset$ for a map f which is either (i) a critical finite branched covering or (ii) a sub-hyperbolic semi-rational branched covering, we can use curves in Γ_c (up to homotopy) dividing the two-sphere S^2 into finitely many topological surfaces which are topological two-spheres removing finite number of disks as follows.

Let E be the post-critical set P_f in the case (i) or the set defined in Equation (2) in the case (ii). Consider the Riemann surface $\widehat{\mathbb{C}} \setminus E$. Suppose $A_{0,i}$ $(i = 1, \ldots, n)$ are a collection of disjoint annuli whose core curves are γ_i $(i = 1, \ldots, n)$, respectively. Set

$$A_0 = \bigcup_{i=1}^{n_0} A_{0,i} \quad \text{and} \quad n_0 = n.$$

Let

$$A_1 = \bigcup_{i=1}^{n_1} A_{1,i}$$

be the union of preimage of elements of A_0 such that every element of A_1 is homotopic to some element in A_0 rel E since Γ_c is a stable and full multi-curve. Up to homotopy, we can assume that

1) every curve γ_i is the core curve of the annulus $A_{0,i}$;
2) every $A_{1,k}$ is a component of the preimage of some $A_{0,j}$ and homotopic to some $A_{0,i}$, denote by $A_{1,ji,\alpha}$;
3) the union $\cup_{j,\alpha} A_{1,ji,\alpha}$ of elements of A_1 which are homotopic to γ_i, denote by $B_{1,i,i}$, is contained inside $A_{0,i}$;
4) the two outmost annuli of $B_{1,i,i}$ share their outer boundary curves with $A_{0,i}$; and
5) restricted to a boundary curve χ of $A_{0,i}$, the map $f : \chi \to f(\chi)$ is conjugated to $z \mapsto z^d : \mathbb{S}^1 \to \mathbb{S}^1$ for some d. Note that $f(\chi)$ is a boundary of $A_{0,j}$.

Each component $A_{0,i}$ of A_0 is called a thin part and each component of $\overline{\mathbb{C}} \backslash A_0$ is called a thick part.

Let $\mathfrak{P}_0 = \{P_i^0\}_{i=1}^{n_0}$ be the collection of all thick parts. The pull-back of all thick parts by f^k is denoted as $\mathfrak{P}_k = f^{-k}(\mathfrak{P}_0)$. Each element of $\mathfrak{P}_k$ belongs to one and only one of the following four classes:

i) Disk component: if it is a topological disk D and $D \cap P_f = \emptyset$.
ii) Punctured disk component: if it is a topological disk P and $\sharp(P \cap P_f) = 1$.
iii) Annulus component: if it is an annulus A and $A \cap P_f = \emptyset$.
iv) Complex component: if it is not in i), ii), and iii).

Since all elements of Γ_c are non-peripheral and non-homotopic to each other, $P_0^0, \ldots, P_n^0$ are all complex components.

For each $P_i^0 \in \mathfrak{P}_0$ and any k, since P_i^0 is a component of $\widehat{\mathbb{C}} \setminus A_0$ satisfying $1) - 5)$ above, there exists an unique component of $\mathfrak{P}_k$, denote by P_i^k such that each component of ∂P_i^k is either peripheral or some component of ∂P_i^0 and such that each component of ∂P_i^0 is some component of ∂P_i^k. This gives a relationship between P_i^0 and P_i^k. We use $P_i^0 \approx P_i^k$ to denote this relation. Therefore, P_i^k and $f(P_i^k)$ are complex components and, furthermore, if $f(P_i^k) = P_j^{k-1}$, then $f(P_i^l) = P_j^{l-1}$ for any l. Thus, by using the action of the branched covering f, for each P_i^0, we have a unique P_i^k such that $P_i^0 \approx P_i^k$ and then $f(P_i^k) = P_j^{k-1}$ and then a unique P_j^0 such that $P_j^0 \approx P_j^k$. Let $n = n_0$. This defines a map $\tau : \{0, 1, \ldots, n\} \to \{0, 1, \ldots, n\}$ such that $\tau(i) = j$ if $f(P_i^k) = P_j^{k-1}$.

Under the relationship $\approx$, each component of $\mathfrak{P}_0$ is eventually periodic and at least one is periodic. Thus we conclude that

Decomposition. *Suppose f is a post-critically finite or sub-hyperbolic semi-rational branched covering not combinatorially equivalent or CLH-equivalent to a rational map. Then f can be decomposed into thin parts and thick parts according to the canonical obstruction. Furthermore, an equivalent relationship can be introduced on thick parts such that every thick part is eventually periodic and at least one is periodic.*

Suppose $P_0^0 \in \mathfrak{P}_0$ is a periodic component of periodic $k > 0$. Suppose $\gamma_1, \ldots, \gamma_p$ are boundary curves of P_0^0 and $\gamma_1, \ldots, \gamma_p, \beta_1, \ldots, \beta_q$ are boundary curves of P_0^k where $\beta_j (j = 1, \ldots, q)$ are peripheral curves. For any β_j, it must be a component of $f^{-k}(\gamma_i)$ for some γ_i. For any γ_i, it must be a component of $f^{-k}(\gamma_l)$ for some γ_l. Denote

$$P_0^0 \setminus P_0^k = \bigcup_{j=1}^{q} D(\beta_j), \quad \overline{\mathbb{C}} \setminus P_0^0 = \bigcup_{i=1}^{p} D(\gamma_i).$$

Let

$$d_{\beta_j} = deg(f^k : \beta_j \to \gamma_i), \quad d_{\gamma_i} = deg(f^k : \gamma_i \to \gamma_l = f^k(\gamma_i)).$$

Define a new branched covering by

$$\widetilde{f} = \widetilde{f}_{P_0^0} = \begin{cases} f^k, & z \in P_0^k \\ \varphi_j \circ z^{d_{\beta_j}} \circ \psi_j, & z \in D(\beta_j) \quad (j = 1, \ldots, q) \\ \varphi_i \circ z^{d_{\gamma_i}} \circ \psi_i, & z \in D(\gamma_i) \quad (i = 1, \ldots, p) \end{cases} \tag{4}$$

where ψ_j, φ_j^{-1} are homeomorphisms from $D(\beta_j)$ and $D(\gamma_i)$ to unit disk $\mathbb{D}$, respectively, and ψ_i, φ_i^{-1} are homeomorphisms from $D(\gamma_i)$ and $D(\gamma_l) = D(f^k(\gamma_i))$ to unit disk $\mathbb{D}$, respectively, such that $\widetilde{f}$ is continuous. Marking a point in each $D(\gamma_i)$, say z_i, we set

$$P_{\widetilde{f}} = P_f|_{P_0^0} \cup (\bigcup_{j=1}^{q} z_j).$$

If $D(\beta_j)$ contains a point, say z^*, belonging to P_f and $f^k(\beta_j) = \gamma_i$, we can select φ_j, φ_i and ψ_j, ψ_i such that $\widetilde{f}(z^*) = z_i$. Also, if $f^k(\gamma_i) = \gamma_l$, we can select $\varphi_i, \varphi_l, \psi_i, \psi_l$ such that $\widetilde{f}(z_i) = z_l$. So $\widetilde{f}(P_{\widetilde{f}}) \subseteq P_{\widetilde{f}}$.

Extension. *Any periodic thick part P_i^0 of period k can be extended to a topological two-sphere with finitely many marked points and the restriction $f^k|P_i^k$ can be extended to either a critically finite type branched covering or a sub-hyperbolic semi-rational type branched covering $\widetilde{f} = \widetilde{f}_{P_i^0}$ such that*

$$\widetilde{f}|P_i^k = f^k|P_i^k.$$

Theorem 5.1 ([Selinger])**.** *Suppose f is a post-critically finite branched covering which is not combinatorially equivalent to a rational map. Then every $\widetilde{f}$ associated to a hyperbolic orbifold is combinatorially equivalent to a unique rational map (up to conjugation of an automorphism of the Riemann sphere).*

Remark 5.1. *Actually, the paper* [Selinger] *contains more. In the post-critically finite case, let $E = P_f$, then the Riemann surface $\widehat{\mathbb{C}} \setminus E$ is a punctured sphere with finitely many punctures. The Teichmüller space $T(E)$ is the same as the Teichmüller space $T(\widehat{\mathbb{C}} \setminus E)$ of Riemann surfaces with basepoint $\widehat{\mathbb{C}} \setminus E$. Thus, one can define its augmented Teichmüller space $\overline{T}(E)$, which is the space of all Teichmüller equivalent classes but allows an equivalent class with a representation of continuous maps from $\widehat{\mathbb{C}} \setminus E$ to a Riemann surface with nodes that are allowed to send a whole non-trivial closed annulus or a whole non-trivial closed curve in the complement of marked points to a node. From* [Masur]*, we know that the closure of the Teichmüller space $T(E)$ under the Weil-Petersson metric is its augmented Teichmüller space. Selinger proved that σ_f from $T(E)$ into itself can be extended to a continuous self map of $\overline{T}(E)$ such that it either has a fixed point in $T(E)$ or a fixed point in the boundary $\partial\overline{T}(E) = \overline{T}(E) \setminus T(E)$.*

Remark 5.2. *In* [Bonnot-Yampolsky]*, another approach is given to prove Theorem 5.1 by applying a result in* [Minsky] *and a result in* [Hassinsky].

In our latest study, we worked on the geometrization of a sub-hyperbolic semi-rational type branched covering as follows.

Theorem 5.2 ([Cheng-Jiang]). *Suppose f is a sub-hyperbolic semi-rational branched covering which is not CLH-equivalent to a rational map. Then every $\widetilde{f} = \widetilde{f}_{P_i^0}$ is either a post-critically finite type branched covering or a sub-hyperbolic semi-rational type branched covering. In the post-critically finite type case, if the orbifold associated to $\widetilde{f}$ is hyperbolic, then $\widetilde{f}$ is combinatorially equivalent to a rational map; in the sub-hyperbolic semi-rational type case, $\widetilde{f}$ is CLH-equivalent to a rational map. Moreover, in the both cases, the realized rational map is unique up to conjugation of an automorphism of the Riemann sphere.*

Unless in the post-critically finite case, in the sub-hyperbolic semi-rational branched covering case, we work on an infinite-dimensional Teichmüller space of the Riemann sphere minus a set of finitely many points and a set of finitely many topological disks. A major work in [Cheng-Jiang] is to overcome this difficulty. First, we proved the following proposition in [Cheng-Jiang], which is useful in general.

Proposition 4. *Suppose R_1 is a Riemann surface, which is the Riemann sphere minus a set E consisting of finite number of points and finite number of disks, with complex structure $\tau_1 = [\mu_1]$. Suppose $\gamma_1, \ldots, \gamma_n$ are non-peripheral non-homotopic simple closed curves on R_1 with $l_{\tau_1}(\gamma_i) < \varepsilon$ (ε sufficiently small). Let R_2 be a Riemann surface with complex structure $\tau_2 = [\mu_2]$ obtained from R_1 by cutting along γ_i and capping every hole by a puncture disk. If there exists a constant $K > 0$ such that for every non-peripheral simple closed curve β other than $\gamma_1, \ldots, \gamma_n$, $l_{\tau_1}(\beta) \geq K$, then there exists a constant $\widetilde{K} = \widetilde{K}(K, \varepsilon) > 0$ such that for every non-peripheral simple closed curve $\widetilde{\beta}$ of R_2, $l_{\tau_2}(\widetilde{\beta}) \geq \widetilde{K}$.*

Secondly, for any initial conformal structure $\tau_0 = [\mu_0]_E$ on the Riemann sphere, by using the decomposition and the extension, we have a conformal structure $\widetilde{\tau}_0 = [\widetilde{\mu}_0]$ which is μ_0 on P_i^k and 0 on other places. Then we have two iterated sequences $\{\tau_n = \sigma_f^n(\tau_0)\}$ and $\{\widetilde{\tau}_n = \sigma_{\widetilde{f}}^n(\tau_0)\}$. Compare these two iterated sequences and other three modified sequences of conformal structures (refer to [Cheng-Jiang, Proposition 5]) and then compare the geometry of punctured Riemann spheres with infinitely many punctures with the geometry of punctured Riemann spheres with finitely many punctures (refer to [Zhang-Jiang, Lemma 5.6] and [Cheng-Jiang, Lemma 5 and Proposition 4]). After we constructed these five different but related punctured Riemann spheres, we then used some method and idea in [Douady-Hubbard, Selinger] to complete the proof.

Remark 5.3. *From Theorem 5.2, we know* $\widetilde{f}$ *has bounded geometry as defined for* $\{\widetilde{\tau}_n\}$. *However, we are still interested in a direct proof of the bounded geometry property for* $\widetilde{f}$. *With this direct proof, we can have a proof of Theorem 5.2 using our framework in the previous section.*

Remark 5.4. *In the sub-hyperbolic semi-rational case,* E *defined in Equation (2) has finitely many points and finitely many disks. The Riemann surface* $\widehat{\mathbb{C}}\setminus E$ *is the Riemann sphere with finitely many points and finitely many disks removed. The Teichmüller space* $T(E)$ *is biholomorphically equivalent to the Teichmüller space* $T(\widehat{\mathbb{C}}\setminus E)$ *of Riemann surfaces with basepoint* $\widehat{\mathbb{C}}\setminus E$ *times the open unit ball* $M(E)$ *of the complex Banach space* $\mathcal{L}^\infty(E)$. *It is an interesting problem to understand how to study the augmented Teichmüller space and how to generalize the result in* [Masur] *for this case. After that we could study a full version of the result in* [Selinger] *in the sub-hyperbolic semi-rational case. One possibility for us is to define* $T_0(E)$ *as follows. For each disk component* D_i *of* E, *consider a ring* A_i *attaching to it such that* $D_i\cup A_i$ *is a bigger disk and such that* $\{D_i\cup A_i\}$ *and all point components of* E *are pairwise disjoint. Let* $\tilde{E}=E\cup\cup_i A_i$. *Consider* $M_0(\mathbb{C})=\{\mu\in M(\mathbb{C})\mid \mu|\tilde{E}=0\}$. *Define* $T_0(E)=\{[\mu]_E\mid \mu\in M_0(\mathbb{C})\}$ *as the space of all* E*-equivalence classes of elements in* $M_0(\mathbb{C})$. *We would like to understand the augmented Teichmüller space* $\overline{T}_0(E)$. *It is still an interesting project for us.*

Acknowledgement. I would like to thank referees for their very useful comments which I have included in this paper. I would like to thank all of my collaborators mentioned in this paper who spent a lot of time with me to discuss this interesting subject. This research is partially supported by the collaboration grant (#199837) from the Simons Foundation, the CUNY collaborative incentive research grant (#1861), and awards from PSC-CUNY. This research is also partially supported by the collaboration grant (#11171121) from the NSF of China and a collaboration grant from Academy of Mathematics and Systems Science and the Morningside Center of Mathematics at the Chinese Academy of Sciences.

Department of Mathematics, Queens College of CUNY, Flushing, NY 11367 and Department of Mathematics, CUNY Graduate School, New York, NY 10016 . Email: yunping.jiang@qc.cuny.edu

6. Transcendental

Tao Chen, Yunping Jiang, and Linda Keen

Dedicated to Professor John Milnor on his 80^{th} birthday

We believe that the framework described in the above sections also applies to the characterization problem for families of quasi-entire and quasi-meromorphic functions with certain finiteness properties. At present, we are interested in the characterization of meromorphic functions with a finite number of asymptotic values but no critical points and the characterization of general exponential functions of the form $P(z)e^{Q(z)}$, where $P(z)$ and $Q(z)$ are two polynomials.

More precisely, in [Chen-Jiang-Keen;1], we define two model spaces of maps we call $\mathcal{AV}2$ and $\mathcal{AV}3$. These are universal covering maps of the two-sphere with two or three removed points that we model on the space of meromorphic functions with two or three asymptotic values and no critical points. We call the spaces of meromorphic functions $\mathcal{M}2$ and $\mathcal{M}3$. Examples of elements in $\mathcal{M}2$ are the tangent map $\lambda \tan z$ and the exponential map $e^{\beta z}$. Nevanlinna [Nevanlinna], characterized meromorphic maps with p asymptotic values and no critical values as functions whose Schwarzian derivative is a polynomial of degree $p-2$. Thus, a meromorphic map in $\mathcal{M}2$ has constant Schwarzian derivative and a meromorphic map in $\mathcal{M}3$ has a linear function as its Schwarzian derivative. For a more detailed description, the reader is referred to our paper [Chen-Jiang-Keen;1],

The bounded geometry and canonical obstruction conditions can also be defined for post-singularly finite maps in $\mathcal{AV}2$ and $\mathcal{AV}3$. We are now working on a proof of the following statement that characterizes post-singularly finite maps in $\mathcal{M}2$ and $\mathcal{M}3$:

A post-singularly finite map in $\mathcal{AV}2$ or in $\mathcal{AV}3$ is combinatorially equivalent to a post-singularly finite transcendental meromorphic function in

$\mathcal{M}2$ or $\mathcal{M}3$ if and only if it has bounded geometry. The realization is unique up to conjugation by an affine map of the plane.

The proof follows the framework described in the previous sections. The new ingredient here is the addition of a topological constraint which we define in order to control the holomorphic maps obtained by the Thurston iteration process and show they remain in a compact subset of the parameter space.

In [Chen-Jiang-Keen;2], we study the characterization of entire post-singularly finite (p, q)-*exponential maps* of the form $E(z) = P(z)e^{Q(z)}$ with P and Q polynomials of degrees $p \geq 0$ and $q \geq 0$ respectively and $p + q \geq 1$. We use the notation $\mathcal{E}_{p,q}$ for this family. Again, we define a model space $\mathcal{T}E_{p,q}$ of *topological exponential maps of type* p, q. These are infinite degree branched coverings with a single finite asymptotic value, normalized to be at the origin, and modeled on the functions $E(z) = P(z)e^{Q(z)}$. For more detailed description, the reader is referred to our paper [Chen-Jiang-Keen;2].

The bounded geometry and canonical obstruction conditions can also be defined for post-singularly finite maps in $\mathcal{T}E_{p,q}$. We are now working on a proof of the following statement that characterizes post-singularly finite maps in $\mathcal{T}E_{p,q}$:

A post-singularly finite map in $\mathcal{T}E_{p,q}$ is combinatorially equivalent to a post-singularly finite entire map of the form Pe^Q if and only if it has bounded geometry.

Again the new ingredient is the addition of a topological constraint which we need in order to control the holomorphic maps obtained by the Thurston iteration process and show they remain in a compact subset of the parameter space.

The framework described in the above sections can also be extended to transcendental maps in another direction. We can define sub-hyperbolic semi-holomorphic topological exponential maps and sub-hyperbolic semi-holomorphic topological meromorphic maps with two asymptotic values (recall Definition 3.1). Some results have straightforward extensions to these maps and conjectures similar to those above also make sense in this context.

Our framework for rational functions involved the proof of the equivalence of statements (1)-(4) in the preceding sections. At present, for a post-singularly finite and post-critically finite map or a general map in $\mathcal{AV}2$, $\mathcal{AV}3$, or $\mathcal{T}E_{p,q}$, it is premature for us to make any statement about the equivalence between (1) with f as a transcendental function, and (4), that f has no Thurston obstruction. It is also premature for us to make any statement about the equivalence between (2) for a transcendental f with bounded geometry, and (4). The reason is the following: If E has infinitely many components, the definition of a Thurston obstruction in Section 2 may not make sense because when the degree of f is infinite, a non-peripheral closed curve γ on $\widehat{\mathbb{C}} \setminus E$, $f^{-1}(\gamma)$ may contain infinitely many non-peripheral closed curves. If E has finitely many components in the complex plane and ∞ is an isolated component, one may be able to define a Thurston obstruction formally, in a manner similar to that in Section 2, because even if the map f has infinite degree, all but finitely many preimages of a non-peripheral curve γ on $\widehat{\mathbb{C}} \setminus E$ are peripheral or null homotopic rel E. Even in this case, however, if one would like to prove (4) $\Rightarrow$ (1) by following the idea of the proof in [Douady-Hubbard], one needs first to find a new proof of Lemma 5.2 in [Douady-Hubbard] which is the key to the proof of Thurston's Theorem (Theorem 3.1) and whose proof crucially depends on the finiteness of the degree of f. Furthermore, in order to prove (4) $\Rightarrow$ (2) by following the ideas in this framework, one needs first to find a new proof of Theorem 7.1 in [Douady-Hubbard], which only works for maps of finite degree, and which plays an important role in our proof of (4) $\Rightarrow$ (2) in Theorems 4.1 and 4.2. Note that when a Thurston obstruction can be appropriately defined, (1) $\Rightarrow$ (4) can be proved just by following the proof in [Douady-Hubbard, Theorem 4.1] and the proof of Theorem 3.3 in [McMullen]; (2) $\Rightarrow$ (4) is currently being studied in [Chen-Jiang-Keen;1, Chen-Jiang-Keen;2].

In [Hubbard-Schleicher-Shishikura], the authors use a Levy cycle condition rather than a Thurston obstruction to characterize the non-existence of convergence. This is because, by the universal covering property of the map, every Thurston obstruction necessarily contains a Levy cycle. In fact, for general quasi-meromorphic and quasi-entire functions, a 'canonical Thurston obstruction' is more natural than a 'Levy cycle'. We are working on a characterization problem for f a post-singularly finite function in either $\mathcal{AV}2$, $\mathcal{AV}3$ or $\mathcal{TE}_{p,q}$ in terms of canonical Thurston obstructions. The delicate part of the problem is the proof that (3) $\Rightarrow$ (2) (see Problem 4).

Acknowledgement: We would like to thank the referee whose careful reading and suggestions have much improved it. The second and the third authors are partially supported by PSC-CUNY awards. The second author is also partially supported by the collaboration grant (#199837) from the Simons Foundation and the CUNY collaborative incentive research grant (#1861). This research is also partially supported by the collaboration grant (#11171121) from the NSF of China and a collaboration grant from Academy of Mathematics and Systems Science and the Morningside Center of Mathematics at the Chinese Academy of Sciences.

Tao Chen, Department of Mathematics, CUNY Graduate School, New York, NY 10016. Email: chentaofdh@gmail.com

Yunping Jiang, Department of Mathematics, Queens College of CUNY, Flushing, NY 11367 and Department of Mathematics, CUNY Graduate School, New York, NY 10016. Email: yunping.jiang@qc.cuny.edu

Linda Keen, Department of Mathematics, Lehman College of CUNY, Bronx, NY 10468 and Department of Mathematics, CUNY Graduate School, New York, NY 10016. Email: LINDA.KEEN@lehman.cuny.edu

References

[Ahlfors-Bers] L. V. Ahlfors and L. Bers, Riemann's mapping theorem for variable metrics. *Annals of Math. (2)* **72** (1960), 385-404. MR0115006 (22 #5813).

[Beck-Jiang-Mitra-Shiga] M. Beck, Y. Jiang, S. Mitra, and H. Shiga, Extending holomorphic motions and monodromy. Annales Academiæ Scientiarum Fennicæ Mathematica Vol. **37** (2012), 53-67. MR2920423.

[Bonnot-Yampolsky] S. Bonnot and M. Yampolsky, Geometrization of post-critically finite branched coverings. ArXiv: 1102.2247v2.

[Brown] D. Brown, Using spider theory to explore parameter spaces. Ph.D Dissertation, Cornell University 2001. MR2702308

[Chen-Jiang] T. Chen and Y. Jiang, Canonical Thurston obstructions for sub-hyperbolic semi-rational branched coverings. Conformal Geometry and Dynamics, Volume **17** (2013), 6-25.

[Chen-Jiang-Keen;1] T. Chen, Y. Jiang and L. Keen, Bounded geometry and families of meromorphic functions with two and three asymptotic values. Work in progress.

[Chen-Jiang-Keen;2] T. Chen, Y. Jiang, and L. Keen, Bounded geometry and characterization of post-singularly finite (p, q)-exponential maps. Work in progress.

[Cheng-Jiang] T. Cheng and Y. Jiang, Geometrization of sub-hyperbolic semi-rational branched coverings. arXiv:1207.1292v1

[Cui-Jiang] G. Cui and Y. Jiang, Geometrically finite and semi-rational branched coverings of the two-sphere. Transactions of the American Mathematical Society Vol. **363**, Number 5, May 2011, 2701-2714. MR2763733 (2012e:37090)

[Cui-Jiang-Sullivan;M] G. Cui, Y. Jiang and D. Sullivan, On geometrically finite branched covering maps. *Manuscript*, 1996.

[Cui-Jiang-Sullivan] G. Cui, Y. Jiang and D. Sullivan, On geometrically finite branched covering maps-I. Locally combinatorial attracting and II. Realization of rational maps (refer to its rewritten version [Cui-Tan]). Complex Dynamics and Related Topics, New Studies in Advanced Mathematics, 2004, The International Press, 1-14 & 15-29. MR2504307 (2011c:37092) & MR2504308 (2011c:37093)

[Cui-Tan] G. Cui and L. Tan, A characterization of hyperbolic rational maps. Invent Math. **183** (2011), 451-516. MR2772086 (2012c:37088)

[Douady-Hubbard] A. Douady and J. Hubbard, A proof of Thurston's topological characterization of rational functions. Acta. Math. **171** (1993), No. 2, 263-297. MR1251582 (94j:58143)

[Earle-McMullen] C. J. Earle and C. McMullen, Quasiconformal isotopies. Holomorphic Functions and Moduli, Vol. I (Berkeley, CA, 1986), 143154, Math. Sci. Res. Inst. Publ., 10, Springer, New York, 1988. MR0955816 (89h:30028)

[Earle-Mitra] C.J. Earle and S. Mitra, Variation of moduli by holomorphic motions. Contemp. Math. **256** (2000), 39-67. MR1759669 (2001f:30031)

[Epstein] A. Epstein, Towers of finite type complex analytic maps. Ph.D Dissertation, The CUNY Graduate Center, 1993. MR2690048

[Epstein-Keen-Tresser] A. Epstein, L. Keen, and C. Tresser, The set of maps $F_{a,b} :\mapsto x + a + \frac{b}{2\pi}\sin(2\pi x)$ with any given rotation interval is contractible. Commun. Math. Phys. **173** (1995), 313-333. MR1355627 (97e:58129)

[Gardiner-Jiang-Wang] F. Gardiner, Y. Jiang, and Z. Wang, Holomorphic motions and related topics. *Geometry of Riemann Surfaces*, London Mathematical Society Lecture Note Series No. 368, 2010, 156-193. MR2665009 (2011j:37087)

[Gardiner-Lakic] F. Gardiner and N. Lakic, Quasiconformal Teichmüller Theory. AMS, Providence, RI, 2000. MR1730906 (2001d:32016)

[Godillon] S. Godillon, Construction de fractions rationnelles à dynamique prescrite. Ph.D. Dissertation, Université de Cergy-Pontoise, Mai 2010.

[Hassinsky] P. Haissinsky, Deformation loaclisee de surfaces de Riemann. *Publ. Math.* Vol. **49** (2005), 249-255. MR2140209 (2006a:30047)

[Hinkkanen] A. Hinkkanen, Uniformly quasiregular semigroups in two dimensions. Annales Academiæ Scientiarum Fennicæ Mathematica, Volumen **21**, 1996, 205-222. MR1375517 (96m:30029)

[Hubbard] J. Hubbard, Teichmüler Theory and Applications to Geometry, Topology, and Dynamics, Volume I: Teichmüller theory. Matrix Edition, Ithaca, NY 14850. MR2245223 (2008k:30055)

[Hubbard-Schleicher-Shishikura] J. Hubbard, D. Schleicher, and M. Shishikura, Exponential Thurston maps and limits of quadratic differentials. Journal of the American Mathematical Society, Vol. **22** (2009), 77-117. MR2449055 (2010c:37100)

[Jiang-Mitra-Wang] Y. Jiang, S. Mitra, Z. Wang, Liftings of holomorphic maps into Teichmülller space. Kodai Mathematical Journal, **32** (2009), No. 3, 544-560. MR2582017 (2010m:32015)

[Jiang-Mitra] Y. Jiang and S. Mitra, Some applications of universal holomorphic motions. Kodai mathematical Journal Vol. **30**, No. 1 (2007), 85-96. MR2319079 (2008c:32019)

[Jiang-Mitra-Shiga] Y. Jiang, S. Mitra, H. Shiga, Quasiconformal motions and isomorphisms of continuous families of Mobius groups. Israel Journal of Mathematics, **188** (2012), 177-194.

[Levy] S. Levy, Critically finite rational maps. Ph. D Dissertation, Princeton University, 1985. MR2634168

[Lieb] G. Lieb, Holomorphic motions and Teichmüller space, Ph.D Dissertation, Cornell University 1990. MR2638376

[Masur] H. Masur, Extension of the Weil-Petersson metric to the boundary of Teichmüller space, Duke Math. J. **43** (1976), no. 3, 623-635. MR 0417456 (54 #5506).

[McMullen] C. McMullen, Complex dynamics and Renormalization. Ann. of Math. Studies, **79**, 1994. MR1312365 (96b:58097)

[Minsky] Y. Minsky, Extremal length estimates and product regions in Teichmüller space. *Duke Math. J.*, Vol. **83** (1996), 249-286. MR1390649 (97b:32019)

[Mitra] S. Mitra, Extensions of holomorphic motions. Israel Journal of Mathematics Volume **159**, Number 1, 277-288. MR2342482 (2009f:32025)

[Milnor] J. Milnor, Dynamics in One Complex Variable, Introductory Lectures. Vieweg, 2nd Edition, 2000. MR1721240 (2002i:37057)

[Nevanlinna] R. Nevanlinna, Analytic Functions. Translated from the second German edition by Phillip Emig. Die Grundlehren der mathematischen Wissenschaften, Band 162 Springer-Verlag, New York-Berlin 1970. MR0279280 (43 #5003)

[Pilgrim] K. M. Pilgrim, Cannonical Thurston obstruction. Advances in Mathematics, **158** (2001), 154-168. MR1822682 (2001m:57004)

[Rees] M. Rees, A partial description of the parameter space of rational maps of degree two, Part 2. Proc. London Math. Soc. (1995) s3-70(3): 644-690. MR1317518 (96g:58161)

[Selinger] N. Selinger, Thurston's pullback map on the argmented Teichmüller space and applications. Invent Math. DOI: 10.1007/s00222-011-0362-3.

[Sullivan;1] D. Sullivan, The ergodic theory at infinity of an arbitrary discrete group of hyperbolic motions. Riemann Surfaces and Related Topics, Proceedings of the 1978 Stony Brook Conference. Ann. of Math. Stud. **97** (1981), 465-496. MR0624833 (83f:58052)

[Sullivan;2] D. Sullivan, Conformal dynamical systems. Geometric dynamics, Lecture note in Math. 1007 (1983), Springer, Berlin, 725-752. MR0730296 (85m:58112)

[Sullivan;3] D. Sullivan, 11 lectures on Thurston's geometric theorem and Thurston's critical finite theorem #1-6 and #26 from 1981 to 1986. Einstein Chair Seminar Video Tape Collections, CUNY Graduate Center.

[Tan] L. Tan, Matings of quadratic polynomials. Ergod. Th. Dynam. Sys., **12** (1992), 589-620. MR1182664 (93h:58129)

[Thurston] W. Thurston, The combinatorics of iterated rational maps. Preprint Princeton University, Princeton, N.J. 1983.

[Tukia] P. Tukia, On two-dimensional quasiconformal groups. Ann. Acad. Sci. Fenn. Ser. A. I. Math. **5**, 1980, 73-78. MR0595178 (82c:30031)

[Wang] X. Wang, A decomposition theorem for Herman maps. arXiv:1203.5563v1.

[Zhang-Jiang] G. Zhang and Y. Jiang, Combinatorial characterization of sub-hyperbolic rational maps. Advances in Mathematics, **221** (2009), 1990-2018. MR2522834 (2010h:37095)

[Zhang;1] G. Zhang, Topological models of polynomials of simple Siegel disk type. Ph.D Dissertation, CUNY Graduate Center 2002.

[Zhang;2] G. Zhang, Dynamics of Siegel rational maps with prescribed combinatorics. arXiv:0811.3043v1.