Characterization of Meromorphic Functions with Two Asymptotic Values *

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Abstract

In this paper we study the topological class of universal covering maps from the plane to the sphere with two removed points; we call the elements topological transcendental maps with two asymptotic values and we denote the space by $\mathcal{AV}2$. The space of meromorphic transcendental maps in $\mathcal{AV}2$ is denoted as $\mathcal{M}2$, which are all meromorphic transcendental functions with constant Schwarzian derivative. We follow the framework given in [Ji] to study the characterization of a map in $\mathcal{M}2$. We prove in this paper that for an element $f \in \mathcal{AV}2$ with finite post-singular set, the following statements are equivalent: (1) f is combinatorially equivalent to a meromorphic transcendental map g in $\mathcal{M}2$; (2) f has bounded geometry; (3) f has no Thurston obstruction; (4) f has no Levy cycle; (5) The canonical Thurston obstruction is empty.

1 Introduction

Thurston asked the question "when can we realize a given branched covering map as a holomorphic map in such a way that the post-critical sets correspond?" and answered it for post-critically finite degree d branched covers of the sphere [T, DH]. His theorem is that a postcritically finite degree $d \geq 2$ branched covering of the sphere, with hyperbolic orbifold, is either combinatorially equivalent to a rational map or there is a topological obstruction, now called a "Thurston obstruction". The rational map is unique up to conjugation by a Möbius transformation.

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Thurston's theorem is proved by defining an appropriate Teichmüller space of rational maps and a holomorphic self map of this space. Iteration of this map converges if and only if no Thurston obstruction exists. This method does not naturally extend to transcendental maps because the proof uses the finiteness of both the degree and the post-critical set in a crucial way. Hubbard, Schleicher, and Shishikura [HSS] generalized Thurston's theorem to a special infinite degree family they call "exponential type" maps. In that paper, the authors study the limiting behavior of quadratic differentials associated to the exponential functions with finite post-singular set. They use a Levy cycle condition (a special case of Thurston's topological condition) to characterize when it is possible to realize a given exponential type map with finite post-singular set as an exponential map by combinatorial equivalence. The main purpose of this paper is to use a different approach based on the framework expounded in [Ji] (see also [JZ, CJ]) to understand the characterization problem for a generalization of this family of infinite degree maps.

In this paper we define a class of maps called topological transcendental maps with two asymptotic values which we denote by $\mathcal{AV}2$. The elements in this class are universal covering maps from the plane to the sphere with two removed points. A meromorphic transcendental map g in $\mathcal{AV}2$ is a meromorphic function with constant Schwarzian derivative and we denote the space of all meromorphic functions with constant Schwarzian derivative by $\mathcal{M}2$. Our main result in this paper gives a full characterization of an $f \in \mathcal{AV}2$ combinatorially equivalent to a map $g \in \mathcal{M}2$.

Theorem 1 (Main Theorem). *Suppose $f \in \mathcal{AV}2$ is a post-singularly finite map. Then the following statements are equivalent:*

(1) f in $\mathcal{AV}2$ is combinatorially equivalent to a unique map g in $\mathcal{M}2$ (up to automorphisms of the Riemann sphere);

(2) f has bounded geometry;

(3) f has no Thurston obstruction;

(4) f has no Levy cycle;

(5) f has no canonical Thurston onstruction.

Our techniques involve adapting the Thurston iteration scheme to our situation. We work with a fixed normalization. There are two important parts to the proof of the main theorem. The first part is to prove that the bounded geometry condition implies the iterates remain in a compact subset of the Teichmüller space. This analysis depends on defining a topological condition that constrains the iterates. The second part is to use the compactness of the iterates to prove that the iteration scheme converges in the Teichmüller space. This part of the proof involves an analysis of quadratic differentials associated to our functions.

The paper is organized as follows. In §2 we review the properties of meromorphic functions with two asymptotic values that constitute the space $\mathcal{M}_2$. In §3, we define the family $\mathcal{AV}2$ that consists of topological maps modeled on maps in $\mathcal{M}_2$ and show that $\mathcal{M}_2 \subset \mathcal{AV}2$. In §4 we define *combinatorial equivalence* between maps in $\mathcal{AV}2$ and in §5 define the Teichmüller space T_f for a map $f \in \mathcal{AV}2$. In §6, we introduce the induced map σ_f from the Teichmüller space T_f into itself; this is the map that defines the Thurston iteration scheme. In §7, we define the concept of *bounded geometry* and in §8 we prove the necessity of the bounded geometry condition in the main theorem. In §9, we give the proof of the sufficiency assuming the iterates remain in a compact subset of T_f. In §10.1, we define a topological property of the post-singularly finite map f in $\mathcal{AV}2$ in terms of the winding number of a certain closed curve. We prove that the winding number is unchanged during iteration of the map σ_f and so provides a topological constraint on the iterates. Finally, in §10.2, we show how the bounded geometry condition together with this topological constraint implies the functions remain in a compact subset of T_f under the iteration to complete the proof of the main theorem.

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2 The Space $\mathcal{M}_2$

In this section we define the space of meromorphic functions $\mathcal{M}_2$. It is the model for the more general space of topological functions $\mathcal{AV}2$ that we define in the next section. We need some standard notation and definitions:

$\mathbb{C}$ is the complex plane, $\hat{\mathbb{C}}$ is the Riemann sphere and $\mathbb{C}^*$ is complex plane punctured at the origin.

Definition 1. *Given a meromorphic function g, the point v is a* logarithmic singularity for the map g^{-1} *if there is a neighborhood U_v and a component V of $g^{-1}(U_v \setminus \{v\})$ such that the map $g : V \to U_v \setminus \{v\}$ is a holomorphic universal covering map. The point v is also called an* asymptotic value *for g and V is called an* asymptotic tract for g. *A point may be an asymptotic value for more than one asymptotic tract. An asymptotic value may be an omitted value.*

Definition 2. *Given a meromorphic function g, the point v is an* algebraic singularity for the map g^{-1} *if there is a neighborhood U_v such that for every component V_i of $g^{-1}(U_v)$ the map $g : V_i \to U_v$ is a degree d_{V_i} branched covering map and $d_{V_i} > 1$ for finitely many components $V_1, \dots V_n$. For these components, if $c_i \in V_i$ satisfies $g(c_i) = v$ then $g'(c_i) = 0$; that is c_i is a* critical point of g *for $i = 1, \dots, n$ and v is a* critical value*.*

Note that by a theorem of Hurwitz, if a meromorphic function is not a homeomorphism, it must have at least two singular points (i.e., critical points and asymptotical values) and, by the big Picard theorem, no transcendental meromorphic function $g : \mathbb{C} \to \hat{\mathbb{C}}$ can omit more than two values.

The space $\mathcal{M}_2$ consists of meromorphic functions whose only singular values are its omitted values. More precisely,

Definition 3. *The space $\mathcal{M}_2$ consists of meromorphic functions $g : \mathbb{C} \to \hat{\mathbb{C}}$ with exactly two asymptotic values and no critical values.*

2.1 Examples

Examples of functions in $\mathcal{M}_2$ are the exponential functions $\alpha e^{\beta z}$ and the tangent functions $\alpha \tan i\beta z = i\alpha \tanh \beta z$ where α, β are complex constants.

The asymptotic values for the exponential functions above are $\{0, \infty\}$; the half plane $\Re \beta z < 0$ is an asymptotic tract for 0 and the half plane $\Re \beta z > 0$ is an asymptotic tract for infinity. The asymptotic values for the tangent functions above are $\{\alpha i, -\alpha i\}$ and the asymptotic tract for αi is the half plane $\Im \beta z > 0$ while the asymptotic tract for $-\alpha i$ is the half plane $\Im \beta z < 0$.

2.2 Nevanlinna's Theorem

To find the form of the most general function in $\mathcal{M}_2$ we use a special case of a theorem of Nevanlinna [N].

Theorem 2 (Nevanlinna)**.** *Every meromorphic function g with exactly p asymptotic values and no critical values has the property that its Schwarzian derivative is a polynomial of degree $p - 2$. That is*

$$S(g) = \left(\frac{g''}{g'}\right)' - \frac{1}{2}\left(\frac{g''}{g'}\right)^2 = a_{p-2}z^{p-2} + \dots a_1 z + a_0. \tag{1}$$

Conversely, for every polynomial $P(z)$ of degree $p-2$, the solution to the Schwarzian differential equation $S(g) = P(z)$ is a meromorphic function with p asymptotic values and no critical values.

It is easy to check that $S(\alpha e^{\beta z}) = -\frac{1}{2}\beta^2$ and $S(\alpha \tan \frac{i\beta}{2} z) = -\frac{1}{2}\beta^2$.

To find all functions in $\mathcal{M}_2$, let $\beta \in \mathbb{C}$ be constant and consider the Schwarzian differential equation

$$S(g) = -\beta^2/2 \tag{2}$$

and the related second order linear differential equation

$$w'' + \frac{1}{2}S(g)w = w'' - \frac{\beta^2}{4}w = 0. \tag{3}$$

It is straightforward to check that if w_1, w_2 are linearly independent solutions to equation (3), then $g_\beta = w_2/w_1$ is a solution to equation (2).

Normalizing so that $w_1(0) = 1, w_1'(0) = -1, w_2(0) = 1, w_2'(0) = 1$ and solving equation (3), we have $w_1 = e^{-\frac{\beta}{2}z}, w_2 = e^{\frac{\beta}{2}z}$ as linearly independent solutions and $g_\beta(z) = e^{\beta z}$ as the solution to equation (2). An arbitrary solution to equation (2) then has the form

$$\frac{Aw_2 + Bw_1}{Cw_2 + Dw_1},\ A, B, C, D \in \hat{\mathbb{C}},\ AD - BC = 1 \tag{4}$$

and its asymptotic values are $\{A/C, B/D\}$.

Remark 1. *The asymptotic values are distinct and omitted.*

Remark 2. *If $B = C = 0, AD = 1, A = \sqrt{\alpha}$ we obtain the exponential family $\{\alpha e^{\beta z}\}$ with asymptotic values at 0 and ∞. If $A = -B = \sqrt{\frac{\alpha i}{2}}$, $C = D = \sqrt{-\frac{i}{2\alpha}}$ we obtain the tangent family $\{\alpha \tan \frac{i\beta}{2} z\}$ whose asymptotic values $\{\pm \alpha i\}$ are symmetric with respect to the origin.*

Remark 3. *Note that in the solutions of $S(g) = -\beta^2/2$ what appears is e^β, not β; this creates an ambiguity about which branches of the logarithm of e^β correspond to the solution of equation (2). In section 9.2.2 we address this ambiguity in our situation. We show that the topological map we start with determines a topological constraint which in turn, defines the appropriate branch of the logarithm for each of the iterates in our iteration scheme.*

Remark 4. *One of the basic features of the Schwarzian derivative is that it satisfies the following cocycle relation: if f, g are meromorphic functions then*

$$S(g \circ f(z)) = S(g(f))f'(z)^2 + S(f(z)).$$

In particular, if T is a Möbius transformation, $S(T(z)) = 0$ and $S(T \circ g(z)) = S((g(z))$ so that post-composing by T doesn't change the Schwarzian.

In our dynamical problems the point at infinity plays a special role and the dynamics are invariant under post-composition by an affine map. Thus, we may assume that all the solutions have one asymptotic value at 0 and that they take the value 1 at 0.

Since this is true for $g_\beta(z) = e^{\beta z}$, any solution with this normalization has the form [1]

$$g_{\alpha,\beta}(z) = \frac{\alpha g_\beta(z)}{(\alpha - \frac{1}{\alpha})g_\beta(z) + \frac{1}{\alpha}} \tag{5}$$

where α is an arbitrary value in $\mathbb{C}^*$. The second asymptotic value is $\lambda = \frac{\alpha}{\alpha - \frac{1}{\alpha}}$. It takes values in $\mathbb{C} \setminus \{0, 1\}$. The point at infinity is an essential singularity for all these functions.

The parameter space $\mathcal{P}$ for these functions is the two complex dimensional space

$$\mathcal{P} = \{\alpha, \beta \in \mathbb{C}^*\}.$$

The parameters define a natural complex structure for the space $\mathcal{M}_2$. The subspace of *entire* functions in $\mathcal{M}_2$ is the one dimensional subspace of $\mathcal{P}$ defined by fixing $\alpha = 1$ and varying β;

$$g_\beta(z) = e^{\beta z}.$$

The tangent family has symmetric asymptotic values. Renormalized, it forms another one dimensional subspace of $\mathcal{P}$. This is defined by fixing $\alpha = \sqrt{2}$ and varying β;

$$g_{\frac{1}{\sqrt{2}},\beta}(z) = 1 + \tanh\frac{\beta}{2}z = \frac{\sqrt{2}e^{\beta z}}{\frac{1}{\sqrt{2}}e^{\beta z} + \frac{1}{\sqrt{2}}}.$$

These functions have asymptotic values at $\{0, 2\}$ and $g_{\frac{1}{\sqrt{2}},\beta}(0) = 1$.

Definition 4. *For $g_{\alpha,\beta}(z) \in \mathcal{M}_2$, the set $\Omega = \{0, \lambda\}$ of asymptotic values is the set of singular values. The* post-singular set *P_g is defined by*

$$P_g = \overline{\bigcup_{x \in \Omega} \cup_{n\geq 0} g^n(x)} \cup \{\infty\}.$$

Note that we include the point at infinity separately in P_g because whether or not it is an asymptotic value, it is an essential singularity and its forward orbit is not defined. The asymptotic values are in P_g and, since 0 and λ are omitted and $g_{\alpha,\beta}(0) = 1 \in P_g$, $\#P_g \geq 3$.

[1] Notice that $g_{\alpha,\beta}$ is obtained from g_β by a Möbius transformation with determinant 1.

3 The Space $\mathcal{AV}2$

We now want to consider the topological structure of functions in $\mathcal{M}_2$ and define $\mathcal{AV}2$ to be the set of maps with the same topology.

Definition 5. *Let X be a simply connected open surface and let S^2 be the 2-sphere. Let $f_{a,b} : X \to S^2 \setminus \{a, b\}$ be an unbranched covering map; that is, a universal covering map. If Y is also a simply connected open surface we say the pair $(X, f^1_{a,b})$ is equivalent to the pair $(Y, f^2_{c,d})$ if and only if there is a homeomorphism $h : X \to Y$ such that $f^2_{c,d} \circ h = f^1_{a,b}$. An equivalence class of such classes is called a* 2-asymptotic value map *and the space of these pairs is denoted by $\mathcal{AV}2$.*

Let $(X, f_{a,b})$ be a representative of a map in $\mathcal{AV}2$. By abuse of notation, we will often suppress the dependence on the equivalence class and identify X with $S^2 \setminus \{\infty\}$ and refer to $f_{a,b}$ as an element of $\mathcal{AV}2$.

By definition $f_{a,b}$ is a local homeomorphism and satisfies the following conditions:

For $v = a$ or $v = b$, let $U_v \subset X$ be a neighborhood of v whose boundary is a simple closed curve that separates a from b and contains v in its interior.

1. $f^{-1}_{a,b}(U_v \setminus \{v\})$ is connected and simply connected.
2. The restriction $f_{a,b} : f^{-1}_{a,b}(U_v \setminus \{v\}) \to (U_v \setminus \{v\})$ is a regular covering of a punctured topological disk whose degree is infinite.
3. $f^{-1}(\partial U_v)$ is an open curve extending to infinity in both directions.

In analogy with meromorphic functions we say

Definition 6. *v is called a* logarithmic singularity *of $f^{-1}_{a,b}$ or, equivalently, an* asymptotic value *of $f_{a,b}$. The domain $V_v = f^{-1}_{a,b}(U_v \setminus \{v\})$ is called an* asymptotic tract for *v.*

Definition 7. $\Omega_f = \{a, b\}$ is the set of singular values of $f_{a,b}$.

Endow S^2 with the standard complex structure so that it is identified with $\hat{\mathbb{C}}$. By the classical uniformization theorem, for any pair $(X, f_{a,b})$, there is a map $\pi : \mathbb{C} \to X$ such that $g_{a,b} = f_{a,b} \circ \pi$ is meromorphic. It is called *the meromorphic function associated to $f_{a,b}$*.

By Nevanlinna's theorem $S(g(z))$ is constant and moreover,

Proposition 1. *If $g(z) \in \mathcal{M}_2$ with $\Omega_g = \{a, b\}$ then $g(z) = g_{a,b}(z) \in \mathcal{AV}2$ and, conversely, if $g_{a,b} \in \mathcal{AV}2$ is meromorphic then $g_{a,b} \in \mathcal{M}_2$.*

Proof. Any $g(z) \in \mathcal{M}_2$ is a universal cover $g : \mathbb{C} \to \hat{\mathbb{C}} \setminus \Omega_g$ and so belongs to $\mathcal{AV}2$. Conversely, if $g_{a,b} \in \mathcal{AV}2$, it is meromorphic and its only singular values are the omitted values $\{a, b\}$; it is thus in $\mathcal{M}_2$. □

We define the post-singular set for functions in $\mathcal{AV}2$ just as we did for functions in $\mathcal{M}_2$.

Definition 8. *For $f = f_{a,b} \in \mathcal{AV}2$, the* post-singular set P_f *is defined by*

$$P_f = \overline{\bigcup_{n \geq 0} f^n(\Omega_f)} \cup \{\infty\}$$

Note that under the identification of S^2 with the Riemann sphere and X with the complex plane, $S^2 \setminus X$ is the point at infinity and it has no forward orbit although it may be an asymptotic value. We therefore include it in P_f.

Post-composition of $f_{a,b}$ with an affine transformation T results in another map in $\mathcal{AV}2$. In what follows, therefore, we will always assume $a = 0$ and the second asymptotic value, λ, depending on T and b, is determined by the condition $f(0) = 1$.

We will be concerned only with functions in $\mathcal{AV}2$ such that P_f is finite. Such functions are called *post-singularly finite*.

4 Combinatorial Equivalence

In this section we define combinatorial equivalence for functions in $\mathcal{AV}2$. Choosing representatives $(X, f_{a,b})$ of the $\mathcal{AV}2$-equivalence classes, we may assume X is always $S^2 \setminus \{\infty\}$ and $\{0, b\}$ are the singular points for all the functions so we will omit the subscripts denoting the omitted points in the definitions below.

Definition 9. *Suppose $(X, f_1), (X, f_2)$ are representatives of two post-singularly finite functions in $\mathcal{AV}2$, chosen as above. We say that they are* combinatorially equivalent *if there are two homeomorphisms ϕ and ψ of S^2 onto itself fixing $\{0, \infty\}$ such that $\phi \circ f_2 = f_1 \circ \psi$ on X and $\phi^{-1} \circ \psi$ is isotopic to the identity of S^2 rel P_{f_1}.*

The commutative diagram for the above definition is

$$\begin{tikzcd} X \arrow[r, "\psi"] \arrow[d, "f_1"] & X \arrow[d, "f_2"] \\ S^2 \arrow[r, "\phi"] & S^2 \end{tikzcd}$$

5 Teichmüller Space T_f.

Let $\mathbf{M} = \{\mu \in L^\infty(\hat{\mathbb{C}}) \mid \|\mu\|_\infty < 1\}$ be the unit ball in the space of all measurable functions on the Riemann sphere. Each element $\mu \in \mathbf{M}$ is called a Beltrami coefficient. For each Beltrami coefficient μ, the Beltrami equation,

$$w_{\bar{z}} = \mu w_z$$

has a unique quasiconformal solution w^μ which maps $\hat{\mathbb{C}}$ to itself and fixes $0, 1, \infty$. Moreover, w^μ depends holomorphically on μ.

Let f be a post-singularly finite function in $\mathcal{AV}2$ with singular set $\Omega_f = \{0, \lambda\}$ and postsingular set P_f. By definition, we have $\#(\Omega_f) = 2$ and $\#(P_f) > 2$. Since post-composition by an affine map is in the equivalence class of f we may always choose a representative such that $\{f(0) = 1\} \subset P_f$; we assume we have always made this choice. It follows that $\lambda \neq 1$ so we always have $\{0, 1, \lambda, \infty\} \subset P_f$.

The Teichmüller space $T(P_f)$ is defined as follows. Given Beltrami differentials $\mu, \nu \in \mathbf{M}$ we say that μ and ν are equivalent, and denote this by $\mu \sim \nu$, if w^μ and w^ν fix $0, 1, \infty$ and $(w^\mu)^{-1} \circ w^\nu$ is isotopic to the identity map of $\hat{\mathbb{C}}$ rel P_f. We set $T_f = T(P_f) = \mathbf{M}/\sim = \{[\mu]\}$.

There is an obvious isomorphism between T_f and the classical Teichmüller space $Teich(R)$ of Riemann surfaces with basepoint $R = \hat{\mathbb{C}} \setminus P_f$. It follows that T_f is a finite-dimensional complex manifold so that the Teichmüller distance d_T and the Kobayashi distance d_K on T_f coincide. It also follows that there are always locally quasiconformal maps in the equivalence class of f; we always assume we have chosen one such as our representative.

6 Induced Holomorphic Map σ_f.

For any post-singularly finite f in $\mathcal{AV}2$, there is an induced map $\sigma = \sigma_f$ from T_f into itself given by:

$$\sigma([\mu]) = [f^*\mu],$$

where

$$\tilde{\mu}(z) = f^*\mu(z) = \frac{\mu_f(z) + \mu((f(z))\theta(z)}{1 + \overline{\mu_f(z)}\mu(f(z))\theta(z)}, \quad \mu_f = \frac{f_{\bar{z}}}{f_z}, \quad \theta(z) = \frac{\overline{f_z}}{f_z}. \tag{6}$$

Because σ is a holomorphic map we have

Lemma 1. *For any two points τ and τ' in T_f,*

$$d_T\Big(\sigma(\tau), \sigma(\tau')\Big) \leq d_T(\tau, \tau').$$

The next lemma follows directly from the definitions.

Lemma 2. *A post-singularly finite f in $\mathcal{AV}2$ is combinatorially equivalent to a meromorphic map in $\mathcal{M}_2$ if and only if σ has a fixed point in T_f.*

Remark 5. *If $\#(P_f) = 3$, then T_f consists of a single point. This point is trivially a fixed point for σ so the main theorem holds. We therefore assume that $\#(P_f) \geq 4$ in the rest of the paper.*

7 Bounded Geometry.

Let the base point of T_f be the hyperbolic Riemann surface $R = \hat{\mathbb{C}} \backslash P_f$ equipped with the standard complex structure $[0] \in T_f$. For τ in T_f, denote by R_τ the hyperbolic Riemann surface R equipped with the complex structure τ.

A simple closed curve $\gamma \subset R$ is called non-peripheral if each component of $\hat{\mathbb{C}} \setminus \gamma$ contains at least two points of P_f. Let γ be a non-peripheral simple closed curve in R. For any $\tau = [\mu] \in T_f$, let $l_\tau(\gamma)$ be the hyperbolic length of the unique closed geodesic homotopic to γ in R_τ.

For any $\tau_0 \in T_f$, let $\tau_n = \sigma^n(\tau_0)$, $n \geq 1$.

Definition 10 (Hyperbolic version)**.** *We say f has bounded geometry if there is a constant $a > 0$ and a point $\tau_0 \in T_f$ such that $l_{\tau_n}(\gamma) \geq a$ for all $n \geq 0$ and all non-peripheral simple closed curves γ in R.*

The iteration sequence $\tau_n = \sigma_f^n \tau_0 = [\mu_n]$ determines a sequence of subsets of $\hat{\mathbb{C}}$

$$P_n = w^{\mu_n}(P_f), \quad n = 0, 1, 2, \cdots .$$

Since all the maps w^{μ_n} fix $0, 1, \infty$, it follows that $0, 1, \infty \in P_n$.

Definition 11 (Spherical Version)**.** *We say f has bounded geometry if there is a constant $b > 0$ and a point $\tau_0 \in T_f$ such that*

$$d_{sp}(p_n, q_n) \geq b$$

for all $n \geq 0$ and $p_n, q_n \in P_n$, where

$$d_{sp}(z, z') = \frac{|z - z'|}{\sqrt{1 + |z|^2}\sqrt{1 + |z'|^2}}$$

is the spherical distance on $\hat{\mathbb{C}}$.

Note that $d_{sp}(z,\infty) = \frac{|z|}{\sqrt{1+|z|^2}}$. Away from infinity the spherical metric and Euclidean metrics are equivalent. Precisely, in any bounded $K \subset \mathbb{C}$, there is a constant $C > 0$ which depends only on K such that

$$C^{-1} d_{sp}(x,y) \leq |x-y| \leq C d_{sp}(x,y) \quad \forall x, y \in K.$$

The following simple lemma justifies using the term "bounded geometry" in both of the definitions above for f.

Lemma 3. *Consider the hyperbolic Riemann surface $\hat{\mathbb{C}} \setminus X$ equipped with the standard complex structure where X is a finite subset such that $0, 1, \infty \in X$. Let $m = \#(X) \geq 3$. Let $a > 0$ be a constant. If every simple closed geodesic in $\hat{\mathbb{C}} \setminus S$ has hyperbolic length greater than a, then there is a constant $b = b(a,m) > 0$ such that the spherical distance between any two distinct points in S is bounded below by b.*

Proof. If $m = 3$ there are no non-peripheral simple closed curves so in the following argument we always assume that $m \geq 4$. Let $X = \{x_1, \cdots, x_{m-1}\}$ and $x_m = \infty$ and let $|\cdot|$ denote the Euclidean metric on $\mathbb{C}$.

Suppose $0 = |x_1| \leq \cdots \leq |x_{m-1}|$. Let $M = |x_{m-1}|$. Then $|x_2| \leq 1$, and we have

$$\prod_{2 \leq i \leq m-2} \frac{|x_{i+1}|}{|x_i|} = \frac{|x_{m-1}|}{|x_2|} \geq M.$$

Hence

$$\max_{2 \leq i \leq m-2} \left\{ \frac{|x_{i+1}|}{|x_i|} \right\} \geq M^{\frac{1}{m-3}}.$$

Let

$$A_i = \{ z \in \mathbb{C} \;\Big|\; |x_i| < z < |x_{i+1}| \}$$

and let $\mathrm{mod}(A_i) = \frac{1}{2\pi} \log \frac{|x_{i+1}|}{|x_i|}$ be its modulus. Then for some integer $2 \leq i_0 \leq m_0 - 2$ if follows that

$$\mathrm{mod}\ (A_{i_0}) \geq \frac{\log M}{2\pi(m-3)}.$$

Denote the minimum length of closed curves γ in A_{i_0}, measured with respect to the hyperbolic metric on A_{i_0}, by $\|\gamma\|_{A_{i_0}}$. Because A_{i_0} is a round annulus, the core curve realizes this minimum and we can compute its hyperbolic length as $\|\gamma\|_{A_{i_0}} = \frac{\pi}{\mathrm{mod}\ (A_{i_0})}$.

Since $A_{i_0} \subset \hat{\mathbb{C}} \setminus S$, the hyperbolic density on A_{i_0} is smaller than the hyperbolic density on $\hat{\mathbb{C}} \setminus S$. Therefore, if $l_{\tau_n}(\gamma)$ denotes the length of the shortest geodesic in the homotopy class of γ with respect to the hyperbolic metric on $\hat{\mathbb{C}} \setminus S$, we have $l_{\tau_n}(\gamma) \leq \|\gamma\|_{A_{i_0}}$. This implies that

$$\frac{\pi}{l_{\tau_n}(\gamma)} \geq \quad \text{mod } (A_{i_0}) \geq \frac{\log M}{2\pi(m-3)}.$$

Thus

$$\log M \leq \frac{2\pi^2(m-3)}{l_{\tau_n}} \leq \frac{2\pi^2(m-3)}{a}.$$

This implies that

$$M \leq M_0 = e^{\frac{2\pi^2(m-3)}{a}}.$$

Thus the spherical distance between ∞ and any finite point in X has a positive lower bound M_0 which depends only on a and m.

Next we show that the spherical distance between any two finite points in X has a positive lower bound dependent only on a and m. By the equivalence of the spherical and Euclidean metrics in a bounded set in the plane, it suffices to prove that $|x-y|$ is greater than a constant b for any two finite points in X.

First consider $J_0(z) = 1/z$ which preserves hyperbolic length with $0, 1, \infty \in J_0(X)$. The above argument implies that $1/|x_i| \leq M_0$ for any $2 \leq i \leq m-1$. This implies that $|x_i| \geq 1/M_0$ for any $2 \leq i \leq m-1$. Similarly, for any $x_i \in X$ for $2 \leq i \leq m-1$, consider $J_i(z) = z/(z-x_i)$ which again preserves hyperbolic length. The above argument implies that $|x_j/|x_j - x_i| \leq M_0$ for any $2 \leq j \neq i \leq m-1$. This in turn implies that $|x_j - x_i| \geq 1/M_0^2$ for any $2 \leq j \neq i \leq m-1$ which proves the lemma. $\square$

8 The Canonical Thurston Obstruction.

If f has no bounded geometry, then for any $\tau_0 \in T_f$, we have a sequence $\{n_i\}_{i=1}^{\infty}$ of positive integers and a sequence of non-peripheral simple closed curves in R such that

$$l_{\tau_{n_i}}(\gamma_{n_i}) \to 0 \quad \text{as} \quad i \to \infty.$$

Now we consider the set

$$\Gamma_c = \{\gamma \subset R \mid l_{\tau_n}(\gamma) \to 0 \text{ as } n \to \infty\}$$

where γ is a non-peripheral simple curve.

Definition 12. *If $\Gamma_c \neq \emptyset$, we called it the canonical Thurston obstruction. Otherwise, we say f has no canonical Thurston obstruction.*

9 The proof of $(1) \Longleftrightarrow (2)$ in Theorem 1.

9.1 Proof of $(1) \Longrightarrow (2)$.

If f is combinatorially equivalent to $g \in \mathcal{M}2$, then σ_f has a unique fixed point τ_0 so that $\tau_n = \tau_0$ for all n. The complex structure on $\hat{\mathbb{C}} \setminus P_f$ defined by τ_0 induces a hyperbolic metric on it. The shortest geodesic in this metric gives a lower bound on the lengths of all geodesics so that f satisfies the hyperbolic definition of bounded geometry.

9.2 Proof of $(1) \Longleftarrow (2)$.

9.2.1 Proof assuming compactness.

The proof of of our main theorem (Theorem 1) is more complicated and needs some preparatory material. There are two parts: one is a compactness argument and the other is a fixed point argument. From a conceptual point of view, the compactness of the iterates is very natural and simple. From a technical point of view, however, it is not at all obvious. Once one has compactness, the proof of the fixed point argument is quite standard (see [Ji]) and works for much more general cases. We postpone the compactness proof to the next two sections and here give the fixed point argument.

Given $f \in \mathcal{AV}2$ and given any $\tau_0 = [\mu_0] \in T_f$, let $\tau_n = \sigma^n(\tau_0) = [\mu_n]$ be the sequence generated by σ. Let w^{μ_n} be the normalized quasiconformal map with Beltrami coefficient μ_n. Then

$$g_n = w^{\mu_n} \circ f \circ (w^{\mu_{n+1}})^{-1} \in \mathcal{M}2$$

since it preserves μ_0 and hence is holomorphic. Thus iterating σ, the "Thurston iteration", determines a sequence $\{g_n\}_{n=0}^{\infty}$ of maps in $\mathcal{M}2$ and a sequence of subsets $P_{f,n} = w^{\mu_n}(P_f)$. Note that $P_{f,n}$ is not, in general, the post-singular set P_{g_n} of g_n. If it were, we would have a fixed point of σ.

Suppose f is a post-singularly finite map in $\mathcal{AV}2$. For any $\tau = [\mu] \in T_f$, w^μ denotes a representative normalized quasiconformal map fixing $0, 1, \infty$ with Beltrami differential μ; let T_τ and T^*_τ denote the respective tangent space and cotangent space of T_f at τ. Then T^*_τ coincides with the space $\mathcal{Q}_\mu$ of integrable meromorphic quadratic differentials $q = \phi(z)dz^2$ on $\hat{\mathbb{C}}$. Integrability means that the norm of q, defined as

$$||q|| = \int_{\hat{\mathbb{C}}} |\phi(z)| dz d\overline{z}$$

is finite. The finiteness implies that q may only have poles at points of $w^\mu(P_f)$ and that these poles are simple.

Set $\tilde{\tau} = \sigma(\tau) = [\tilde{\mu}]$. By abuse of notation, we write $f^{-1}(P_f)$ for $f^{-1}(P_f \setminus \{\infty\}) \cup \{\infty\}$. We have the following commutative diagram:

$$\begin{array}{ccc} \hat{\mathbb{C}} \setminus f^{-1}(P_f) & \xrightarrow{w^{\tilde{\mu}}} & \hat{\mathbb{C}} \setminus w^{\tilde{\mu}}(f^{-1}(P_f)) \\ \downarrow f & & \downarrow g_{\mu,\tilde{\mu}} \\ \hat{\mathbb{C}} \setminus P_f & \xrightarrow{w^{\mu}} & \hat{\mathbb{C}} \setminus w^{\mu}(P_f). \end{array}$$

By the definition of σ, $\tilde{\mu} = f^*\mu$ so that the map $g = g_{\mu,\tilde{\mu}} = w^{\mu} \circ f \circ (w^{\tilde{\mu}})^{-1}$ defined on $\mathbb{C}$ is meromorphic. By remark 1, $g(z)$ is in $\mathcal{M}_2$ and in particular, $g'(z) \neq 0$.

Let $\sigma_* = d\sigma : T_\tau \to T_{\tilde{\tau}}$ and $\sigma^* : T^*_{\tilde{\tau}} \to T^*_\tau$ be the respective tangent and co-tangent maps of σ. Let η be a tangent vector at τ so that $\tilde{\eta} = \sigma_*\eta$ is the corresponding tangent vector at $\tilde{\tau}$. These tangent vectors can be pulled back to vectors $\xi, \tilde{\xi}$ at the origin in T_f by maps

$$\eta = (w^{\mu})^*\xi \quad \text{and} \quad \tilde{\eta} = (w^{\tilde{\mu}})^*\tilde{\xi}.$$

This results in the following commutative diagram,

$$\begin{array}{ccc} \tilde{\eta} & \xleftarrow{(w^{\tilde{\mu}})^*} & \tilde{\xi} \\ \uparrow f^* & & \uparrow g^* \\ \eta & \xleftarrow{(w^{\mu})^*} & \xi \end{array}$$

Now suppose $\tilde{q}$ is a co-tangent vector in $T^*_{\tilde{\tau}}$ and let $q = \sigma^*\tilde{q}$ be the corresponding co-tangent vector in T^*_τ. Then $\tilde{q} = \tilde{\phi}(w)dw^2$ is an integrable quadratic differential on $\hat{\mathbb{C}}$ whose only poles can be simple and occur at the points in $w^{\tilde{\mu}}(P_f)$; $q = \phi(z)dz^2$ is an integrable quadratic differential on $\hat{\mathbb{C}}$ whose only poles can be simple and occur at the points in $w^{\mu}(P_f)$. This implies that $q = \sigma_*\tilde{q}$ is also the push-forward integrable quadratic differential

$$q = g_*\tilde{q} = \phi(z)dz^2$$

of $\tilde{q}$ by g. This follows from the fact that $w^{\tilde{\mu}}$ takes the tesselation of fundamental domains for f to a tesselation of fundamental domains for g and on each fundamental domain g is a is a homeomorphism onto $\hat{\mathbb{C}} \setminus \{0, \lambda\}$ since $0, \lambda$ are the two asymptotic values of g. The coefficient $\phi(z)$ of q is therefore given by the standard transfer operator $\mathcal{L}$

$$\phi(z) = (\mathcal{L}\tilde{\phi})(z) = \sum_{g(w)=z} \frac{\tilde{\phi}(w)dw^2}{(g'(w))^2}. \tag{7}$$

Since $g'(w) \neq 0$, equation (7) implies the poles of q occur only at the images of the poles of $\tilde{q}$; the integrability implies these poles can only be simple. Therefore, as a meromorphic quadratic differential defined on $\hat{\mathbb{C}}$, q satisfies

$$||q|| \leq ||\tilde{q}||. \tag{8}$$

By formula (7) we have

$$< \tilde{q}, \tilde{\xi} >=< q, \xi >$$

which, together with inequality (8), implies

$$\|\tilde{\xi}\| \leq \|\xi\|.$$

This gives another proof that σ is weakly contracting. We can, however, prove strong contraction.

Lemma 4.

$$||q|| < ||\tilde{q}||$$

and

$$\|\tilde{\xi}\| < \|\xi\|.$$

Proof. Suppose there is a $\tilde{q}$ such that $||q|| = ||\tilde{q}||$ and that Z is the set of poles of $\tilde{q}$. Then, since g has no critical points, the poles of q must be contained in $g(Z)$. Using a change of variables on each fundamental domain we obtain the equaiities

$$\int_{\hat{\mathbb{C}}} |\phi(z)| dz\, d\overline{z} = \int_{\hat{\mathbb{C}}} |\tilde{\phi}(w)|\, dw d\overline{w} = \int_{\hat{\mathbb{C}}} \Big|\frac{\tilde{\phi}(w)}{(g'(w))^2}\Big|\, dz d\overline{z}.$$

The triangle inequality then implies that at every point z the argument of $\frac{\tilde{\phi}(w)}{(g'(w))^2}$ is the same; that is, for each pair w, w' with $g(w) = g(w') = z$, there is a positive real number a_z such that

$$\frac{\tilde{\phi}(w)}{(g'(w))^2} = a_z \frac{\tilde{\phi}(w')}{(g'(w'))^2}.$$

Thus, by formula (7) we see that $||q|| = ||\tilde{q}||$ implies $\phi(z) = \infty$ giving us a contradiction. □

Remark 6. *What we have shown is that $||q|| = ||\tilde{q}||$ implies*

$$g_* q = \phi(g(w)) = a\tilde{q}(w)$$

and therefore all the pre-images of all the poles are poles. That is,

$$g^{-1}(g(Z)) \subset Z \cup \Omega_g.$$

But this is a contradiction because $g^{-1}(g(Z))$ is an infinite set and $Z \cup \Omega_g$ is a finite set.

As an immediate corollary we have strong contraction.

Corollary 1. *For any two points τ and τ' in T_f,*

$$d_T\Big(\sigma(\tau), \sigma(\tau')\Big) < d_T(\tau, \tau').$$

Furthermore,

Proposition 2. *If σ has a fixed point in T_f, then this fixed point must be unique. This is equivalent to saying that a post-singularly finite f in $\mathcal{AV}2$ is combinatorially equivalent to at most one map $g \in \mathcal{M}2$.*

We can now finish the proof of the sufficiency under the assumptions that f has bounded geometry and that the meromorphic maps defined by

$$g_n = w^{\mu_n} \circ f \circ (w^{\mu_{n+1}})^{-1} \tag{9}$$

remain inside a compact subset of $\mathcal{M}2$.

Note that if $P_f = \{0, 1, \infty\}$, then f is a universal covering map of $\mathbb{C}^*$ and is therefore combinatorially equivalent to $e^{2\pi i z}$. Thus in the following argument, we assume that $\#(P_f) \geq 4$. Then, given our normalization conventions and the bounded geometry hypothesis we see that the functions g_n, $n = 0, 1, \ldots$ satisfy the following conditions:

1) $m = \#(w^{\mu_n}(P_f)) \geq 4$ is fixed.

2) $0, 1, \infty, g_n(1) \in w^{\mu_n}(P_f)$.

3) $\{0, 1, \infty\} \subseteq g_n^{-1}(w^{\mu_n}(P_f))$.

4) there is a $b > 0$ such that $d_{sp}(p_n, q_n) \geq b$ for any $p_n, q_n \in w^{\mu_n}(P_f)$.

Any integrable quadratic differential $q_n \in T^*_{\tau_n}$ has, at worst, simple poles in the finite set $P_{n+1,f} = w^{\mu_{n+1}}(P_f)$. Since $T^*_{\tau_n}$ is a finite dimensional linear space, there is a quadratic differential $q_{n,max} \in T^*_{\tau_n}$ with $\|q_{n,max}\| = 1$ such that

$$0 \leq a_n = \sup_{||q_n||=1} \|(g_n)_* q_n\| = \|(g_n)_* q_{n,max}\| < 1.$$

Moreover, by the bounded geometry condition, the possible simple poles of $\{q_{n,max}\}_{n=1}^{\infty}$ lie in a compact set and hence these quadratic differentials lie in a compact subset of the space of quadratic differentials on $\hat{\mathbb{C}}$ with, at worst, simples poles at $m = \#(P_f)$ points.

Let

$$a_{\tau_0} = \sup_{n \geq 0} a_n.$$

Let $\{n_i\}$ be a sequence of integers such that the subsequence $a_{n_i} \to a_{\tau_0}$ as $i \to \infty$. By our assumption of compactness, $\{g_{n_i}\}_{i=0}^{\infty}$ has a convergent subsequence, (for which we use the same notation) that converges to a holomorphic map $g \in \mathcal{M}2$. Taking a further subsequence if necessary, we obtain a convergent sequence of sets $P_{n_i,f} = w^{\mu_{n_i}}(P_f)$ with limit set X. By bounded geometry, $\#(X) = \#(P_f)$ and $d_{sp}(x,y) \geq b$ for any $x, y \in X$. Thus we can find a subsequence $\{q_{n_i,max}\}$ converging to an integrable quadratic differential q of norm 1 whose only poles lie in X and are simple. Now by lemma 4,

$$a_{\tau_0} = ||g_* q|| < 1.$$

Thus we have proved that there is an $0 < a_{\tau_0} < 1$, depending only on b and f and τ_0 such that

$$\|\sigma_*\| \leq \|\sigma^*\| \leq a_{\tau_0}.$$

Let l_0 be a curve connecting τ_0 and τ_1 in T_f and set $l_n = \sigma_f^n(l_0)$ for $n \geq 1$. Then $l = \cup_{n=0}^{\infty} l_n$ is a curve in T_f connecting all the points $\{\tau_n\}_{n=0}^{\infty}$. For each point $\tilde{\tau}_0 \in l_0$, we have $a_{\tilde{\tau}_0} < 1$. Taking the maximum gives a uniform $a < 1$ for all points in l_0. Since σ is holomorphic, a is an upper bound for all points in l. Therefore,

$$d_T(\tau_{n+1}, \tau_n) \leq a\, d_T(\tau_n, \tau_{n-1})$$

for all $n \geq 1$. Hence, $\{\tau_n\}_{n=0}^{\infty}$ is a convergent sequence with a unique limit point τ_∞ in T_f and τ_∞ is a fixed point of σ.

9.2.2 Compactness.

The final step in the proof of (1) $\Leftarrow$ (2) is to show the compactness assumption is valid. In the case of rational maps where the map is a branched covering of finite degree, the bounded geometry condition guarantees compactness, in the case of $f \in \mathcal{AV}2$, however, because the map is a branched covering of infinite degree, we need a further discussion of the topological properties of post-singular maps. We will show that for these maps there is a topologicial constraint that, together with bounded geometry condition guarantees compactness under the iteration process. The point is that this constraint and the bounded geometry condition together control the size of the fundamental domains so that they are neither too small nor too big.

A topological constraint. We start with $f \in \mathcal{AV}2$; recall $\Omega_f = \{0, \lambda\}$ is the set of asymptotic values of f and that we have normalized so that $f(0) = 1$. Suppose that this f is post-singularly finite; that is, P_f is finite so that the orbits $\{c_k = f^k(0)\}_{k=0}^{\infty}$ and $\{c'_k = f^k(\lambda)\}_{k=0}^{\infty}$ are both finite, and thus, preperiodic. Note that neither can be periodic because the asymptotic values are omitted. Consider the orbit of 0. Preperiodicity means there are integers $k_1 \geq 0$ and $l \geq 1$ such that $f^l(c_{k_1+1}) = c_{k_1+1}$. That is,

$$\{c_{k_1+1}, \ldots, c_{k_1+l}\}$$

is a periodic orbit of period l. Set $k_2 = k_1 + l$.

Let γ be a continuous curve connecting c_{k_1} to c_{k_2} in $\mathbb{R}^2$ which is disjoint from P_f, except at its endpoints. Because $f(c_{k_1}) = f(c_{k_2}) = c_{k_1+1}$, the image curve $\delta = f(\gamma)$ is a closed curve. We can choose γ once and for all such that δ separates 0 and λ; that is, so that δ is a non-trivial curve closed curve in $\hat{\mathbb{C}} \setminus \{0, \lambda\}$. The fundamental group $\pi_1(\hat{\mathbb{C}} \setminus \{0, \lambda\}) = \mathbb{Z}$ so the homotopy class $\eta = [\delta]$ in the fundamental group is an integer which essentially counts the number of fundamental domains between c_{k_1} and c_{k_2} and defines a "distance" between the fundamental domains. The integer η depends only on the choice of γ and since γ is fixed, so is η.

We now show that η is an invariant of the Thurston iteration procedure and is thus a topological constraint on the iterates.

Lemma 5. *Given $\tau_0 = [\mu_0] \in T_f$, let $\tau_n = \sigma^n(\tau_0) = [\mu_n]$ be the sequence generated by σ. Let w^{μ_n} be the normalized quasiconformal map with Beltrami coefficient μ_n Let $\gamma_{n+1} = w^{\mu_{n+1}}(\gamma)$, $\delta_n = w^{\mu_n}(\delta)$ and $\lambda_n = w^{\mu_n}(\lambda)$. Then $[\delta_n] \in \pi_1(\hat{\mathbb{C}} \setminus \{0, \lambda_n\}) = \eta$ for all n.*

Proof. The iteration defines the map

$$g_n = w^{\mu_n} \circ f \circ (w^{\mu_{n+1}})^{-1} \in \mathcal{AV}2$$

which is holomorphic since it preserves μ_0.

The continuous curve

$$\gamma_{n+1} = w^{\mu_{n+1}}(\gamma)$$

goes from $c_{k_1,n+1} = w^{\mu_{n+1}}(c_{k_1})$ to $c_{k_2,n+1} = w^{\mu_{n+1}}(c_{k_2})$. The image curve

$$\delta_n = g_n(\gamma_{n+1}) = w^{\mu_n}(f((w^{\mu_{n+1}})^{-1}(\gamma_{n+1}))) = w^{\mu_n}(f(\gamma)) = w^{\mu_n}(\delta)$$

is a closed curve through the point $c_{k_1+1,n} = w^{\mu_n}(c_{k_1+1})$.

From our normalization, it follows that

$$g_n(z) = g_{\alpha_n,\beta_n}(z) = \frac{\alpha_n e^{\beta_n z}}{(\alpha_n - \frac{1}{\alpha_n})e^{\beta_n z} + \frac{1}{\alpha_n}}. \tag{10}$$

and 0 is an omitted value for g_n. Since $\lambda_n = w^{\mu_n}(\lambda)$, it is also omitted for g_n and

$$\lambda_n = \frac{\alpha_n}{\alpha_n - \frac{1}{\alpha_n}} \in P_n. \tag{11}$$

Because

$$w^{\mu_n} : \hat{\mathbb{C}} \setminus \{0, \lambda\} \to \hat{\mathbb{C}} \setminus \{0, \lambda_n\}$$

is a normalized homeomorphism, it preserves homotopy classes and $\eta = [\delta_n] \in \pi_1(\hat{\mathbb{C}} \setminus \{0, \lambda_n\}) = \mathbb{Z}$. Thus the homotopy class of δ_n in the space $\hat{\mathbb{C}} \setminus \{0, \lambda_n\}$ is the same throughout the iteration. $\square$

Bounded geometry implies compactness. By hypothesis f has bounded geometry and by the normalization of f, $\Omega_f = \{0, \lambda\}$, $f(0) = 1$ so that $\{0, 1, \lambda, \infty\} \subset P_f$. Moreover the iterates

$$g_n = w^{\mu_n} \circ f \circ (w^{\mu_{n+1}})^{-1}$$

belong to $\mathcal{M}_2$.

Recall that $P_n = w^{\mu_n}(P_f)$ and because w^{μ_n} fixes $\{0, 1, \infty\}$ for all $n \geq 0$, $\{0, 1, \infty\} \subset P_n$. By equation (10),

$$g_n(1) = w^{\mu_n}(f(1)) = \frac{\alpha_n e^{\beta_n}}{(\alpha_n - \frac{1}{\alpha_n})e^{\beta_n} + \frac{1}{\alpha_n}} \in P_n.$$

so that

$$\{0, 1, \lambda_n, g_n(1), \infty\} \subseteq P_n.$$

If $\#(P_f) = 3$, then $\lambda = \infty$ and $f(1) = 1$. In this case, $\lambda = \lambda_n = \infty$, $g_n(1) = 1$ for all $n \geq 0$ and $\#(P_n) = 3$ so that $g_n(z) = e^{\beta_n z}$. The homotopy class of δ_n is always η, which is its winding number about the origin in the complex analytic sense. Thus $\beta_n = 2\pi i \eta$ for all n and $g_n = e^{2\pi i \eta z}$, which is the fixed under Thurston iteration and trivially lies in a compact subset in $\mathcal{M}2$.

From now on we assume that $\#(P_f) \geq 4$. We first prove the compactness of the iterates in the case that $\lambda = \infty$. By normalization, $\lambda_n = \infty$ and

$$g_n(z) = e^{\beta_n z}$$

for all $n \geq 0$.

Because f has bounded geometry, $g_n(1) \neq 1$ has a definite spherical distance from 1 and the sequence $\{|\beta_n|\}$ is bounded from below; that is, there is a constant $k > 0$ such that

$$k \leq |\beta_n|, \quad \forall n > 0.$$

Now we use the topological constraint to prove that the sequence $\{|\beta_n|\}$ is also bounded from above. We have $g_n'(z) = \beta_n g_n(z)$ and the homotopy class of δ_n is the winding number about the origin in the complex analytic sense, thus

$$\eta = \frac{1}{2\pi i} \oint_{\delta_n} \frac{dw}{w} = \frac{1}{2\pi i} \int_{\gamma_n} \frac{g_n'(z)}{g_n(z)} dz$$

$$= \frac{\beta_n}{2\pi i}(c_{k_2,n+1} - c_{k_1,n+1}).$$

Both $c_{k_2,n+1}, c_{k_1,n+1} \in P_{n+1}$, so bounded geometry implies the constant $k > 0$ above can be chosen so that

$$|c_{k_2,n+1} - c_{k_1,n+1}| \geq k.$$

Combining this with the formula for η we get

$$|\beta_n| \le \frac{\eta}{2\pi|c_{k_2,n+1} - c_{k_1,n+1}|} \le \frac{\eta}{2\pi k}.$$

and thus deduce that $\{g_n(z) = e^{\beta_n z}\}$ forms a compact subset in $\mathcal{M}2$.

Now let us prove compactness of the iterates when $\lambda \neq \infty$. In this case, since

$$\lambda_n = \frac{\alpha_n}{\alpha_n - \frac{1}{\alpha_n}} \in P_{n+1}$$

has a definite spherical distance from 0, 1, and ∞, bounded geometry implies there are two constants $0 < k < K < \infty$ such that

$$k \le |\alpha_n|, \;\; |\alpha_n - 1| \le K, \quad \forall\, n \ge 0.$$

In this case, we have that $g_n(1) \neq 1$ too. Since $g_n(1) \in P_{n+1}$, bounded geometry implies that the constant k can be chosen such that

$$k \le |\beta_n|, \;\; \forall n > 0.$$

Again we use the topological constraint to prove that $\{|\beta_n|\}$ is also bounded from above. Let

$$M_n(z) = \frac{\alpha_n z}{(\alpha_n - \frac{1}{\alpha_n})z + \frac{1}{\alpha_n}}$$

so that $g_n(z) = M_n(e^{\beta_n z})$. The map $M_n : \hat{\mathbb{C}} \setminus \{0, \infty\} \to \hat{\mathbb{C}} \setminus \{0, \lambda_n\}$ is a homeomorphism so it induces an isomorphism from the fundamental group $\pi_1(\hat{\mathbb{C}} \setminus \{0, \infty\})$ to the fundamental group $\pi_1(\hat{\mathbb{C}} \setminus \{0, \lambda_n\})$. Thus, η is the homotopy class $[\tilde{\delta}_n]$ where $\tilde{\delta}_n = M_n^{-1}(\delta_n)$.

Note that $\tilde{\delta}_n$ is the image of γ_{n+1} under $\widetilde{g}_n(z) = e^{\beta_n z}$. Since $\widetilde{\delta}_n$ is a closed curve in $\hat{\mathbb{C}} \setminus \{0, \infty\}$, η is the winding number of $\tilde{\delta}_n$ about the origin in the complex analytic sense, and we can compute

$$\eta = \frac{1}{2\pi i} \oint_{\widetilde{\delta_n}} \frac{dw}{w} = \frac{1}{2\pi i} \int_{\gamma_{n+1}} \frac{\tilde{g}_n'(z)}{\widetilde{g}_n(z)} dz = \frac{\beta_n}{2\pi i}(c_{k_2,n+1} - c_{k_1,n+1}).$$

As above, $c_{k_2,n+1}, c_{k_1,n+1} \in P_{n+1}$, and by bounded geometry there is a constant $k > 0$ such that

$$|c_{k_2,n+1} - c_{k_1,n+1}| \ge k,$$

so that

$$|\beta_n| \le \frac{\eta}{2\pi|c_{k_2,n+1} - c_{k_1,n+1}|} \le \frac{\eta}{2\pi k}.$$

This inequality proves that $\{g_n(z) = g_{\alpha_n,\beta_n}(z)\}$ forms a compact subset in $\mathcal{M}2$.

Finally, we have shown that in all cases the sequence $\{g_n\}$ is a compact subset in $\mathcal{M}2$. This combined with the proof in subsection 9.2.1 completes the proof of (1) $\Longleftarrow$ (2) in Theorem 1.

10 The Thurston Obstruction

Consider a simple closed curve $\gamma \in \widehat{\mathbb{C}} \setminus P_f$. If γ separates the asymptotic values 0 and λ, then $f^{-1}(\gamma)$ contains a single component which is an open curve with both its endpoints at ∞ and f. If γ does not separate 0 and λ, then $f^{-1}(\gamma)$ consists of an infinite number of simple closed curves in $\widehat{\mathbb{C}} \setminus P_f$ and f is a homeomorphism on each component.

Definition 13. *A simple closed curve $\gamma \subset \widehat{\mathbb{C}} \setminus P_f$ is called* non-peripheral *if every component of $\widehat{\mathbb{C}} \setminus \gamma$ contains at least two different points of P_f; it is* peripheral *if one of the components of $\widehat{\mathbb{C}} \setminus P_f$ contains one or no points of P_f.*

Since $P_f \setminus \{\infty\}$ is a finite set, it is contained in finitely many fundamental domains for f. Let D_f be the closure of the union of these fundamental domains together with together with their adjacent domains and all the domains that lie inbetween so that D_f is a connected set containing a finite number fundamental domains. This number depends on f and the deployment of the points of P_f. If γ is non-peripheral and does not separate 0 and λ, every closed component δ of $f^{-1}(\gamma)$ is contained in one or two fundamental domains and each non-peripheral δ must be contained in D_f. It follows that there are only finitely many non-peripheral closed components in $f^{-1}(\gamma)$; this number depends only on the number of fundamental domains in D_f and the number of points in P_f.

Definition 14. *A multi-curve*

$$\Gamma = \{\gamma_1, \cdots, \gamma_n\}$$

is a set of pairwise disjoint, non-homotopic, and non-peripheral curves on $\widehat{\mathbb{C}} \setminus P_f$.

Definition 15. *A multi-curve Γ is called f-stable if for any $\gamma \in \Gamma$, every non-peripheral component of $f^{-1}(\gamma)$ is homotopic to some element of Γ in $\widehat{\mathbb{C}} \setminus P_f$.*

Let $\mathbb{R}^\Gamma$ be the real vector space generated by a multi-curve Γ. We define the Thurston linear transformation,

$$f_\Gamma : \mathbb{R}^\Gamma \to \mathbb{R}^\Gamma$$

as follows: for any $\gamma_j \in \Gamma$, consider all the components of $f^{-1}(\gamma_j)$ homotopic to γ_i in $\widehat{\mathbb{C}} \setminus E$, and denote them by $\gamma_{i,j,\alpha}$. Let a_{ij} be the number of elements in the set $\{\gamma_{i,j,\alpha}\}$. Then

$$f_\Gamma(\gamma_j) = \sum_i a_{ij}\gamma_i.$$

Let A_Γ be the matrix of the linear transformation of f_Γ so that a_{ij} is the ij^{th}-entry of A_Γ. There may be infinitely many different multi-curves, however,

Lemma 6. *There are only finitely many possible matrices corresponding to Thurston transformations. The number depends on the number n_f of fundamental domains in D_f and the number p of points in P_f.*

Proof. Any multi-curve in $\hat{\mathbb{C}} \setminus P_f$ consists of at most $p-3$ simple closed curves so the matrix A_Γ has at most $(p-3)^2$ entries. Each entry is of the form $\#\{\alpha \mid \gamma_{i,j,\alpha}\} \leq n_f(p-3)$ so that the number of distinct matrices is $n_f(p-3)^3$. $\square$

Remark 7. *Note if γ belongs to an f-stable multi-curve, the degree of the map $f : \gamma \mapsto f(\gamma)$ is bounded by n_f.*

Proposition 3. *If Γ is an f-stable multi-curve, then the linear transformation f_Γ commutes with iteration; that is,*

$$(f^n)_\Gamma = (f_\Gamma)^n.$$

Proof. To show that the iteration is well defined we need to show that we don't get any new non-peripheral cuves. To see this, suppose γ is peripheral so that one component of its complement in $\widehat{\mathbb{C}} \setminus (P_f)$ is either a disk or a punctured disk. Denote this component by Ω. If Ω is a disk, then every component of $f^{-1}(\Omega)$ is also a disk in $\widehat{\mathbb{C}} \setminus P_f$. If Ω is a punctured disk and if the puncture is 0 or λ, then $f^{-1}(\Omega)$ is a disk with ∞ as a boundary point. If the puncture is any other point in E, then $f^{-1}(\Omega)$ consists of infinitely many components, each either contained in a single fundamental domain or overlapping two adjacent fundamental domains. Each component is either a disk or a punctured disk. Thus any component of $f^{-1}(\gamma)$ is peripheral.

Now let B_Γ be the matrix of the linear transformation f^2_Γ and denote its kj^{th}-entry by b_{kj}. Then

$$\begin{aligned}
b_{kj} &= \sum_{i,\alpha,\beta} \frac{1}{\deg(f^2|_{\gamma_{k,\{i,j,\alpha\},\beta}})} = \sum_{i,\alpha,\beta} \frac{1}{\deg(f|_{\gamma_{k,\{i,j,\alpha\},\beta}})} \frac{1}{\deg(f|_{\gamma_{i,j,\alpha}})} \\
&= \sum_{i,\alpha} \sum_{\beta} \frac{1}{\deg(f|_{\gamma_{k,i,\beta}})} \frac{1}{\deg(f|_{\gamma_{i,j,\alpha}})} = \sum_{i} \sum_{\alpha} a_{ki} \frac{1}{\deg(f|_{\gamma_{i,j,\alpha}})} \\
&= \sum_{i} a_{ki} a_{ij}
\end{aligned}$$

It follows that $(f^2)_\Gamma = (f_\Gamma)^2$. Inductively, we have $(f^n)_\Gamma = (f_\Gamma)^n$. $\square$

Remark 8. *Proposition 3 is not true for a multi-curve Γ which is not f-stable. The computation in the proof of Proposition 3 shows, however, that $(f^n)_\Gamma \geq (f_\Gamma)^n$ for any Γ.*

The *spectral radius of* A_Γ, denoted by either λ_Γ or $\lambda(A_\Gamma)$, is the maximum of the absolute values of all its eigenvalues. Since the entries of A_Γ are all non-negative numbers, the Perron-Frobenius Theorem guarantees that $\lambda(A_\Gamma)$ is positive and an eigenvalue for A_Γ whose corresponding eigenvector has non-negative coefficients.

Definition 16. *An f-stable multi-curve Γ is called a* Thurston obstruction *if the leading eigenvalue $\lambda(A_\Gamma) \geq 1$.*

Suppose Γ is a not necessarily f-stable multi-curve. We can rearrange $A = A_\Gamma$ into the form

$$A = \begin{pmatrix} A_{11} & 0 & \cdots & 0 \\ A_{21} & A_{22} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ A_{s1} & A_{s2} & \cdots & A_{ss} \end{pmatrix}$$

where all the blocks A_{jj} are either irreducible or 0. Moreover,

$$\lambda(A) = \sup_j \lambda(A_{jj}).$$

Let Γ_j denote the subset of curves in Γ corresponding to the j-th block in the decomposition above so that $A_{jj} = A_{\Gamma_j}$; Γ_j is called an *irreducible component* of Γ. It may not be f-stable. Let

$$\Gamma' = \cup_j \Gamma_j$$

where the union runs over all j such that $\lambda(A_{jj}) \geq 1$.

Definition 17. *Suppose Γ is an f-stable multi-curve and let $\gamma \in \Gamma$. If there exists a $\gamma' \in \Gamma'$ and an integer $k \geq 0$ such that γ is homotopic to a component $f^{-k}(\gamma')$ then the least such integer k is called the* depth of γ with respect to Γ. *The set of all curves in Γ with finite depth is denoted by Γ_0 and the set $\Gamma \setminus \Gamma_0$ is denoted by Γ_∞.*

Lemma 7. *If Γ is a Thurston obstruction, then Γ_0 is also a Thurston obstruction. In particular after applying a permutation,*

$$A_\Gamma = \begin{pmatrix} A_{\Gamma_\infty} & 0 \\ \bigstar & A_{\Gamma_0} \end{pmatrix}$$

where $\lambda(A_{\Gamma_\infty}) < 1$ and $\lambda(A_\Gamma) = \lambda(A_{\Gamma_0}) \geq 1$.

Proof. We claim that $\Gamma' \subset \Gamma_0$. To see this, take $\gamma \in \Gamma_0$; then there exists an integer k and an element $\gamma' \in \Gamma'$ such that γ is homotopic to a component of $f^{-k}(\gamma')$. It follows that any non-peripheral component $\widetilde{\gamma}$ of $f^{-1}(\gamma)$ is homotopic to a component of $f^{-(k+1)}(\gamma')$. Moreover since Γ is f-stable, there exists an element

$\gamma_i \in \Gamma$ which is homotopic to $\widetilde{\gamma}$. It follows that any non-peripheral component of $f^{-1}(\gamma)$ is homotopic to some $\gamma_i \in \Gamma$ whose depth with respect to Γ is at most $k+1$, and therefore $\gamma_i \in \Gamma_0$.

Now because $\Gamma' \subset \Gamma_0$, it follows that

$$\lambda(A_{\Gamma_\infty}) < 1 \quad \text{and} \quad \lambda(A_{\Gamma_0}) = \lambda(A_\Gamma) \geq 1$$

and A_Γ must be of the specified form. □

Since there are only finitely many possible Thurston matrices we have

Proposition 4. *There is a positive integer M_f depending on the number p of points in P_f and the number n_f of fundamental domains in D_f such that for any f-stable multi-curve Γ in $\widehat{\mathbb{C}} \setminus P_f$, the depth of any $\gamma \in \Gamma_0$ is less than or equal to M_f.*

Proposition 5. *Given an irreducible multi-curve Γ in $\widehat{\mathbb{C}} \setminus P_f$ (not necessarily f-stable) with $\lambda(f_\Gamma) \geq 1$, let $\mathbf{v}$ be the unique positive eigenvector of A_Γ corresponding to $\lambda(f_\Gamma) \geq 1$ with biggest coordinate 1. Then there exists a number $0 < \beta \leq 1$, depending only on p and n, such that the smallest coordinate of $\mathbf{v}$ is bounded below by β.*

Proof. Since there are only finitely many possible matrices for all the multi-curves, there are finitely many positive eigenvectors $\mathbf{v}$ with the biggest coordinate 1 corresponding to the maximal positive eigenvalues of the matrices. The eigenspace corresponding to the positive maximal eigenvalue of an irreducible non-negative matrix is one dimensional and is generated by a vector $\mathbf{v}$ all of whose entries are positive. Normalizing $\mathbf{v}$ so that its biggest coordinate is 1, determines the smallest coordinate and since there are only finitely many such vectors, there is a lower bound β on the smallest coordinate. □

11 Proof of (3) $\Longleftrightarrow$ (4)

Suppose $\Gamma = \{\gamma_i\}_{i=1}^n$ is a Thurston obstruction. Then we have a vector

$$\mathbf{v} = \sum_{i=1}^n b_i \gamma_i, \quad b_i \geq 0, \quad \sum_{i=1}^n b_i \neq 0,$$

such that $A_\Gamma \mathbf{v} = \lambda \mathbf{v}$. Suppose $b_1 \neq 0$. This implies that there is a least j such that $a_{1j} b_j \neq 0$. Thus $f^{-1}(\gamma_1)$ has a non-peripheral component γ' homotopic to γ_j such that $f : \gamma' \to \gamma_1$ is a homeomorphism. Up to homotopy, we may just as well assume $\gamma' = \gamma_j$ and relabel so that $j = 2$. By exactly the same argument, there is a γ_j for γ_2

which we can take as γ_3. Since Γ contains a finite number of elements, eventually, we get a set of curves forming a cycle

$$\{\gamma_1, \gamma_2, \cdots, \gamma_m\}, \quad m \le n,$$

such that $\gamma_{i+1 \pmod m}$ is a component of $f^{-1}(\gamma_i)$ and $f : \gamma_{i+1 \pmod m} \to \gamma_i$ is a homeomorphism. Such a cycle is called a *Levy cycle*. Thus we have shown that every Thurston obstruction contains a Levy cycle and every Levy cycle can be expanded to a Thurston obstruction. This completes the proof of the implication (3) $\Longleftrightarrow$ (4) of Theorem 1.

12 Proof of $(1) \Longrightarrow (3)$

A holomorphic map can have no Thurston obstruction or Levy cycle. This can be proved by using the Frame Mapping Theorem in Teichmüller space (see Jenkins [Jen] and Strebel [Str]). The argument is given in detail in [DH, Theorem 4.1].

13 Proof of $(2) \Longleftarrow (3)$

Much of the material in this section is adapted from [DH].

Any $\tau = [\mu] \in T_f$ represents a complex structure on $\widehat{\mathbb{C}}$ which makes $\widehat{\mathbb{C}} \setminus P_f$ into a hyperbolic Riemann surface R_τ that is Teichmüller equivalent to $\widehat{\mathbb{C}} \setminus w^\mu(P_f)$. For any simple closed curve γ on $\widehat{\mathbb{C}} \setminus P_f$, let

$$l(\gamma, \tau) \tag{12}$$

be the hyperbolic length of the unique simple closed geodesic homotopic to $w^\mu(\gamma)$ in $\widehat{\mathbb{C}} \setminus w^\mu(P_f)$.

Remark 9. *The definition of $l(\gamma, \tau)$ does not depend on $\mu \in \tau$.*

Define

$$w(\gamma, \tau) = -\log l(\gamma, \tau). \tag{13}$$

The next lemma is proved in [DH].

Lemma 8. *The function $T_f \to \mathbb{R}$ given by $\tau \mapsto w(\gamma, \tau)$ is Lipschitz, with Lipschitz constant 2.*

Represent the fundamental group of R_τ as a subgroup of $PSL(2, \mathbb{R})$ so that a lift of the simple closed geodesic homotopic to $w^\mu(\gamma)$ is the imaginary axis. The cyclic subgroup keeping the imaginary axis fixed is generated by $z \mapsto e^{l(\gamma,\tau)}$ and a fundamental domain for this subgroup is conformally equivalent to an annulus $A(\gamma, \tau)$ with modulus

$$\operatorname{mod}(A(\gamma, \tau)) = \frac{\pi}{2} \cdot \frac{1}{l(\gamma, \tau)}. \tag{14}$$

Moreover, it is standard (see [DH] for example) that if $l(\gamma, \tau)$ is less than a universal constant $l^* = \log 3 + 2\sqrt{2}$, R_τ contains an embedded annulus $a(\gamma, \tau)$ whose modulus $m(l)$ is a continuous and decreasing function of $l = l(\gamma, \tau)$ and satisfies

$$\frac{\pi}{2l} - 1 < m(l) < \frac{\pi}{2l}.$$

It is also standard that simple closed geodesics whose lengths are less than this constant are disjoint and that there are disjoint annuli with these as core curves. We shall always tacitly assume that the core curves of the annuli we work with have length less than this constant.

Rewriting, we have

$$\operatorname{mod}(A(\gamma, \tau)) - 1 < \operatorname{mod}(a(\gamma, \tau)) < \operatorname{mod}(A(\gamma, \tau)) \tag{15}$$

Before proceeding further, we define two concepts , state their properties and a technical lemma.

Definition 18. *Let κ be a real number. A sequence $\{a_n\}_{n=0}^{\infty}$ of real numbers is called* κ-quasi-nondecreasing *if for all $n_1 < n_2$ we have $a_{n_2} - a_{n_1} \geq \kappa$. A sequence is called* quasi-nondecreasing *if it is κ-quasi-nondecreasing for some κ.*

It is easy to check the following propositions and lemma.

Proposition 6. *Suppose $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ are two sequences. If $\{a_n\}_{n=0}^{\infty}$ is κ-quasi-nondecreasing and $|a_n - b_n| < r$ for all n, then $\{b_n\}$ is $(\kappa - 2r)$-quasi-nondecreasing.*

Proposition 7. *Suppose $\{a_n\}$ is quasi-nondecreasing and unbounded. Then $a_n \to +\infty$ as $n \to +\infty$.*

Lemma 9. *If $t \geq 1$, then $\log(t+1) - 1 < \log t$.*

Now, note that if $w(\gamma, \tau) \geq \log \frac{4}{\pi}$, then

$$1 \leq \operatorname{mod}(A(\gamma, \tau)) - 1.$$

Taking logarithms of all the terms in inequality (15), applying Lemma 9 and Equation (14), yields

$$\log(\pi/2) - 1 + w(\gamma, \tau) < \log \operatorname{mod}(a(\gamma, \tau)) < \log(\pi/2) + w(\gamma, \tau). \tag{16}$$

It follows that if $w(\gamma,\tau) \geq \log \frac{4}{\pi}$,[2]

$$|\log \operatorname{mod}(a(\gamma,\tau)) - w(\gamma,\tau)| < \log \pi/2. \tag{17}$$

Given a multi-curve Γ, denote the vectors of moduli $(\operatorname{mod}(A(\gamma,\tau)))$ and $(\operatorname{mod}(a(\gamma,\tau)))$ by $\operatorname{mod}(A(\Gamma,\tau))$ and $\operatorname{mod}(a(\Gamma,\tau))$ respectively. Define

$$\underline{\operatorname{mod}}(A(\Gamma,\tau)) = \min_{\gamma\in\Gamma}\{\operatorname{mod}(A(\gamma,\tau))\}$$

and

$$\underline{\operatorname{mod}}(a(\Gamma,\tau)) = \min_{\gamma\in\Gamma}\{\operatorname{mod}(a(\gamma,\tau))\}.$$

Lemma 10. *Let β be the constant in Proposition 5. Let Γ be an irreducible multi-curve Γ such that $\lambda_{A_\Gamma} > 1$. If $\tau_0 \in T_f$ and $\tau_n = \sigma_f^n(\tau_0)$, $n \geq 1$, then*

(i) $\underline{mod}(A(\Gamma,\tau_n)) \geq \beta\underline{mod}(a(\Gamma,\tau_0))$, and

(ii) $\underline{mod}(a(\Gamma,\tau_n)) \geq \beta\underline{mod}(a(\Gamma,\tau_0)) - 1$.

Proof. Given n, denote the matrix corresponding to the linear transformation $f_\Gamma^n : \mathbb{R}^\Gamma \to \mathbb{R}^\Gamma$ by B. It is easy to see that $B \geq A_\Gamma^n$.

Let $\mathbf{v}$ be the unique positive eigenvector with $|\mathbf{v}| = 1$ corresponding to the eigenvalue λ_{A_Γ}. Let $\mathbf{1}$ denote the vector all of whose coordinates are 1. Then, given $\tau_0 \in T_f$,

$$\operatorname{mod}(a(\Gamma,\tau_0)) \geq \underline{\operatorname{mod}}(a(\Gamma,\tau_0))\mathbf{1} \geq \underline{\operatorname{mod}}(a(\Gamma,\tau_0))\mathbf{v}.$$

For any $n \geq 1$ and $\tau_n = \sigma_f^n(\tau_0)$, let $\gamma_{i,j,\alpha}^n$ be the components of $f^{-n}(\gamma_j)$ homotopic to γ_i, and $A_{i,j,\alpha}^n$ be the component $f^{-n}(a(\gamma_j,\tau_0))$, whose core curve is $\gamma_{i,j,\alpha}$ and set $d_{i,j,\alpha}^n = \deg f^n|_{\gamma_{i,j,\alpha}^n}$. Then

$$\operatorname{mod}(a(\gamma_{i,j,\alpha}^n,\tau_n)) = \operatorname{mod}(a(\gamma_j,\tau_0))/d_{i,j,\alpha}^n,$$

The annuli $a(\gamma_{i,j,\alpha}^n,\tau_n)$ are disjoint and homotopic to the the curve γ_i. These can be lifted to the covering space of the annulus $A(\gamma_i,\tau_n)$ where, using the Grötzsch inequality we obtain

$$\sum_{\alpha,j} \operatorname{mod}(a(\gamma_{i,j,\alpha}^n,\tau_n)) \leq \operatorname{mod}(A(\gamma_i,\tau_n)).$$

Consequently, we have

$$\begin{aligned}\operatorname{mod}(A(\Gamma,\tau_n)) &\geq \operatorname{mod}(a(\Gamma,\tau_n)) \geq B\operatorname{mod}(a(\Gamma,\tau_0))\\ &\geq A_\Gamma^n\operatorname{mod}(a(\Gamma,\tau_0)) \geq A_\Gamma^n\,\underline{\operatorname{mod}}(a(\Gamma,\tau_0))\mathbf{v}\\ &\geq \underline{\operatorname{mod}}(a(\Gamma,\tau_0))\mathbf{v} \geq \beta\,\underline{\operatorname{mod}}(a(\Gamma,\tau_0))\mathbf{1}\end{aligned}$$

Hence for all $\gamma \in \Gamma$, we have $\underline{\operatorname{mod}}(A(\gamma,\tau_n)) \geq \beta\,\underline{\operatorname{mod}}(a(\Gamma,\tau_0))$.

The second conclusion follows from the first and inequality (15). ◻

[2]The constants chosen in this section are not necessarily the best possible. For example, any constant greater than 1 would do here.

Lemma 11. *If $a,\ b > 0$, $0 < \beta \leq 1$, and $e^a \geq \beta e^b - 1$, then $a - b \geq \log \beta - 1$.*

Proof. If $\beta e^b - 1 \geq 1$, then by Lemma 9, we have

$$\log(\beta e^b - 1) \geq \log(\beta e^b) - 1 \geq \log \beta + b - 1.$$

Hence by the assumption $e^a \geq \beta e^b - 1$, we have $a - b \geq \log \beta - 1$.

If $\beta e^b - 1 < 1$, then $b < \log 2 - \log \beta$. Since $a > 0$,

$$a - b > 0 - b = -b > \log \beta - \log 2 > \log \beta - 1.$$

□

For $\tau \in T_f$ and a multi-curve Γ, define

$$\underline{w}(\Gamma, \tau) = \min_{\gamma \in \Gamma} w(\gamma, \tau) \quad \text{and} \quad w(\Gamma, \tau) = \max_{\gamma \in \Gamma} w(\gamma, \tau).$$

Lemma 12. *Let Γ be an irreducible multi-curve such that the $\lambda_{A_\Gamma} \geq 1$. Suppose $\underline{w}(\Gamma, \tau_0) \geq \log(6/\beta) + \log \pi$ for some $\tau_0 \in T_f$. Then setting $\tau_n = \sigma_f^n(\tau_0)$, the sequence $\{\underline{w}(\Gamma, \tau_n)\}$, is $(\log \beta - 1 - 2 \log \pi)$-quasi-nondecreasing.*

Proof. For $\underline{w}(\Gamma, \tau_0) \geq \log(6/\beta) + \log \pi > \log \frac{4}{\pi}$, by (16), we have

$$\log \underline{\mathrm{mod}}(a(\Gamma, \tau_0)) \geq \log(6/\beta) \text{ or } \underline{\mathrm{mod}}(a(\Gamma, \tau_0)) \geq 6/\beta$$

and in particular, $\beta\, \underline{\mathrm{mod}}(a(\Gamma, \tau_0)) - 1 \geq 4$.

Applying Lemma 10, it follows that

$$\underline{\mathrm{mod}}(a(\Gamma, \tau_n)) \geq 4 \tag{18}$$

for all n and therefore that the elements of the sequence $y_n = \log \underline{\mathrm{mod}}(a(\Gamma, \tau_n))$ are all positive. Choose arbitrary integers $n_2 > n_1 \geq 0$, and set $a = y_{n_2},\ b = y_{n_1}$ and $n = n_2 - n_1$ so that by Lemma 10, $e^a \geq \beta e^b - 1$. Lemma 11, then implies $a - b \geq \log \beta - 1$, so that the sequence $\{y_n\}$ is $(\log \beta - 1)$-quasi-nondecreasing.

The hypothesis $\underline{w}(\Gamma, \tau_0) \geq \log(6/\beta) + \log \pi$ and inequalities (15) and (16), yield $\underline{\mathrm{mod}}(A(\Gamma, \tau_0)) \geq 4$ and then that $\underline{w}(\Gamma, \tau_n) \geq \log(4/\pi)$. Since $\mathrm{mod}(a(\gamma, \tau_n))$ is continuous and decreasing with respect to $l(\gamma, \tau_n)$, the minima are attained for the same $\gamma \in \Gamma$ as n varies. That is,

$$\underline{\mathrm{mod}}(a(\Gamma, \tau_n)) = \mathrm{mod}(a(\gamma, \tau_n)) \quad \text{and} \quad \underline{w}(\Gamma, \tau_n) = w(\gamma, \tau_n)$$

and therefore, by inequality (17)

$$|y_n - \underline{w}(\Gamma, \tau_n)| < \log \pi.$$

Thus $\underline{w}(\Gamma, \tau_n)$ is $(\log \beta - 1 - 2 \log \pi)$-quasi-nondecreasing. □

Lemma 13. *Let $k \geq 1$ be an integer, and set $D = d_{\mathcal{T}}(\tau_0, \tau_1)$. If γ_1, γ_2 are non-peripheral simple closed curves in $\widehat{\mathbb{C}} \setminus P_f$ such that some component of $f^{-k}(\gamma_1)$ is homotopic to γ_2, then*

$$w(\gamma_2, \tau_0) \geq w(\gamma_1, \tau_0) - k(\log n_f + 2D).$$

Proof. Consider $Y = f^{-k}(R_{\tau_0}) \subset R_{\tau_k}$. Then $f^k : \gamma_2 \to \gamma_1$ is a covering map of degree at most n_f^k. This implies that $l_Y(\gamma_2) \leq d^k l(\gamma_1, \tau_0)$. Since the inclusion map $Y \hookrightarrow R_{\tau_k}$ is length-decreasing, it follows that $l(\gamma_2, \tau_k) \leq n_f^k l(\gamma_1, \tau_0)$ and therefor that $w(\gamma_2, \tau_k) \geq w(\gamma_1, \tau_0) - k \log n_f$. Since $d_T(\tau_i, \tau_{i+1}) \leq d_T(\tau_0, \tau_1) = D$ and since the map $\gamma \mapsto w(\gamma, \tau_0)$ is a Lipschitz function with constant 2, we get

$$w(\gamma_2, \tau_0) \geq w(\gamma_2, \tau_k) - 2kD \geq w(\gamma_1, \tau_0) - k(2D + \log n_f).$$

□

Recall that p is the number of points in P_f.

Lemma 14. *Suppose Γ is an irreducible multi-curve. Then for all $\gamma_i,\ \gamma_j \in \Gamma$, and all $\tau \in T_f$,*

$$|w(\gamma_i, \tau) - w(\gamma_j, \tau)| \leq (p-3)(\log n_f + 2D).$$

Proof. Since Γ is irreducible, there is an integer $q \leq |\Gamma| \leq p - 3$ such that γ_i is homotopic to a preimage of $f^{-q}(\gamma_j)$. By Lemma 13,

$$w(\gamma_i, \tau) \geq w(\gamma_j, \tau) - (p-3)(\log n_f + 2D).$$

The lemma follows by symmetry. □

Let

$$A = -\log\log(3 + 2\sqrt{2}). \tag{19}$$

In the next proposition, β is the constant in Proposition 5 and M_f is the constant in Proposition 4.

Proposition 8. *Given an f-stable multi-curve Γ, suppose again that $\underline{w}(\Gamma, \tau_0) \geq \log(6/\beta) + \log \pi$. Write $\Gamma = \Gamma' \sqcup \Gamma''$, where Γ' is the union of the irreducible components Γ_j of Γ for which $\lambda(A_{\Gamma_j}) \geq 1$. Then*

(1) for all $\gamma \in \Gamma'$, $\{w(\gamma, \tau_n)\}_{n \geq 0}$ is κ-quasi-nondecreasing, with $\kappa = \log \beta - 1 - 2\log \pi - 2(p-3)(\log n_f + 2D)$;

(2) for all $\gamma \in \Gamma''$ and all $n \geq 0$,

$$w(\gamma, \tau_n) \geq \underline{w}(\Gamma', \tau_n) - M_f(\log n_f + 2D).$$

(3) There exists a constant J_A, such that if $\underline{w}(\Gamma, \tau_0) \geq J_A$, then for every $\gamma \in \Gamma$ and $n \geq 0$, $w(\gamma, \tau_n) \geq A$

Proof. For (1), let Γ_j be an irreducible component of Γ for which $\lambda_{A_{\Gamma_j}} \geq 1$. By hypothesis, $\underline{w}(\Gamma, \tau_0) \geq \log(6/\beta) + \log \pi$ so we can apply Lemma 12 to deduce that $\{\underline{w}(\Gamma_j, \tau_n)\}$ is $(\log \beta - 1 - 2\log \pi)$-quasi-nondecreasing.

Applying Lemma 14 and Property 6 to Γ_j we have that for each $\gamma \in \Gamma_j$, the sequence $\{w(\gamma, \tau_n)\}$ is $\kappa = (\log \beta - 1 - 2\log \pi - 2(p-3)(\log n_f + 2D))$-quasi-nondecreasing.

For (2), note that by Lemma 13 and (1), we have that for all $\gamma \in \Gamma''$, and all $i \geq 0$,

$$w(\gamma, \tau_n) \geq \min_{\gamma' \in \Gamma'}\{w(\gamma', \tau_n)\} - M_f(\log n_f + 2D).$$

For (3) set $J_A = \max\{\kappa + 2M_f(\log n_f + 2D), \log(6/\beta) + \log \pi\}$. Now by (1) and (2) of this lemma, if for any $\gamma \in \Gamma$ and all $n \geq 0$, $\underline{w}(\Gamma, \tau_0) \geq J_A$, then $\underline{w}(\Gamma, \tau_0) \geq J_A$ and $w(\gamma, \tau_n) \geq A$. $\square$

For $\tau \in T_f$, let

$$L_\tau = \{w(\gamma, \tau) \mid \gamma \text{ is a closed curve in } \widehat{\mathbb{C}} \setminus P_f\},$$

and set

$$w(\tau) = \sup L_\tau.$$

For any τ, there are only finitely many non-trivial closed curves γ in $\widehat{\mathbb{C}} \setminus P_f$ whose hyperbolic length $l(\gamma, \tau)$ is shorter than $l^* = \log(3+2\sqrt{2})$; it follows that $[A, \infty) \setminus L_\tau$ is a set of finitely many intervals. For any $J > 0$, let (a, b) be the leftmost interval in $[A, \infty) \setminus L_\tau$ of length J, and set

$$\Gamma_{J,\tau} = \{\gamma \in \widehat{\mathbb{C}} \setminus P_f \mid w(\gamma, \tau) \geq b\} \tag{20}$$

where γ runs over all simple closed curves in $\widehat{\mathbb{C}} \setminus P_f$.

Given $\tau = [\mu] \in T_f$, let $\tau' = \sigma_f(\tau) = [\mu']$ and $D = d(\tau, \tau')$. Set $f_\tau = w^\mu f (w^{\mu'})^{-1}$ and let $P_\tau = w^\mu(P_f)$, $P_{\tau'} = w^{\mu'}(P_f)$, and $P''_\tau = f_\tau^{-1}(P_\tau)$.

Lemma 15. *We have*

(1) If $J \geq \log n_f + 2D$ and if $\Gamma_{J,\tau} \neq \emptyset$, then $\Gamma_{J,\tau}$ is f-stable.

(2) Any simple closed geodesic on $\widehat{\mathbb{C}} \setminus P''_\tau$ of length less than $n_f e^{-b}$ is homotopic to a component of $f_\tau^{-1}(w^\mu(\gamma))$ for some $\gamma \in \Gamma_{J,\tau}$.

Proof. (1) Since $b > -\log l^*$, $e^{-b} < l^*$, and all the curves in $\Gamma_{J,\tau}$ are pairwise disjoint. If $\gamma \in \Gamma_{J,\tau}$, and γ' is a component of $f^{-1}(\gamma)$ such that $f : \gamma' \to \gamma$ is a map of degree $d_\alpha \leq n_f$, then

$$l_{\widehat{\mathbb{C}}\setminus P''_\tau}(w^{\mu'}(\gamma')) = d_\alpha l(\gamma, \tau).$$

Because $P_{\tau'} \subset P''_{\tau_\tau}$, it follows that

$$l(\gamma', \tau') < l_{\widehat{\mathbb{C}}\setminus P''_\tau}(w^{\mu'}(\gamma')) = d_\alpha l(\gamma, \tau) \leq n_f l(\gamma, \tau).$$

This in turn implies that

$$w(\gamma', \tau') > w(\gamma, \tau) - \log n_f.$$

By the assumption $w(\gamma, \tau) > b$ and by Lemma 8, we have

$$w(\gamma', \tau) > w(\gamma', \tau') - 2D > w(\gamma, \tau) - \log n_f - 2D > b - J > a$$

and therefore that γ' must be in $\Gamma_{J,\tau}$. This implies that $\Gamma_{J,\tau}$ is f-stable.

(2) For any $\gamma \in \Gamma_{J,\tau}$, and any non-peripheral component $w^{\mu'}(\gamma')$ of $f_\tau^{-1}(w^\mu(\gamma))$, (1) implies $l_{\widehat{\mathbb{C}}\setminus P''}(\gamma') < n_f e^{-b}$.

On the other hand, if $\alpha' \subset \widehat{\mathbb{C}} \setminus P''_\tau$ is a simple closed geodesic of length $< e^{-b}$, then $\alpha = f_\tau(\alpha')$ is a geodesic in $\widehat{\mathbb{C}} \setminus P_\tau$ of length $n_f e^{-b}$. Since $n_f e^{-b} < 2l*$, α must be simple. Then $\gamma = (w^\mu)^{-1}(\alpha)$ is a simple closed curve in $\widehat{\mathbb{C}} \setminus P_f$ such that $l(\gamma, \tau) \leq n_f e^{-b}$ or equivalently $w(\gamma, \tau) > b - \log n_f$. Now $b - a = J \geq \log n_f$ so $\gamma \in \Gamma_{J,\tau}$. □

Lemma 16. *Let X be Riemann surface, $P \subset X$ be a finite set, with $\sharp P = p > 0$. Set $X' = X \setminus P$, and choose $L < \log(3 + 2\sqrt{2})$. Let γ be simple closed geodesic on X, and $\gamma'_1, \cdots, \gamma'_s$ be the closed geodesics of X' homotopic to γ in X and of length $< L$. Set $l = l_X(\gamma)$, $l'_i = l_{X'}(\gamma'_i)$. Then*

(1) $s \leq p + 1$;

(2) for all i, $l'_i > l$;

(3) $1/l - 2/\pi - (p+1)/L < \sum_i 1/l'_i \leq 1/l + (p+1)/\pi$.

See [DH] for a proof.

Lemma 17. *There is a constant $C > 0$ depending only on p and d such that for any f-stable multi-curve Γ with $\lambda_\Gamma < 1/2$. Then for any $\tau \in T_f$, if $\underline{w}(\Gamma, \tau) \geq C$, then*

$$w(\Gamma, \sigma_f(\tau)) < w(\Gamma, \tau).$$

Proof. Suppose

$$\Gamma = \{\gamma_1, \cdots, \gamma_n\}.$$

Let

$$\mathbb{R}^\Gamma = \{\mathbf{v} = a_1\gamma_1 + \cdots + a_n\gamma_n\}$$

with the supremum norm

$$|\mathbf{v}| = \max_{1\le i\le n} |a_i|.$$

Then

$$|A| = \sup_{|\mathbf{v}|=1} |A\mathbf{v}|$$

be the corresponding norm for any matrix $A : \mathbb{R}^\Gamma \to \mathbb{R}^\Gamma$.

For any $\tau = [\mu]$, let $\tau' = \sigma_f(\tau) = [\mu']$ and $P_\tau = w^\mu(P_f)$ and $P_{\tau'} = w^{\mu'}(P_f)$. We have the following commuting diagram

$$\begin{array}{ccc} (\hat{\mathbb{C}}, P_f) & \xrightarrow{w^{\mu'}} & (\mathbb{C}, P_{\tau'}) \\ \downarrow f & & \downarrow g \\ (\hat{\mathbb{C}}, P_f) & \xrightarrow{w^{\mu}} & (\mathbb{C}, P_\tau) \end{array}$$

where g is a rational map. Let $P'' = g^{-1}(P_\tau)$. Define v_1,$v_2 \in \mathbb{R}^\Gamma$ by

$$v_1 = \begin{pmatrix} \frac{1}{l(\gamma_1,\tau)} \\ \vdots \\ \frac{1}{l(\gamma_n,\tau)} \end{pmatrix} \text{ and } v_2 = \begin{pmatrix} \frac{1}{l(\gamma_1,\tau')} \\ \vdots \\ \frac{1}{l(\gamma_n,\tau')} \end{pmatrix}.$$

Let $B > 0$ be a constant such that $L = de^{-B} < \log(3 + 2\sqrt{2}$. For any $\gamma_j \in \Gamma$, let $\gamma_{i,j,\alpha}$ be the components of $f^{-1}(\gamma_j)$ homotopic to γ_i. If $\underline{w}(\Gamma, \tau) > B$, then for any $\gamma_j \in \Gamma$, by Lemma 15, it follows that the hyperbolic length of all the components of $g^{-1}(\alpha_j)$ in $\widehat{\mathbb{C}} \setminus P''$ is less than $L = de^{-B}$, where α_j is the simple closed geodesic homotopic to γ_j in $\widehat{\mathbb{C}} \setminus P_\tau$.

Since $g : \widehat{\mathbb{C}} \setminus P'' \to \widehat{\mathbb{C}} \setminus P_\tau$ is a covering, we have

$$\sum_{j,\alpha} \frac{1}{l_{\widehat{\mathbb{C}}\setminus P''}(w^{\mu'}(\gamma_{i,j,\alpha}))} = [A_\Gamma v_1]_i.$$

By Lemma 16, for each i, we have

$$\frac{1}{l(\gamma_i, \tau')} < [A_\Gamma v_1]_i + \frac{1}{\pi} + \frac{pd}{L}.$$

So

$$|v_2| < \frac{1}{2}|v_1| + \frac{1}{\pi} + \frac{pd}{L}. \tag{21}$$

If

$$|v_1| > 2(\frac{1}{\pi} + \frac{pd}{L}),$$

we claim $|v_2| < |v_1|$. If not,

$$|v_2| \geq \frac{1}{2}|v_1| + \frac{1}{2}|v_1| \geq \frac{1}{2}|v_1| + \frac{1}{\pi} + \frac{pd^m}{L}.$$

It is a contraction to (21). The contradiction implies that if we choose

$$C = \max\left\{\log 2\Big(\frac{1}{\pi} + \frac{pd}{L}\Big), B\right\},$$

then for any $\tau \in T_f$, if $w(\Gamma, \tau) \geq C$, then

$$w(\Gamma, \sigma_f(\tau)) < w(\Gamma, \tau).$$

$\square$

In the rest of this section, we will use $\max\{-\log\log(3+2\sqrt{2}), C\}$ to replace the constant A in (19), that is, we consider

$$A = \max\{-\log\log(3+2\sqrt{2}), C\}. \tag{22}$$

Recall the definition of $\Gamma_{J,\tau}$ in (20) for a $\tau \in T_f$.

Since there is only finite many possible matrices for the all possible f-stable multi-curves Γ and there is no Thurston obstruction, there is a number $m \geq 1$, depending on d and p, such that $\lambda(A_\Gamma^m) < 1/2$ for any Γ.

We prove the theorem by contradiction. Suppose f has no bounded geometry. Take a $\tau_0 \in T_f$, let $\tau_n = \sigma_f^n(\tau_0)$. Since f has no bounded geometry, it follows that there exists a sequence of non-peripheral curves $\gamma_{n_j} \in \mathbb{C} \setminus P_f$, such that

$$w(\gamma_{n_j}, \tau_{n_j}) \to 0, \quad \text{as} \quad j \to \infty. \tag{23}$$

Let $D = d_{\mathcal{T}}(\tau_0, \tau_1)$. Let $J > C + 2mD$. Let $\Gamma_n = \Gamma_{J,\tau_n}$. By (23), we know there exists an n such that $\Gamma_n \neq \emptyset$. From the definition of Γ_n, we have that $\underline{w}(\Gamma_n, \tau_n) > C + 2mD$. Let $C(J)$ be an increasing function of J. Then we have infinitely many n such that

$$w(\Gamma_n, \tau_n) \geq C(J).$$

Let n_0 be the smallest such integer. Since σ_f is contracting, then by using Lemma 8, we have that

$$\underline{w}(\Gamma_{n_0}, \tau_{n_0 - m}) \geq C.$$

Lemma 17 implies that

$$w(\Gamma_{n_0}, \tau_{n_0}) < w(\Gamma, \tau_{n_0-m}) < C(J),$$

which is a contradiction. This contradiction implies that f must have bounded geometry. This completes the proof of (2) $\Longleftarrow$ (3) in Theorem 1.

13.1 Proof of (2) $\Longleftrightarrow$ (5) in Theorem 1.

We first prove (2) $\Longrightarrow$ (5). It is relatively easy. Since f has bounded geometry, there are a constant $a > 0$ and a $\tau_0 \in T_f$ such that $l(\gamma, \tau_n) \geq a$ for all simple closed curves in $\widehat{\mathbb{C}} \setminus P_f$. By the Lipschitz property (Lemma 8) of the map $\tau \to -\log l(\gamma, \tau)$, for any $\tau_0' \in T_f$, let $d = d_{\mathcal{T}}(\tau_0, \tau_0')$. Then $d_{\mathcal{T}}(\tau_n, \tau_n') \leq d_{\mathcal{T}}(\tau_0, \tau_0') \leq d$. This implies that

$$|\log l(\gamma, \tau_n) - \log(\gamma, \tau_n')| \leq 2d_{\mathcal{T}}(\tau_n, \tau_n') \leq 2d.$$

Thus $l(\gamma, \tau_n')$ can not tend to zero as n goes to ∞.

Now we prove (2) $\Longleftarrow$ (5). Since $\lambda(A_{\Gamma_\infty}) < 1$, we have a $k > 0$ such that $\lambda(A^k_{\Gamma_\infty}) < 1/2$.

Proposition 9. *Let $\tau_0 \in T_f$ and $\tau_n = \sigma_f^n(\tau_0)$ for $n > 0$. There exists a constant $C(J) > 0$ depending on p, d, ϵ_0, $D = d_{\mathcal{T}}(\tau_0, \tau_1)$ and $J \geq m(\log d + 2D + 1)$ such that if $w(\tau_0) > C(J)$, then $\Gamma = \Gamma_{J,\tau_0} \neq \emptyset$ is a stable multi-curve. Moreover, if $\Gamma_\infty \neq \emptyset$, then*

$$w(\Gamma_\infty, \tau_k) \leq w(\Gamma_\infty, \tau_0).$$

Lemma 18. *Let $J \geq m(\log d + 2D + 1)$. Suppose $w_(\tau_0) < C(J)$ and suppose $\Gamma = \Gamma_{J,\tau_m} \neq \emptyset$ for some $m \geq 0$. Let $E(J) = C(J) + 2mD$. If $\Gamma_\infty \neq \emptyset$, then for all n,*

$$w(\Gamma_\infty, \tau_n) < E(J).$$

Moreover, if $w(\gamma, \tau_m) \geq E(J)$, then $\gamma \in \Gamma_0$.

Proof. We prove the first inequality by contradiction. Suppose there is an $n > 0$ such that $w(\Gamma_\infty, \tau_n) \geq C(J) + 2mD$. Suppose n_0 is the first integer having this property. Then we have $w(\Gamma_\infty, x_{n_0 - m}) \geq C(J)$. Then by Proposition 9 and the fact that n_0 is the first integer such that $w(\Gamma_\infty, \tau_{n_0}) \geq C(J) + 2mD$, we have

$$w(\Gamma_\infty, \tau_{n_0}) \leq w(\Gamma_\infty, \tau_{n_0 - m}) < C(J) + 2mD.$$

This is a contradiction.

If $w(\gamma, \tau_m) \geq E(J) > C(J) \geq A + (p-3)J$, then $\gamma \in \Gamma_{J,\tau_k} = \Gamma$ since there are at most $p - 3$ simple closed curves in $\widehat{\mathbb{C}} \setminus P_f$ such that $w(\gamma, \tau_m) > A$. But $\gamma \notin \Gamma_\infty$ because of the first conclusion and the assumption. Therefore, $\gamma \in \Gamma_0$. $\square$

We prove it by contradiction. Suppose f has no bounded geometry, that is there exist an $\tau_0 \in T_f$ such that there exists γ_k and τ_{n_k} with $w(\gamma_k, \tau_{n_k}) \to \infty$. Let $D = d_{\mathcal{T}}(\tau_0, \tau_1)$.

We can find a $J_0 > C + 2mD$ such that $w(\tau_0) < C(J_0)$. Without lost of generality, we assume $J_0 = J_A + |A|$, where J_A is the number in Proposition 8. Since $C(J)$ is increasing with J, then $w(\tau_0) < C(J)$ for all $J \geq J_0$.

Fix $J \geq J_0$. We have some n such that $\Gamma_n = \Gamma_{J,\tau_n} \neq \emptyset$. Then $\underline{w}(\Gamma_n, \tau_n) > C + 2mD$. Write

$$\Gamma_n = \Gamma_{n,\infty} \cup \Gamma_{n,0}.$$

If $\Gamma_{n,\infty} \neq \emptyset$, similar to the proof of Lemma 17, there exists an integer m depending on p and d such that

$$w(\Gamma_{n,\infty}, \tau_m) < w(\Gamma_{n,\infty}, \tau_0).$$

Furthermore, we claim that for all $n \geq m$,

$$w(\Gamma_{n,\infty}, \tau_n) \leq C(J) + 2mD.$$

The claim implies that if $w(\gamma, \tau_k) > C(J) + 2mD$, then $w(\gamma, \tau_k) > A + (p-3)J$, then $\gamma \in \Gamma_{n_k,0}$.

We prove the claim by contradiction. Suppose there is an $n \geq 0$ such that $w(\Gamma_{n,\infty}, \tau_n) > C(J) + 2mD$. Suppose n_0 is the first integer having this property. From the definition of Γ_{n_0}, we know that $\underline{w}(\Gamma_{n_0}) > C + 2mD$. Thus, $\underline{w}(\Gamma_{n_0,\infty}) > C + 2mD$ and $\underline{w}(\Gamma_{n_0,\infty}, \tau_{n_0} - m) > C$ by Lemma 8. But, similar to the proof of Lemma 17,

$$w(\Gamma_{n_0,\infty}, \tau_{n_0}) \leq w(\Gamma_{n_0,\infty}, \tau_{n_0} - m) \leq C(J) + 2mD.$$

This is a contradiction. The contradiction implies the claim. This further implies that $\Gamma_{n_k,0} \neq \emptyset$ for k large. Thus Γ_{n_k} is a Thurston obstruction.

Define

$$\Gamma_J = \cup_{n \geq n_0} \Gamma_{J,\tau_n,0} \quad \text{and} \quad \mathcal{G} = \cup_{J \geq J_0} \Gamma_J.$$

Then $\mathcal{G}$ is a multi-curve.

Since $w(\gamma_k, \tau_{n_k}) \to \infty$, given any fixed $J \geq J_0$, $w(\gamma_k, \tau_{n_k}) \geq C(J) + 2mD$ for infinitely many k, Hence $\gamma_k \in \Gamma_J \subset \mathcal{G}$ infinitely often. Since $\mathcal{G}$ is finite, for some $\gamma \in \mathcal{G}$, we have $\gamma_k = \gamma$ for infinitely many k. Hence the set

$$\Gamma_u = \{\gamma \mid w(\gamma, \tau_n) \text{ is unbounded}\} \subset \mathcal{G}$$

is nonempty.

We first claim that $\Gamma_u = \mathcal{G} = \cap_{J \geq J_0} \Gamma_J$. We prove this claim as follows.

The inclusion $\cap_{J \geq J_0} \Gamma_J \subset \Gamma_u$ is clear. To see the other inclusion, let $\gamma \in \Gamma_u$. Given $J \geq J_0$, there exists some n such that $w(\gamma, \tau_n) > C(J) + 2mD$. By Lemma 18, $\gamma \in \Gamma_{J,\tau_n,0} \subset \Gamma_J$. Then $\gamma \in \cap_{J \geq J_0} \Gamma_J$.

Next we claim that $\Gamma_u = \Gamma_{J_c}$ for some $J_c \geq J_0$. We give a proof of this claim by using contradiction as follows: Otherwise, since $\Gamma_u = \cap_{J \geq J_0} \Gamma_J$, for all $J \geq J_0$, there exists a $\gamma_J \in \Gamma_J$ and $\gamma_J \notin \Gamma_u$. Since $\mathcal{G}$ is finite, there exists a γ, such that $\gamma = \gamma_J \in \Gamma_J$ for infinitely many J, while $\gamma \notin \Gamma_u$. This is a contradiction, since $\gamma \in \Gamma_J$ for infinitely many J implies that the sequence $\{w(\gamma, \tau_n)\}$ is unbounded. The contradiction proves the claim.

Now we have that $\Gamma_u = \cup_{n\geq n_0}\Gamma_{J_c,\tau_n,0}$ for some $J_c \geq J_0$. For every n such that $\Gamma = \Gamma_{J_c,\tau_n,0}$ is nonempty, then $\Gamma = \Gamma' \sqcup \Gamma''$. Applying Proposition 8, we know that if $\gamma' \in \Gamma'$, then the sequence $\{w(\gamma', \tau_n)\}$ is both unbounded and quasi-nondecreasing, so $w(\gamma', \tau_n) \to \infty$. (2) of Proposition 8 implies that $w(\gamma, \tau_n) \to \infty$ for all $\gamma \in \Gamma$. This implies that $\Gamma_{J_c,\tau_n,0} \subset \Gamma_c$ and thus $\Gamma_u = \Gamma_c$.

Finally we claim that for some $n = n_c$ we have $\Gamma_c = \Gamma_{J_c,\tau_{n_c},0}$. This last claim implies that Γ_c is a Thurston obstruction. We prove this last claim as follows:

Since $\Gamma_u = \Gamma_c = \cup_n\Gamma_{J_c,\tau_n,0}$, the inclusion $\Gamma_{J_c,\tau_n,0} \subset \Gamma_c \subset \mathcal{G}$ holds for all n. Since there are finitely many elements in $\mathcal{G}$, there exists an n_c so large that for all $\gamma \in \Gamma_c$ and all $n \geq n_c$ we have $w(\gamma, \tau_n) > C(J_c) + 2mD$. By Lemma 18, we have $\gamma \in \Gamma_{J_c,\tau_{n_c},0}$. Thus, $\Gamma_c = \Gamma_{J_c,n_c,0}$.

Finally, Γ_c depends only on f and not on the initial point τ_0, since for any γ, the map $\tau \mapsto w(\gamma, \tau)$ is Lipschitz with the Lipschitz constant two and σ_f is contracting. The theorem follows.

Corollary 2. *For any $\tau \in T_f$, there is a number $L = L(\tau) > 0$, such that for any simple closed curve $\gamma \notin \Gamma_c$, then $l(\gamma, \sigma_f^n(\tau)) \geq L$ for all n.*

Proof. Suppose $\tau_n = \sigma_f^n(\tau)$. Let γ be an arbitrary non-peripheral, essential, simple closed curve. If $w(\gamma, \tau_n) > C(J_c) + 2mD$ for some n, then $\gamma \in \Gamma_{J_c,\tau_n,0}$. The proof of the above theorem show that $\gamma \in \Gamma_c$. Thus if $\gamma \notin \Gamma_c$, $w(\gamma, \tau_n) \leq C(J_c) + 2mD$ for all n. Let $L = e^{-C(J_c)-2mD}$. Then $l(\gamma, \tau_n) \geq L$ for all n if $\gamma \notin \Gamma_c$. □

14 Proof of $(3) \Longrightarrow (5)$ in Theorem 1.

Suppose $\Gamma_c \neq \emptyset$. First we prove that Γ_c is a multi-curve. For any $\gamma_1, \gamma_2 \in \Gamma_c$, there is a τ_0 and $n > 0$ such that $l(\gamma_i, \tau_n) \leq \log(3 + 2\sqrt{2})$ for $i = 1, 2$, where $\tau_n = \sigma_f^n(\tau_0)$. By the collar lemma, γ_1 and γ_2 are disjoint. Moreover, it follows that Γ_c contains at most $p - 3$ simple closed non-peripheral curves.

Next we prove Γ_c is f-stable. For any $\gamma \in \Gamma_c$, let γ' be a non-peripheral simple closed curve in $f^{-1}(\gamma)$. Then $f : \gamma' \to \gamma$ is a covering map of degree $d' \leq d = \deg f$. For any $\tau_0 \in T_f$ and any $n > 0$, let $\tau_n = \sigma_f^n(\tau_0) = [\mu_n]$. Let $\tau_{n+1} = [\mu_{n+1}]$. Then

$$g_n = w^{\mu_n} \circ f \circ (w^{\mu_{n+1}})^{-1} : \widehat{\mathbb{C}} \setminus w^{\mu_{n+1}}(f^{-1}(P_f)) \to \widehat{\mathbb{C}} \setminus w^{\mu_n}(P_f)$$

is a holomorphic covering. This implies that $l(\gamma', \tau_{n+1})$ is less than the hyperbolic length of γ' measured in $\widehat{\mathbb{C}} \setminus w^{\mu_{n+1}}(f^{-1}(P_f))$ which is $d'l(\gamma, \tau_n) \leq dl(\gamma, \tau_n)$. Thus, $l(\gamma', \tau_{n+1}) \to 0$ as $n \to \infty$. so we have that $\gamma' \in \Gamma_c$. Therefore, Γ_c is f-stable.

Now we prove that $\lambda(A_{\Gamma_c}) \geq 1$ as follows. If the last inequality does not hold, we have a $m > 0$ such that $\lambda(A_{\Gamma_c}^m) < 1/2$. Let $J > C + 2mD$, we can take a $\tau \in T_f$ such that $\underline{w}(\Gamma_c, \tau_n) > C + 2mD$ for all n, where $D = d_{\mathcal{T}}(\tau, \sigma_f(\tau))$. Let $C(J)$ be large enough such that $w(\Gamma_c, \tau) < C(J)$. By the definition of Γ_c, it follows that there exists an n such that $w(\Gamma_c, \tau_n) > C(J)$. Let n_0 be first such integer. By Lemma 17, Since σ_f is contracting, it follows that $\underline{w}(\Gamma_c, \tau_{n_0-m}) \geq C$ by Lemma 8. From the fact that n_0 is the first integer n such that $w(\Gamma_c, \tau_n) \geq C(J)$, it follows that

$$w(\Gamma_c, \tau_{n_0}) < w(\Gamma_c, \tau_{n_0-m}) < C(J),$$

which contradicts the assumption that $w(\Gamma_c, \tau_{n_0}) \geq C(J)$. So we have that $\lambda(A_{\Gamma_c}) \geq 1$. Therefore Γ_c is a Thurston obstruction. This completes the proof of (3) $\Longrightarrow$ (5) in Theorem 1.

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